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GENERALIZED PARTON DISTRIBUTIONS IN TERMS OF LIGHT-CONE WAVE FUNCTIONS

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The matrix elements of local operators such as electromagnetic current, energy momentum tensor, angular momentum, and generalized parton distributions have exact representations in terms of light-cone Fock state wavefunctions of bound states such as hadrons. We present formulae which express the form factors and the generalized parton distributions in terms of the light-cone wavefunctions.

1. Introduction

The light-cone expansion is constructed by quantizing QCD at fixed light-cone time $\tau = t + z$ and forming the invariant light-cone Hamiltonian: $H_{LC}^{QCD} = P^+P^- - \vec{P}_\perp^2$ where $P^\pm = P^0 \pm P^z$. The proton state, for example, satisfies: $H_{LC}^{QCD} \,|\, \psi_p\rangle = M_p^2 \,|\, \psi_p\rangle$. The expansion of the proton eigensolution $|\, \psi_p\rangle$ on the color-singlet $B = 1$, $Q = 1$ eigenstates $\{|\, n\rangle\}$ of the free Hamiltonian $H_{LC}^{QCD}(g = 0)$ gives the light-cone Fock expansion [1,2]:

$$\left|\psi_p(P^+, \vec{P}_\perp)\right\rangle = \sum_n \prod_{i=1}^{n} \frac{\mathrm{d}x_i \, \mathrm{d}^2\vec{k}_{\perp i}}{\sqrt{x_i}\, 16\pi^3} \; 16\pi^3 \delta\left(1 - \sum_{i=1}^{n} x_i\right) \delta^{(2)}\left(\sum_{i=1}^{n} \vec{k}_{\perp i}\right) \times\psi_n(x_i, \vec{k}_{\perp i}, \lambda_i) \left| n;\, x_i P^+, x_i \vec{P}_\perp + \vec{k}_{\perp i}, \lambda_i \right\rangle . \qquad (1)$$

The light-cone momentum fractions $x_i = k_i^+/P^+$ and $\vec{k}_{\perp i}$ represent the relative momentum coordinates of the QCD constituents. The physical transverse momenta are $\vec{p}_{\perp i} = x_i \vec{P}_\perp + \vec{k}_{\perp i}$. The λ_i label the light-cone spin projections s^z of the quarks and gluons along the quantization direction z. The physical gluon polarization vectors $\epsilon^\mu(k,\ \lambda = \pm 1)$ are specified in light-cone gauge by the conditions $k \cdot \epsilon = 0,\ \eta \cdot \epsilon = \epsilon^+ = 0$. The n-particle states are normalized as

$$\left\langle n;\, p_i'^+, \vec{p}'_{\perp i}, \lambda_i' \;\middle|\, n;\, p_i^+, \vec{p}_{\perp i}, \lambda_i \right\rangle = \prod_{i=1}^{n} 16\pi^3 p_i^+ \delta(p_i'^+ - p_i^+)\ \delta^{(2)}(\vec{p}'_{\perp i} - \vec{p}_{\perp i})\ \delta_{\lambda_i' \lambda_i} \, . \qquad (2)$$

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The light-cone wavefunctions $\psi_n(x_i, \vec{k}_{\perp i}, \lambda_i)$ are universal, process independent, and thus control all hadronic reactions. Given the light-cone wavefunctions, one can compute the moments of the helicity distributions measurable in polarized deep inelastic experiments [2]. Exclusive semi-leptonic B-decay amplitudes involving timelike currents [3], electromagnetic and gravitational form factors of hadrons [4], and generalized parton distributions appearing in deeply virtual Compton scattering can also be evaluated exactly in terms of light-cone wavefunctions [5].

2. An Example of Light-Cone Fock State Decomposition and Wavefunction

In the language of light-cone quantization, the electron anomalous magnetic moment $a_e = \alpha/2\pi$ is due to the one-fermion one-gauge boson Fock state component of the physical electron. We employ light-cone gauge $A^+ = 0$ so that the gauge boson polarizations are physical. The light-cone fractions $x_i = k_i^+/P^+$ are positive: $0 < x_i \leq 1$, $\sum_i x_i = 1$. We adopt a non-zero boson mass λ for the sake of generality.

The two-particle Fock state for an electron with $J^z = +\frac{1}{2}$ has four possible spin combinations:

$$\begin{aligned}
&\left|\Psi^{\uparrow}_{\rm two\ particle}(P^+, \vec{P}_\perp = \vec{0}_\perp)\right\rangle \qquad (3)\\
&= \int \frac{\mathrm{d}^2\vec{k}_\perp \mathrm{d}x}{\sqrt{x(1-x)}16\pi^3}\Big[\ \psi^{\uparrow}_{+\frac{1}{2}\,+1}(x,\vec{k}_\perp)\left|+\frac{1}{2}\,+1\,;\ xP^+\,,\ \vec{k}_\perp\right\rangle \\
&\qquad +\psi^{\uparrow}_{+\frac{1}{2}\,-1}(x,\vec{k}_\perp)\left|+\frac{1}{2}\,-1\,;\ xP^+\,,\ \vec{k}_\perp\right\rangle \\
&\qquad +\psi^{\uparrow}_{-\frac{1}{2}\,+1}(x,\vec{k}_\perp)\left|-\frac{1}{2}\,+1\,;\ xP^+\,,\ \vec{k}_\perp\right\rangle \\
&\qquad +\psi^{\uparrow}_{-\frac{1}{2}\,-1}(x,\vec{k}_\perp)\left|-\frac{1}{2}\,-1\,;\ xP^+\,,\ \vec{k}_\perp\right\rangle\ \Big]\ ,
\end{aligned}$$

where the two-particle states $|s^z_{\rm f}, s^z_{\rm b};\ xP^+, \vec{k}_\perp\rangle$ are normalized as in (2). Here $s^z_{\rm f}$ and $s^z_{\rm b}$ denote the z-component of the spins of the constituent fermion and boson, respectively. The wavefunctions can be evaluated explicitly in QED perturbation theory using the rules given in Refs. [2,4,6]:

$$\left\{\begin{array}{l}
\psi^{\uparrow}_{+\frac{1}{2}\,+1}(x,\vec{k}_\perp) = -\sqrt{2}\frac{(-k^1+{\rm i}k^2)}{x(1-x)}\,\varphi\ ,\\
\psi^{\uparrow}_{+\frac{1}{2}\,-1}(x,\vec{k}_\perp) = -\sqrt{2}\frac{(+k^1+{\rm i}k^2)}{1-x}\,\varphi\ ,\\
\psi^{\uparrow}_{-\frac{1}{2}\,+1}(x,\vec{k}_\perp) = -\sqrt{2}(M-\frac{m}{x})\,\varphi\ ,\\
\psi^{\uparrow}_{-\frac{1}{2}\,-1}(x,\vec{k}_\perp) = 0\ ,
\end{array}\right. \tag{4}$$

where

$$\varphi = \varphi(x, \vec{k}_\perp) = \frac{\frac{e}{\sqrt{1-x}}}{M^2 - \frac{\vec{k}_\perp^2 + m^2}{x} - \frac{\vec{k}_\perp^2 + \lambda^2}{1-x}} \ . \tag{5}$$

Similarly, the wavefunctions for an electron with $J^z = -\frac{1}{2}$ are given by

$$\begin{cases} \psi^{\downarrow}_{+\frac{1}{2}\,+1}(x, \vec{k}_\perp) = 0 \ , \\ \psi^{\downarrow}_{+\frac{1}{2}\,-1}(x, \vec{k}_\perp) = -\sqrt{2}\ (M - \frac{m}{x})\, \varphi \ , \\ \psi^{\downarrow}_{-\frac{1}{2}\,+1}(x, \vec{k}_\perp) = -\sqrt{2}\ \frac{-k^1 + ik^2}{1-x}\, \varphi \ , \\ \psi^{\downarrow}_{-\frac{1}{2}\,-1}(x, \vec{k}_\perp) = -\sqrt{2}\ \frac{k^1 + ik^2}{x(1-x)}\, \varphi \ . \end{cases} \tag{6}$$

The coefficients of φ in (4) and (6) are the matrix elements of $\frac{\overline{u}(k^+, k^-, \vec{k}_\perp)}{\sqrt{k^+}} \gamma \cdot \epsilon^* \frac{u(P^+, P^-, \vec{P}_\perp)}{\sqrt{P^+}}$ which are the numerators of the wavefunctions corresponding to each constituent spin s^z configuration. The two boson polarization vectors in light-cone gauge are $\epsilon^\mu = (\epsilon^+ = 0\ , \epsilon^- = \frac{2\vec{\epsilon}_\perp \cdot \vec{k}_\perp}{k^+}, \vec{\epsilon}_\perp)$ where $\vec{\epsilon} = \vec{\epsilon}_\perp^{\,\uparrow,\downarrow} = \mp(1/\sqrt{2})(\widehat{x} \pm i\widehat{y})$. The polarizations also satisfy the Lorentz condition $k \cdot \epsilon = 0$. In the above we have generalized the framework of QED by assigning a mass M to the external electrons, but a different mass m to the internal electron lines and a mass λ to the internal photon line [6]. The idea behind this is to model the structure of a composite fermion state with mass M by a fermion and a vector constituent with respective masses m and λ. In next sections we will express the form factors in terms of the example light-cone wavefunctions presented in this section in order to show the meaning of the formulae clearly. These formulae can be generalized to those expressed in terms of general light-cone wavefunctions [4,5].

3. Electromagnetic Form Factors

For a spin-$\frac{1}{2}$ composite system, the Dirac and Pauli form factors $F_1(q^2)$ and $F_2(q^2)$ are defined by

$$\langle P'|J^\mu(0)|P\rangle \ = \ \overline{u}(P') \left[F_1(q^2)\gamma^\mu + F_2(q^2)\frac{i}{2M}\sigma^{\mu\alpha}q_\alpha \right] u(P) \ , \tag{7}$$

where $q^\mu = (P' - P)^\mu$ and $u(P)$ is the bound state spinor. In the light-cone formalism the Dirac and Pauli form factors are conveniently identified from the spin-conserving and spin-flip vector current matrix elements of the J^+ current [6]:

$$\left\langle P + q, \uparrow \left| \frac{J^+(0)}{2P^+} \right| P, \uparrow \right\rangle = F_1(q^2) \ , \tag{8}$$

$$\left\langle P+q,\uparrow \left| \frac{J^+(0)}{2P^+} \right| P,\downarrow \right\rangle = -(q^1-\mathrm{i}q^2)\frac{F_2(q^2)}{2M} \ . \tag{9}$$

We use the light-cone frame: $q = (q^+,\vec{q}_\perp,q^-) = \left(0,\vec{q}_\perp,\frac{-q^2}{P^+}\right)$, $P = (P^+,\vec{P}_\perp,P^-) = \left(P^+,\vec{0}_\perp,\frac{M^2}{P^+}\right)$, where $q^2 = -2P\cdot q = -\vec{q}_\perp^{\,2}$. We use the component notation $a = (a^+,\vec{a}_\perp,a^-)$ and our metric is defined by $a^\pm = a^0 \pm a^3$ and $a\cdot b = \frac{1}{2}(a^+b^- + a^-b^+) - \vec{a}_\perp\cdot\vec{b}_\perp$.

We can express the Dirac and Pauli form factors $F_1(q^2)$ and $F_2(q^2)$ using the light-cone wavefunction. From (4) and (8) we have

$$\begin{aligned} F_1(q^2) \ &= \ \left\langle \Psi^\uparrow(P^+,\vec{P}_\perp=\vec{q}_\perp))|\Psi^\uparrow(P^+,\vec{P}_\perp=\vec{0}_\perp)\right\rangle \qquad (10) \\ = \int \frac{\mathrm{d}^2\vec{k}_\perp \mathrm{d}x}{16\pi^3} & \Big[\psi^{\uparrow\ *}_{+\frac{1}{2}\,+1}(x,\vec{k}'_\perp)\psi^{\uparrow}_{+\frac{1}{2}\,+1}(x,\vec{k}_\perp) + \psi^{\uparrow\ *}_{+\frac{1}{2}\,-1}(x,\vec{k}'_\perp)\psi^{\uparrow}_{+\frac{1}{2}\,-1}(x,\vec{k}_\perp) \\ & +\psi^{\uparrow\ *}_{-\frac{1}{2}\,+1}(x,\vec{k}'_\perp)\psi^{\uparrow}_{-\frac{1}{2}\,+1}(x,\vec{k}_\perp)\Big] \ , \end{aligned}$$

where

$$\vec{k}'_\perp = \vec{k}_\perp + (1-x)\vec{q}_\perp \ . \tag{11}$$

The Pauli form factor is obtained from the spin-flip matrix element of the J^+ current. From (4), (6) and (9) we have

$$\begin{aligned} -\frac{(q^1-\mathrm{i}q^2)}{2M}\ F_2(q^2) \ &= \ \left\langle \Psi^\uparrow(P^+,\vec{P}_\perp=\vec{q}_\perp))|\Psi^\downarrow(P^+,\vec{P}_\perp=\vec{0}_\perp)\right\rangle \qquad (12) \\ = \int \frac{\mathrm{d}^2\vec{k}_\perp \mathrm{d}x}{16\pi^3} & \Big[\psi^{\uparrow\ *}_{+\frac{1}{2}\,-1}(x,\vec{k}'_\perp)\psi^{\downarrow}_{+\frac{1}{2}\,-1}(x,\vec{k}_\perp) + \psi^{\uparrow\ *}_{-\frac{1}{2}\,+1}(x,\vec{k}'_\perp)\psi^{\downarrow}_{-\frac{1}{2}\,+1}(x,\vec{k}_\perp)\Big]. \end{aligned}$$

4. Generalized Parton Distributions

We begin with the kinematics of virtual Compton scattering

$$\gamma^*(q) + p(P) \rightarrow \gamma(q') + p(P') \ . \tag{13}$$

We specify the frame by choosing a convenient parametrization of the light-cone coordinates for the initial and final proton: $P = \left(\ P^+\ ,\ \vec{0}_\perp\ ,\ \frac{M^2}{P^+}\ \right)$, $P' = \left((1-\zeta)P^+\ ,\ -\vec{\Delta}_\perp\ ,\ \frac{M^2+\vec{\Delta}_\perp^2}{(1-\zeta)P^+}\right)$, where M is the proton mass. The four-momentum transfer from the target is $\Delta = P - P' = \left(\zeta P^+\ ,\ \vec{\Delta}_\perp\ ,\ \frac{t+\vec{\Delta}_\perp^2}{\zeta P^+}\right)$, where $t = \Delta^2$. In addition, overall energy-momentum conservation requires $\Delta^- = P^- - P'^-$, which connects $\vec{\Delta}_\perp^2$, ζ, and t according to $t = 2P\cdot\Delta = -\frac{\zeta^2M^2+\vec{\Delta}_\perp^2}{1-\zeta}$.

In the limit $Q^2 = -q^2 \to \infty$ at fixed ζ and t the Compton amplitude is given by [7,8]

$$M^{IJ}(\vec{q}_\perp, \vec{\Delta}_\perp, \zeta) = \epsilon^I_\mu \, \epsilon^{*J}_\nu \, M^{\mu\nu}(\vec{q}_\perp, \vec{\Delta}_\perp, \zeta) = -e_q^2 \, \frac{1}{2\overline{P}^+} \int_{\zeta-1}^{1} \mathrm{d}x \qquad (14)$$

$$\times \ \{ \ t^{IJ}(x,\zeta) \ \overline{U}(P') \Big[H(x,\zeta,t) \ \gamma^+ \ + \ E(x,\zeta,t) \ \frac{i}{2M} \, \sigma^{+\alpha}(-\Delta_\alpha) \Big] U(P)$$

$$+ \ s^{IJ}(x,\zeta) \ \overline{U}(P') \Big[\widetilde{H}(x,\zeta,t) \ \gamma^+\gamma_5 \ + \ \widetilde{E}(x,\zeta,t) \ \frac{1}{2M} \, \gamma_5(-\Delta^+) \Big] U(P) \ \} \ ,$$

where $\overline{P} = \frac{1}{2}(P' + P)$. For circularly polarized initial and final photons ($I,\ J$ are $\uparrow$ or $\downarrow$)) presented below (6), we have $t^{\,\uparrow\uparrow}(x,\zeta) = t^{\,\downarrow\downarrow}(x,\zeta) = \frac{1}{x-i\epsilon} + \frac{1}{x-\zeta+i\epsilon}$, $s^{\,\uparrow\uparrow}(x,\zeta) = -s^{\,\downarrow\downarrow}(x,\zeta) = \frac{1}{x-i\epsilon} - \frac{1}{x-\zeta+i\epsilon}$, and $t^{\,\uparrow\downarrow}$, $t^{\,\downarrow\uparrow}$, $s^{\,\uparrow\downarrow}$ and $s^{\,\downarrow\uparrow}$ are zero.

Since the coupling of the electromagnetic current $e_q J^+(0)$ on the quark line is identical to the Compton amplitude with $e_q^2 \, t^{IJ}$ replaced simply by the quark charge e_q, one finds

$$\int_{\zeta-1}^{1} \frac{\mathrm{d}x}{1-\frac{\zeta}{2}} \ H(x,\zeta,t) = F_1(t) \ , \qquad \int_{\zeta-1}^{1} \frac{\mathrm{d}x}{1-\frac{\zeta}{2}} \ E(x,\zeta,t) = F_2(t) \ . \qquad (15)$$

Analogous sum rules relate $\widetilde{H}$ and $\widetilde{E}$ with the form factors of the axial vector current $J_5^\mu(y) = \overline{\psi}(y)\gamma^\mu\gamma_5\psi(y)$. The factors $(1-\zeta/2)$ in (15) appear because we use the normalization convention for the Compton form factors which involves $\overline{P}^+$ on the right-hand side of (14), and at the same time parametrize light-cone momentum fractions with respect to $P^+ = (1-\zeta/2)\overline{P}^+$.

In the domain $\zeta < x < 1$, for a general value of ζ between 0 and 1, we can express the generalized parton distributions as the overlap of the light-cone wavefunctions [5]:

$$\frac{\sqrt{1-\zeta}}{1-\frac{\zeta}{2}} \ H_{(2\to2)}(x,\zeta,t) \ - \ \frac{\zeta^2}{4(1-\frac{\zeta}{2})\sqrt{1-\zeta}} \ E_{(2\to2)}(x,\zeta,t) \qquad (16)$$

$$= \int \frac{\mathrm{d}^2\vec{k}_\perp}{16\pi^3} \Big[\psi^{\uparrow\,*}_{+\frac{1}{2}\,+1}(x',\vec{k}'_\perp)\psi^{\uparrow}_{+\frac{1}{2}\,+1}(x,\vec{k}_\perp) + \psi^{\uparrow\,*}_{+\frac{1}{2}\,-1}(x',\vec{k}'_\perp)\psi^{\uparrow}_{+\frac{1}{2}\,-1}(x,\vec{k}_\perp)$$

$$+\psi^{\uparrow\,*}_{-\frac{1}{2}\,+1}(x',\vec{k}'_\perp)\psi^{\uparrow}_{-\frac{1}{2}\,+1}(x,\vec{k}_\perp) \Big] \ ,$$

$$\frac{1}{\sqrt{1-\zeta}} \ \frac{(\Delta^1 - i\Delta^2)}{2M} \ E_{(2\to2)}(x,\zeta,t) \qquad (17)$$

$$= \int \frac{\mathrm{d}^2\vec{k}_\perp}{16\pi^3} \Big[\psi^{\uparrow\,*}_{+\frac{1}{2}\,-1}(x',\vec{k}'_\perp)\psi^{\downarrow}_{+\frac{1}{2}\,-1}(x,\vec{k}_\perp) + \psi^{\uparrow\,*}_{-\frac{1}{2}\,+1}(x',\vec{k}'_\perp)\psi^{\downarrow}_{-\frac{1}{2}\,+1}(x,\vec{k}_\perp) \Big] \ ,$$

where

$$x' = \frac{x-\zeta}{1-\zeta}, \quad \vec{k}'_\perp = \vec{k}_\perp - \frac{1-x}{1-\zeta} \ \vec{\Delta}_\perp \ . \qquad (18)$$

Analogous formulae hold in the domain $\zeta - 1 < x < 0$, where the struck parton in the target is an antiquark instead of a quark. In the domain $0 \leq x \leq \zeta$, the parton number changing $n+1 \to n-1$ ($\Delta n = -2$) off-diagonal transition matrix elements contribute. The formulae for the domain $0 \leq x \leq \zeta$ are given in Ref. [5]. The same situation also occurs in the heavy meson decays since the decay form factors are timelike [3]. In general, when the initial and final hadrons have different values of the + component of momentum, there are parton number changing contributions as well as parton number conserving ones.

5. Conclusions

The light-cone Fock representation allows one to compute the matrix elements of local currents as overlap integrals of the light-cone wavefunctions which are frame independent. In particular, we can evaluate the forward and non-forward matrix elements of the electroweak currents, moments of the deep inelastic structure functions, as well as the electromagnetic form factors. Given the local operators for the energy-momentum tensor $T^{\mu\nu}(x)$ and the angular momentum tensor $M^{\mu\nu\lambda}(x)$, one can directly compute momentum fractions, spin properties, the gravitomagnetic moment, and the form factors $A(q^2)$ and $B(q^2)$ appearing in the coupling of gravitons to composite systems [4,5]. We presented the formulae which express the electromagnetic form factors and generalized parton distributions as overlap integrals of the light-cone wavefunctions. These formulae provided useful physical intuitions in the related processes.

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