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Spontaneous Gravitational Instability of Star Distribution in a nonrotating Galaxy*

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Abstract

Gravitational instability of the distribution of stars in a galaxy is a well-known phenomenon in astrophysics. This work is a preliminary attempt to analyze this effect using the standard tools developed in accelerator physics. The result is first applied to nonrotating gallaxies with spherical and planar symmetries. Extensions to rotating galaxies are not studied here.

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Spontaneous Gravitational Instability

of Star Distribution in a nonrotating Galaxy

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1 Introduction

Consider a distribution of stars in a galaxy described by a distribution density $\rho(\vec{x}, \vec{v}, t)$ in the phase space $(\vec{x}, \vec{v})$. We wish to analyze the stability of this distribution of stars under the influence of their collective gravitational force. To simplify the problem, we will use a flat Euclidean space-time and will consider Newtonian, nonrelativistic dynamics only. The instability does not assume a specific cosmological model other than Newtonian gravity. If this approach turns out successful, a large arsenal of analysis tools can be transported from accelerator physics to this problem.

The instability we are interested in is self-generated, i.e. it occurs spontaneously. In particular, it does not require an initial "seed" fluctuation at the birth of the galaxy. The instability growth pattern as well as its rate of growth are intrinsic properties of the system. This gravitational instability is a wel-known problem; its first analysis was almost a century ago [1]. What we do in the following is to treat the same problem using the standard techniques developed in the study of collective instabilities in circular accelerators [2].

2 Dispersion Relation

Consider a particular star in the galaxy. The equations of motion of this star are

$$\begin{aligned}\dot{\vec{x}} &= \vec{v} \\ \dot{\vec{v}} &= G \int d\vec{v}' \int d\vec{x}' \, \frac{\rho(\vec{x}', \vec{v}', t)(\vec{x}' - \vec{x})}{|\vec{x}' - \vec{x}|^3} \end{aligned} \tag{1}$$

Note that these equations do not depend on the mass of the star under consideration.

Evolution of ρ is described by the Vlasov equation

$$\begin{aligned} & \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial \vec{x}} \cdot \dot{\vec{x}} + \frac{\partial \rho}{\partial \vec{v}} \cdot \dot{\vec{v}} \\ = & \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial \vec{x}} \cdot \vec{v} + \frac{\partial \rho}{\partial \vec{v}} \cdot G \int d\vec{v}' \int d\vec{x}' \, \frac{\rho(\vec{x}', \vec{v}', t)(\vec{x}' - \vec{x})}{|\vec{x}' - \vec{x}|^3} \\ = & 0 \end{aligned} \tag{2}$$

Let the galaxy distribution be given by an unperturbed distribution ρ_0 plus some small perturbation. Let the unperturbed distribution ρ_0 depend only on $\vec{v}$,

$$\rho_0 = \rho_0(\vec{v}) \tag{3}$$

This unperturbed distribution is uniform in $\vec{x}$, i.e. it is uniform in the infinite 3-D space. The function $\rho_0(\vec{v})$ is so far unrestricted, and is to be prescribed externally.

The perturbation around ρ_0 will have some structure in t and in $\vec{x}$. We Fourier decompose this structure and write

$$\rho(\vec{x}, \vec{v}, t) = \rho_0(\vec{v}) + \Delta\rho(\vec{v}) \, e^{-i\omega t + i\vec{k}\cdot\vec{x}} \tag{4}$$

The quantity $\Delta\rho$ is considered to be infinetisimal compared with ρ_0.

Substituting Eq.(4) into Eq.(2) and keeping only first order in $\Delta\rho$ yield

$$-i(\omega - \vec{v}\cdot\vec{k})\Delta\rho(\vec{v}) + G\left(\int d\vec{v}'\,\Delta\rho(\vec{v}')\right)\frac{\partial\rho_0(\vec{v})}{\partial\vec{v}}\cdot\vec{q}(\vec{k}) = 0 \tag{5}$$

where

$$\vec{q}(\vec{k}) \equiv \int d\vec{x}'\,\frac{e^{i\vec{k}\cdot(\vec{x}'-\vec{x})}(\vec{x}'-\vec{x})}{|\vec{x}'-\vec{x}|^3} = \int d\vec{y}\;\frac{e^{i\vec{k}\cdot\vec{y}}\,\vec{y}}{|\vec{y}|^3} \tag{6}$$

is a well-defined quantity depending only on $\vec{k}$; it is the Fourier transform of the Newton kernel, and might be called the graviton propagator. In fact, avoiding the singularity at the origin $\vec{k} = \vec{0}$, it can be shown that

$$\vec{q}(\vec{k}) = \frac{4\pi i}{|\vec{k}|^2}\,\vec{k} \tag{7}$$

Eq.(5) can be rewritten as

$$\Delta\rho(\vec{v}) = -iG\left(\int d\vec{v}'\,\Delta\rho(\vec{v}')\right)\frac{\frac{\partial\rho_0(\vec{v})}{\partial\vec{v}}\cdot\vec{q}(\vec{k})}{\omega - \vec{v}\cdot\vec{k}} \tag{8}$$

Integrating both sides over $\vec{v}$ and canceling out the mutual factor of $\int d\vec{v}'\,\Delta\rho(\vec{v}')$ then gives a dispersion relation that must be satisfied by ω and $\vec{k}$,

$$1 = -iG\int d\vec{v}\frac{\frac{\partial\rho_0(\vec{v})}{\partial\vec{v}}\cdot\vec{q}(\vec{k})}{\omega - \vec{v}\cdot\vec{k}} \tag{9}$$

We need to solve this dispersion relation for a given $\rho_0(\vec{v})$ to find the most unstable pattern of perturbation and its corresponding growth rate, as will be described next. This result, we hope, could say something about the characteristic dimension of galaxies.

3 Uniform Isotropic Galaxy

We next consider an unperturbed distribution that depends only on the magnitude of $\vec{v}$, i.e., let

$$\rho_0 = \rho_0(|\vec{v}|^2) \tag{10}$$

which gives

$$\frac{\partial \rho_0}{\partial \vec{v}} = 2\vec{v}\, \rho_0'(|\vec{v}|^2) \tag{11}$$

This is the case of a uniform isotropic (spherically symmetric) galaxy. Normalization condition is

$$\begin{aligned} \int_0^\infty 4\pi v^2 dv\, \rho_0(v^2) &= \rho_m \\ &= \text{volume mass density of stars} \end{aligned} \tag{12}$$

Substituting Eqs.(7) and (11) into Eq.(9) then gives

$$1 = \frac{8\pi G}{|\vec{k}|^2} \int d\vec{v}\, \rho_0'(|\vec{v}|^2) \frac{\vec{v}\cdot\vec{k}}{\omega - \vec{v}\cdot\vec{k}} \tag{13}$$

Let $\vec{k} = (0, 0, k)$, and choose coordinates so that $\vec{v} = v(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, Eq.(13) becomes, with a change of variable $u = \cos\theta$,

$$1 = \frac{16\pi^2 G}{k} \int_0^\infty v^3 dv\, \rho_0'(v^2) \int_{-1}^{1} du\, \frac{u}{\omega - kvu} \tag{14}$$

One must refrain from performing the integration over u at this time. Proper treatment of the singularity is first necessary. We then follow the standard technique used in accelerator physics on Landau damping [3]. The treatment amounts to adding an infinitesimal positive imaginary part to ω, i.e. $\omega \to \omega + i\epsilon$,

$$\mathcal{I}(\omega, kv) \equiv \int_{-1}^{1} du\, \frac{u}{\omega - kvu}$$

$$
\begin{aligned}
&\to \int_{-1}^{1} du \, \frac{u}{\omega + i\epsilon - kvu} \\
&= \text{P.V.} \int_{-1}^{1} du \, \frac{u}{\omega - kvu} - i \frac{\pi\omega}{k^2 v^2} \, H\left(1 - \left|\frac{\omega}{kv}\right|\right) \\
&= -\frac{2}{kv} - \frac{\omega}{k^2 v^2} \ln \left|\frac{\omega - kv}{\omega + kv}\right| - i \frac{\pi\omega}{k^2 v^2} \, H\left(1 - \left|\frac{\omega}{kv}\right|\right) \qquad (15)
\end{aligned}
$$

where P.V. means taking the principal value of the integral, and $H(x) = 1$ for $x > 0$ and 0 for $x < 0$ is the step function.

To be specific, we next take a uniform distribution of ρ_0,

$$
\rho_0(v^2) = \begin{cases} \frac{3\rho_m}{4\pi v_0^3} & \text{if } v^2 < v_0^2 \\ 0 & \text{otherwise} \end{cases} \qquad (16)
$$

with

$$
\rho_0'(v^2) = -\frac{3\rho_m}{8\pi v_0^4} \delta(v - v_0) \qquad (17)
$$

The quantity v_0^2 is related to the "temperature" of the stars. Substituting Eq.(17) into Eq.(14) gives the dispersion relation

$$
1 = -\frac{6\pi G \rho_m}{k v_0} \mathcal{I}(\omega, k v_0) \qquad (18)
$$

Substituting Eq.(15) into Eq.(18) then gives

$$
\lambda = \frac{1}{2 + x \ln \left|\frac{x-1}{x+1}\right| + i\pi x \, H(1 - |x|)} \qquad (19)
$$

where

$$
\lambda = \frac{6\pi G \rho_m}{k^2 v_0^2} \quad \text{and} \quad x = \frac{\omega}{k v_0} \qquad (20)
$$

4 Stability Condition

We next need to compute the instability growth rate, which is given by the imaginary part of ω, as a function of k. The star distribution $\rho_0(\vec{v})$ is unstable when ω is complex with a positive imaginary part. We need to compute x as a function of λ using Eq.(19) in order to obtain ω as a function of k.

In general x is complex, but at the edge of instability, x is real. The edge of stability can be seen by plotting the RHS of Eq.(19) as x is scanned along the real axis from $-\infty$ to ∞. Fig.1 shows the real and imaginary parts of the RHS of Eq.(19) in such a scan. The horizontal and vertical axes of Fig.1 are the real and imaginary parts of the RHS of Eq.(19) respectively. As x is scanned from $-\infty$ to ∞, the RHS of Eq.(19) traces out a cherry-shaped diagram, including the "stem" of the cherry running from $-\infty$ to 0 along the real axis. If λ lies inside this cherry diagram (including the stem), the galaxy distribution is stable. Since λ is necessarily real and positive, the stability condition therefore reads

$$\lambda < \frac{1}{2} \tag{21}$$

Eq.(21) indicates that a hot universe (high temperature, i.e. large v_0) is more stable than a cold universe. This is expected due to the Landau damping mechanism. It also indicates that the star distribution is unstable for long-wavelength perturbations (small k). The threshold wavelength is given by $2\pi/k_{\rm th}$, where

$$k_{\rm th} = \frac{\sqrt{12\pi G\rho_m}}{v_0} \tag{22}$$

Perturbations with wavelength longer than that corresponding to Eq.(22) are unstable. One might expect that the galaxy will have a dimension of the order of this wavelength because if the galaxy had a larger dimension, it would have broken up due to the instability. There will be more discussions on this point later.

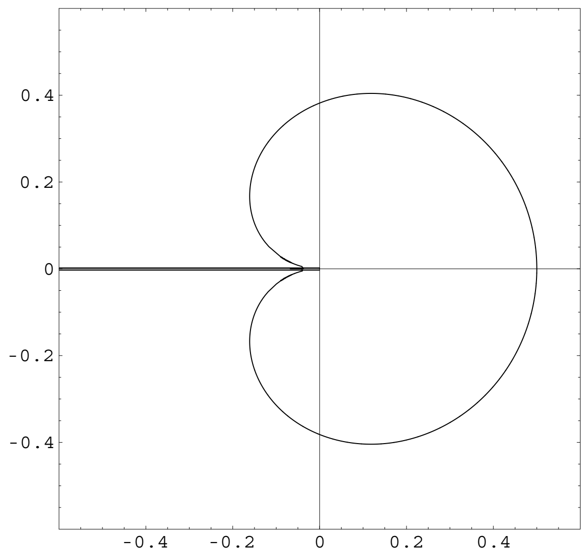

Figure 1: Stability diagram for the galaxy distribution.

5 Spontaneous Gravitational Instability

When $\lambda > 1/2$, ω will be complex. The instability growth rate will be determined by the imaginary part of ω,

$$\tau^{-1} = \mathrm{Im}(\omega) \tag{23}$$

We need to go back to Eq.(19), but modify it slightly for complex ω. Let

$$\frac{\omega}{kv_0} = x + iy, \quad (y > 0) \tag{24}$$

Eq.(19) then becomes

$$\lambda = \frac{1}{2 + \left(\frac{x+iy}{2}\right) \ln\left[\frac{(x-1)^2+y^2}{(x+1)^2+y^2}\right] + (ix - y)\left[\tan^{-1}\left(\frac{x+1}{y}\right) - \tan^{-1}\left(\frac{x-1}{y}\right)\right]} \tag{25}$$

When $y \to 0^+$, we obtain Eq.(19) as it should.

We will need to solve Eq.(25) for x and y for given $\lambda > \frac{1}{2}$. It turns out that in this range there is always one solution with purely imaginary ω, i.e. $x = 0$, and

$$\lambda = \frac{1}{2 - 2y \tan^{-1}\left(\frac{1}{y}\right)} \tag{26}$$

or, written out explicitly,

$$\frac{6\pi G\rho_m}{k^2 v_0^2} = \frac{1}{2 - \frac{2\tau^{-1}}{kv_0} \tan^{-1}\left(\frac{kv_0}{\tau^{-1}}\right)} \tag{27}$$

We need to find τ^{-1} as a function of k. To do so, we first scale the variables by

$$u = \frac{kv_0}{\sqrt{6\pi G\rho_m}}, \quad v = \frac{\tau^{-1}}{\sqrt{6\pi G\rho_m}} \tag{28}$$

and then

$$\frac{1}{u^2} = \frac{1}{2 - 2\left(\frac{v}{u}\right) \tan^{-1}\left(\frac{u}{v}\right)} \tag{29}$$

Fig.2 shows the result.

As seen from Fig.2, the growth rate vanishes ($v = 0$) when $u = \sqrt{2}$, corresponding to $\lambda = 1/2$, i.e. at the stability boundary. This is of course expected. Fig.2 also shows that instability occurs fastest for small u, i.e. small k and large wavelegnth. The growth rate is maximum at $u = 0$ with $v = \sqrt{2/3}$. This means the maximum growth rate occurs for perturbation of infinite wavelength, and is given by

$$(\tau^{-1})_{\max} = \sqrt{4\pi G\rho_m} \tag{30}$$

Note that the growth rate is independent of v_0, even though there is still the condition that the distribution is unstable, i.e. $\lambda > 1/2$, which does depend on

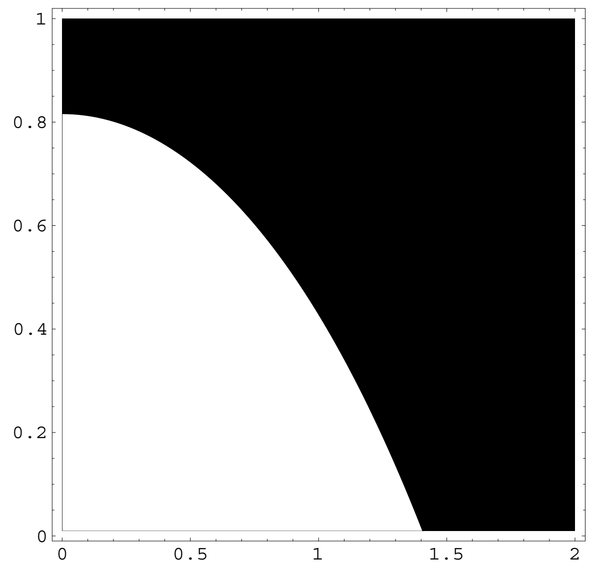

Figure 2: v vs u according to Eq.(29).

v_0 and can be cast into the form (see Eq.(22))

$$k < \frac{\sqrt{3}}{v_0}(\tau^{-1})_{\max} \tag{31}$$

The fastest instability corresponds to $k = 0$, or an instability wavelength of infinity.[†]

According to Eq.(31), all stable galaxies must have a dimension smaller than a critical value, i.e.

$$\text{galaxy dimension} < \frac{2\pi v_0}{\sqrt{12\pi G\rho_m}} \tag{32}$$

[†]This result depends on our assumption of Newtonian dynamics of action-at-a-distance. Perturbation at one location instantly affects locations infinitely far away. If this action-at-a-distance effect is appropriately removed, it is expected that the instability for perturbations with very large wavelengths will be weakened.

The stability is provided through Landau damping. When the temperature $v_0 \to 0$, no galaxies can be stable. Eqs.(30) and (32) are our main results.

6 Numerical Estimates

For a numerical application, we take estimates from the Milky Way,

$$\begin{aligned} \rho_m &= 2 \times 10^{-23} \text{ g/cm}^3 \\ v_0 &= 200 \text{ km/s} \end{aligned}$$

We obtain a maximum growth time of $\tau_{\max} = 7 \times 10^6$ years for perturbations with very large wavelengths. For stability, the galaxy dimension must be smaller than 14000 light-years, which seems to be consistent with the size of the Milky Way.

7 Discussions

- The case studied so far is that of a galaxy with uniform distribution of stars. One direction of generalization is to consider galaxies with a finite spherically symmetric distribution. One attempt was made and included in Appendix A. Our finding here is that a spherically symmetric distribution of the Haissinski type (to be explained in Appendix A) does not exist.

- Appendix B gives an extension to a planar galaxy, still nonrotating. The unperturbed distribution does exist and is given in Appendix B. However, this planar distribution is found, as shown in Appendix C, to be always stable against perturbations that do not involve transverse structures. Any instability of the planar galaxy will therefore have to have a sufficiently complex pattern.

- It is conceivable that the same analysis can be applied to the dynamics of galaxies in a galaxy cluster, instead of stars in a galaxy. In that case, $\rho(\vec{x}, \vec{v}, t)$ describes the distribution of galaxies in the galaxy cluster. We might then take the corresponding numerical values

$$\rho_m = 10^{-31} \text{ g/cm}^3$$

$$v_0 = 1000 \text{ km/s}$$

  We obtain a growth time of $\tau_{\text{max}} = 1 \times 10^{11}$ years. The galaxy cluster dimension should be smaller than 1×10^9 light-years. These values do not seem to be too unreasonable.

- For more detailed applications, we will have to include the rotation of the galaxy into the analysis. The unperturbed distribution will then involve also the angular momentum. The analysis is much more involved but should be straightforward.

- Still further extensions might include the special relativity and general relativity to replace Newtonian gravity and to avoid the "action at a distance" problem.

References

[1] See for example, P.L. Palmer, "Stability of Collisionless Stellar Systems", Kluwer Academic Publishers, The Netherlands, 1994, and references quoted therein.

[2] See for example, Alexander W. Chao, "Physics of Collective Beam Instabilities in High Energy Accelerators", John Wiley & Sons, New York, 1993, and references quoted therein.

[3] L.D. Landau, J. Phys. USSR 10, 25 (1946).

[4] J. Haissinski, Nuovo Cimento 18B, 72 (1973).

Appendix A

So far we have considered the stability of a galaxy whose unperturbed distribution is uniform in the infinite space and is nonrotating. As a first (unsuccessful) attempt of genralization, we will look for an unperturbed distribution that is isotropic, nonrotating, and finite in size. To do so, we first note that Eq.(1) is derivable from a Hamiltonian

$$H = \frac{\vec{v}^2}{2} - G \int d\vec{v}' \int d\vec{x}' \, \frac{\rho(\vec{x}', \vec{v}', t)}{|\vec{x}' - \vec{x}|} \tag{33}$$

We then make the observation that one possible unperturbed distribution is that it is a function of this Hamiltonian, i.e.

$$\rho_0(\vec{x}, \vec{v}) = \rho_0(H) \tag{34}$$

For example, one may choose

$$\rho_0(\vec{x},\vec{v}) = \mathcal{N} e^{-H/\sigma_v^2} = \mathcal{N}\exp\left[-\frac{1}{\sigma_v^2}\left(\frac{\vec{v}^2}{2} - G\int d\vec{v}'\int d\vec{x}'\,\frac{\rho_0(\vec{x}',\vec{v}')}{|\vec{x}'-\vec{x}|}\right)\right] \quad (35)$$

where σ_v is the rms of the magnitude of $\vec{v}$, and is a prescribed input parameter in this model. The quantity $\mathcal{N}$ is a normalization so that integrating ρ_0 over $\vec{x}$ and $\vec{v}$ gives the total mass of the galaxy M. Note that Eqs.(34) and (35) are not a useful ansatz for a rotating galaxy because it assumes a distribution that is isotropic in $\vec{v}$.

Equation (35) is a self-consistent equation for ρ_0. It is equivalent to the Haissinski equation in accelerator physics [4]. Our job is to solve for ρ_0 from Eq.(35). It turns out that the distribution factorizes,

$$\rho_0(\vec{x},\vec{v}) = \frac{e^{-\vec{v}^2/2\sigma_v^2}}{(\sqrt{2\pi}\,\sigma_v)^3}\;\rho_m(\vec{x}) \quad (36)$$

The quantity ρ_m is then the mass volume density of the stars in the galaxy, now a function of $\vec{x}$. Substituting Eq.(36) into Eq.(35) yields self-consistent equation for $\rho_m(\vec{x})$,

$$\rho_m(\vec{x}) = (\sqrt{2\pi}\,\sigma_v)^3\,\mathcal{N}\exp\left[\frac{G}{\sigma_v^2}\int d\vec{x}'\,\frac{\rho_m(\vec{x}')}{|\vec{x}'-\vec{x}|}\right] \quad (37)$$

If we now assume ρ_m is also isotropic, i.e. $\rho_m(\vec{x}) = \rho_m(r)$ in spherical coordinates, then Eq.(37) becomes

$$\rho_m(r) = (\sqrt{2\pi}\,\sigma_v)^3\,\mathcal{N}\exp\left[\frac{4\pi G}{\sigma_v^2}\int_0^\infty r'^2 dr'\,\frac{\rho_m(r')}{\max(r,r')}\right] \quad (38)$$

It turns out that no solution exisits that satisfies Eq.(38) while is also normalizable to a finite total mass of the galaxy. This means that an isotropic unperturbed distribution of the Haissinski type does not exist.

Appendix B

A planar distribution avoids the singularity problem that leads to the failure of a Haissinski type distribution in the spherical case. Use cylindrical coordinates $(\vec{x}_\perp, z)$, and let the unperturbed distribution be independent of $\vec{x}_\perp$ and factorizable in such a way that

$$\rho_0(\vec{x}_\perp, \vec{v}_\perp, z, v_z) = \rho_\perp(\vec{v}_\perp)\, \rho_z(z, v_z) \tag{39}$$

where we demand

$$\int d\vec{v}_\perp\, \rho_\perp(\vec{v}_\perp) = 1 \tag{40}$$

$$\begin{aligned} \int dz \int dv_z\, \rho_z(z, v_z) &= \Sigma \\ &= \text{surface mass density of stars} \end{aligned} \tag{41}$$

This unperturbed distribution is that of an infinite disk of finite thickness.

We will first need the equations of motion,

$$\begin{aligned} \dot{\vec{x}}_\perp &= \vec{v}_\perp \\ \dot{\vec{v}}_\perp &= \vec{0} \\ \dot{z} &= v_z \\ \dot{v}_z &= 2\pi G \int dv'_z \int dz'\, \rho_z(z', v'_z)\, \mathrm{sgn}(z' - z) \end{aligned} \tag{42}$$

Equation (42) is derivable from first principles, as well as from Eq.(1). The corresponding Hamiltonian is

$$\begin{aligned} H_\perp &= \frac{\vec{v}_\perp^2}{2} \\ H_z &= \frac{v_z^2}{2} + 2\pi G \int dv'_z \int dz'\, \rho_z(z', v'_z)\, |z' - z| \end{aligned} \tag{43}$$

We then form the Haissinski ansatz

$$
\begin{aligned}
\rho_\perp &= \frac{1}{2\pi\sigma_{v\perp}^2} e^{-\vec{v}_\perp^2/2\sigma_{v\perp}^2} \\
\rho_z &= \mathcal{N} \exp\left[-\frac{1}{\sigma_{vz}^2}\left(\frac{v_z^2}{2} + 2\pi G \int dv_z' \int dz'\, \rho_z(z', v_z')\, |z' - z|\right)\right]
\end{aligned}
\tag{44}
$$

where $\sigma_{v\perp}$ relates to the transverse temperature, and σ_{vz} relates to the longitudinal temperature. The fact that the transverse and longitudinal motions decouple allows the two different temperatures.

Note that although a gaussian form of $\rho_\perp$ is most natural, this assumption is not compulsary. Any normalized form is acceptable, without affecting our following analysis.

Writing ρ_z as

$$
\rho_z(z, v_z) = \frac{1}{\sqrt{2\pi}\,\sigma_{vz}} e^{-v_z^2/2\sigma_{vz}^2}\, \rho_m(z), \quad \text{with} \quad \int_{-\infty}^{\infty} dz\, \rho_m(z) = \Sigma \tag{45}
$$

then gives the Haissinski equation

$$
\rho_m(z) = \sqrt{2\pi}\,\sigma_{vz}\,\mathcal{N}\, \exp\left[-\frac{2\pi G}{\sigma_{vz}^2} \int dz'\, \rho_m(z')\, |z' - z|\right] \tag{46}
$$

Equation (46) can be manipulated to yield

$$
\left(\frac{\rho_m'}{\rho_m}\right)' + \frac{4\pi G}{\sigma_{vz}^2}\, \rho_m = 0 \tag{47}
$$

where a prime means taking derivative with respect to z. We then make a transformation to the scaled variables u and w,

$$
z = u\, \frac{\sigma_{vz}^2}{G\Sigma}, \quad \rho_m = w\, \frac{G\Sigma^2}{\sigma_{vz}^2} \tag{48}
$$

to obtain

$$
\left(\frac{w'}{w}\right)' + 4\pi w = 0 \tag{49}
$$

where a prime now means taking derivative with respect to u. The Haissinki equation (46) is rewritten as

$$w(u) = \sqrt{2\pi}\,\frac{\sigma_{vz}^3 \mathcal{N}}{G\Sigma^2}\exp\left[-2\pi\int_{-\infty}^{\infty} du'\, w(u')|u'-u|\right] \tag{50}$$

There is also the normalization condition

$$\int_{-\infty}^{\infty} du\, w(u) = 1 \tag{51}$$

as well as the condition that $w'(0) = 0$. The planar unperturbed distribution has an exponential tail in $|z|$. The distribution found numerically by MATHEMATICA is shown in Fig.3.

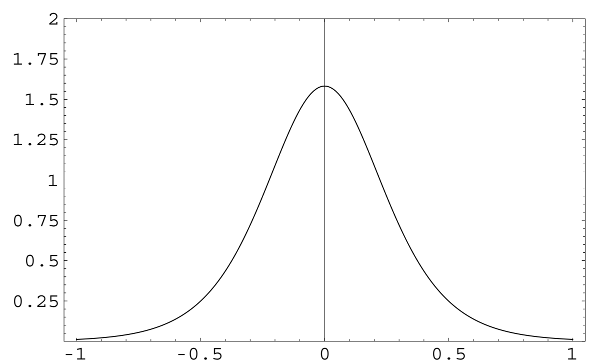

Figure 3: Planar unperturbed star distribution $w(u)$ vs u. $w(0) \approx 1.5822$.

Given the function $w(u)$, the planar unperturbed distribution is summarized as

$$\rho_z(z, v_z) = \frac{G\Sigma^2}{(2\pi)^{1/2}\sigma_{vz}^3}\exp\left(-\frac{v_z^2}{2\sigma_{vz}^2}\right) w\left(\frac{G\Sigma}{\sigma_{vz}^2}\, z\right) \tag{52}$$

$$\rho_0(\vec{v}_\perp, z, v_z) = \frac{G\Sigma^2}{(2\pi)^{3/2}\sigma_{v\perp}^2\sigma_{vz}^3}\exp\left(-\frac{\vec{v}_\perp^2}{2\sigma_{v\perp}^2} - \frac{v_z^2}{2\sigma_{vz}^2}\right) w\left(\frac{G\Sigma}{\sigma_{vz}^2}\, z\right) \tag{53}$$

The thickness of the planar distribution is $\approx \sigma_{vz}^2/G\Sigma$. This thickness corresponds, not surprisingly, to an equipartition of the longitudinal potential and kinetic energies.

Appendix C

To study the gravitational stability of the planar unperturbed distribution Eqs.(52, 53), we need to analyze the behavior of its infinitesimal perturbations. We have examined perturbations of the type

$$\rho(\vec{x}_\perp, \vec{v}_\perp, z, v_z) = \rho_\perp(\vec{v}_\perp)\,[\rho_z(z, v_z) + \Delta\rho(z, v_z, t)] \qquad (54)$$

i.e. the perturbation occurs only in the *longitudinal* (z, v_z) dimension. We found that such perturbations are always stable. Analysis leading to this conclusion is given in the Appendix. Instabilities of a planar galaxy will therefore have to involve the transverse coordinates in forms different from Eq.(54).

The Vlasov equation, to first order in $\Delta\rho$, reads

$$\frac{\partial \Delta\rho}{\partial \tau} + \frac{\partial \Delta\rho}{\partial u} v + \frac{\partial \Delta\rho}{\partial v}\,\frac{w'(u)}{w(u)} - \sqrt{2\pi}\, v e^{-v^2/2} w(u) \int dv' \int du'\, \Delta\rho(u', v')\, \mathrm{sgn}(u' - u) \;=\; 0 \qquad (55)$$

where we have introduced the scaled dimensionless variables

$$u = \frac{G\Sigma}{\sigma_{vz}^2}\, z, \quad v = \frac{v_z}{\sigma_{vz}}, \quad \tau = \frac{G\Sigma}{\sigma_{vz}}\, t \qquad (56)$$

The function w'/w in the third term is the gravitational focusing coming from the unperturbed distribution of the stars.

To proceed, we first try to linearize the problem (thus losing Landau damping) for small u. In doing so, however, to be self-consistent, we must at the same time linearize the unperturbed distribution ρ_z, i.e.

$$
\begin{aligned}
w &\approx w(0) e^{-2\pi w(0) u^2} \\
\rho_z &\approx \frac{G\Sigma^2 w(0)}{\sqrt{2\pi}\,\sigma_{vz}^3}\, e^{-v^2/2-2\pi w(0) u^2}
\end{aligned}
\tag{57}
$$

Substituting Eq.(57) into Eq.(55) gives

$$
\begin{aligned}
&\frac{\partial \Delta\rho}{\partial \tau} - \omega_0 \frac{\partial \Delta\rho}{\partial \phi} - \sqrt{2\pi}\,\omega_0^2 r \sin\phi\ \frac{\omega_0^2}{4\pi}\, e^{-\omega_0^2 r^2/2} \\
&\quad \times \int_0^\infty r' dr' \int_0^{2\pi} d\phi'\, \Delta\rho(r',\phi',\tau)\, \mathrm{sgn}(r'\cos\phi' - r\cos\phi) \;=\; 0
\end{aligned}
\tag{58}
$$

where

$$
u = r\cos\phi, \quad \frac{v}{\omega_0} = r\sin\phi, \quad \omega_0 = \sqrt{4\pi w(0)} \tag{59}
$$

Consider a collective mode

$$
\Delta\rho = e^{-i\Omega\tau} \sum_{m=-\infty}^{\infty} R_m(r) e^{-im\phi} \tag{60}
$$

Charge conservation requires that

$$
\int_0^\infty 4\pi r dr\, R_0(r) = 0 \tag{61}
$$

Eq.(58) becomes

$$
\begin{aligned}
&-i\Omega R_m(r) + im\omega_0 R_m(r) - \frac{\omega_0^4}{4\pi\sqrt{2\pi}}\, r\, e^{-\omega_0^2 r^2/2} \int_0^{2\pi} \sin\phi\, d\phi\, e^{im\phi} \\
&\quad \times \int_0^\infty r' dr' \int_0^{2\pi} d\phi' \sum_{m'=-\infty}^{\infty} R_{m'}(r') e^{-im'\phi'}\, \mathrm{sgn}(r'\cos\phi' - r\cos\phi) \;=\; 0
\end{aligned}
\tag{62}
$$

Integration over ϕ' and after some algebraic manipulations, we obtain

$$
\begin{aligned}
&-i\Omega R_m(r) + im\omega_0 R_m(r) - \frac{\omega_0^4}{\sqrt{2\pi}}\, e^{-\omega_0^2 r^2/2} \int_0^\infty r' dr' \sum_{m'=-\infty}^{\infty} R_{m'}(r') \\
&\quad \times i^{m-m'-1} m \int_{-\infty}^{\infty} \frac{dk}{k^2} J_m(kr) J_{m'}(kr') \;=\; 0
\end{aligned}
\tag{63}
$$

The case of $m = 0$ is a special mode. It is the static eigenmode with

$$\Omega = 0, \quad m = 0 \tag{64}$$

while the corresponding eigenfunction $R_0(r)$ is arbitrary as long as it satisfies Eq.(61).

We now decompose $R_m(r)$ as

$$R_m(r) = \left(\frac{\omega_0 r}{\sqrt{2}}\right)^{|m|} e^{-\omega_0^2 r^2/2} \sum_{n=0}^{\infty} a_{mn}\, L_n^{(|m|)}\left(\frac{\omega_0^2 r^2}{2}\right) \tag{65}$$

where $L_n^{(m)}$'s are the generalized Laguerre polynomials. Using their orthogonality properties, and applying to both sides of Eq.(63) by (for chosen m and n)

$$\int_0^{\infty} r dr \left(\frac{\omega_0 r}{\sqrt{2}}\right)^{|m|} L_n^{(|m|)}\left(\frac{\omega_0^2 r^2}{2}\right) \tag{66}$$

we obtain

$$\begin{aligned} & -i(\Omega - m\omega_0)\, a_{mn} \\ & -\sqrt{2}\,\omega_0 \sum_{\substack{m'=-\infty \\ m+m'=\text{even}}}^{\infty} \sum_{n'=0}^{\infty} a_{m'n'} \frac{m\, i^{|m|-|m'|-1}(|m|+|m'|+2n+2n'-3)!!}{(|m|+n)!n'!\, 2^{2n+2n'+|m|+|m'|}} = 0 \end{aligned} \tag{67}$$

The infinite matrix equation (67) is then solved for the eigenmode frequency Ω. Instability of the perturbations of type Eq.(54) is to be identified with complex solution of Ω, but it is found that all eigenvalues of Ω are real. We conclude that the planar galaxy is stable against longitudinal perturbations of the form (54). The largest "frequency shift" occurs for the $m = 1$ mode with $\Omega/\omega_0 \approx 1.37$. Instabilities, if any, will have to involve transverse dynamics.