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ASYMPTOTIC MULTIPLICITY DISTRIBUTIONS AND ANALOGUE OF THE CENTRAL LIMIT THEOREM*

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Abstract

Under the assumption of gentle behavior of higher cumulants or correlation moments, we discuss how the multiplicity distributions approach the Gaussian (normal) or approximately Gaussian distribution at high energy. This is an analogue of the central limit theorem. A detailed comparison with experiment is made based on this formalism and shows that such an approach may be useful. It is pointed out that if the 2-prong inelastic cross section in the pp reaction is identified with the lower end point of the multiplicity distribution, then a deviation from the Gaussian form is necessary at the present energy. The asymptotic relation

$$\frac{1}{\sqrt{2\pi}} \frac{\sigma_{inel}}{\sigma_m} = \gamma = \sqrt{\overline{(n - \overline{n})^2}}$$

is well satisfied by experimental data, where σ_m and γ stand for the maximum of the topological cross sections and the width of the limiting Gaussian form, respectively. If the ratio of the width γ and the modal multiplicity m approaches a non-vanishing value at infinite energy, then we obtain a scaling of the distribution function, the scaling function being of approximately Gaussian form with the scaling variable n/m.

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I. INTRODUCTION

For a long time, the Poisson distribution has been a favorite model of physicists for describing the high energy multiplicity distribution. Recent experiments,[1] however, indicate a departure from it by exhibiting nonvanishing correlation moments. It has been pointed out, in fact, that the asymptotic multiplicity distribution seems to approach a normal distribution[2-4] as energy increases. Such a phenomenon resembles the central limit theorem in statistics and was proved by Haldane some time ago in the case of a continuous distribution on the interval $(-\infty, \infty)$. The assumption that leads to this result is that the higher cumulants do not grow too fast, a condition which is met by multiperipheral models,[5] field theoretical models,[6] and a gas model.[7]

In this article, we elaborate on the Haldane theorem and present it in a form suitable for analyzing experimental data. In Section II, the central limit theorem is exhibited for the Poisson distribution so as to be useful for discussions of the later sections. Section III presents the definition of various moments and their relationships. In Section IV, we prove the Haldane theorem for a discrete distribution on the interval $(0, \infty)$ and derive the asymptotic expansion formula. Based on the result of Section IV, we discuss the asymptotic limit of the distribution function and various asymptotic relations among the parameters which describe the distribution function. A possibility of scaling of the distribution function is pointed out (Section V). Comparison with experimental data and discussion follow in Sections VI and VII.

II. THE POISSON DISTRIBUTION AND ASYMPTOTIC FORM

It is a result of the central limit theorem[8,9] that the Poisson distribution

$$P_n = \frac{a^n}{\Gamma(n+1)} e^{-a} \tag{2.1}$$

approaches the normal form in the limit $a \longrightarrow \infty$. In order to see this more explicitly, we transform Eq. (2.1) into the form

$$P_n = \frac{1}{\sqrt{2\pi}\,\beta} \exp\left[-\frac{(n-m)^2}{2\gamma^2}\right]\left\{1 + \sum_{k=3}^{\infty} a_k\left(\frac{n-m}{\gamma}\right)^k\right\} \tag{2.2}$$

$$= \frac{1}{\sqrt{2\pi}\,\beta} \exp\left\{-\frac{1}{2}\left(\frac{n-m}{\gamma}\right)^2 + \sum_{k=3}^{\infty} b_k\left(\frac{n-m}{\gamma}\right)^k\right\} \quad . \tag{2.3}$$

This can be done by using the asymptotic expansion of the Gamma function

$$\ln \Gamma(n+1) = n+ \frac{1}{2}\Big)\ln n - n + \frac{1}{2}\ln(2\pi) + \frac{1}{12n} + O(n^{-3}) \quad , \tag{2.4}$$

and the formula

$$\left.\frac{\partial \ln P_n}{\partial n}\right|_{n=m} = 0 \ , \tag{2.5}$$

$$\frac{1}{\gamma^2} = -\left.\frac{\partial^2 \ln P_n}{\partial n^2}\right|_{n=m} \quad , \tag{2.6}$$

$$\frac{1}{\beta} = \sqrt{2\pi}\, P_m \tag{2.7}$$

and

$$b_k = \frac{\gamma^k}{k!} \left.\frac{\partial^k \ln P_n}{\partial n^k}\right|_{n=m} \quad , \ k \geq 3 \ . \tag{2.8}$$

Leaving the details of the calculation to Appendix A, we write down the asymptotic solution for the parameters,

$$
\begin{aligned}
m &= a - \frac{1}{2} - \frac{1}{24\,a} + O(a^{-3}) \quad , \\
\gamma &= \sqrt{a}\left(1 + \frac{1}{48\,a^2} + O(a^{-4})\right), \\
\beta &= \sqrt{a}\left(1 - \frac{1}{24\,a} + \frac{1}{1152\,a^2} + O(a^{-3})\right)
\end{aligned}
\tag{2.9}
$$

and

$$
b_k = \frac{(-1)^{k-1}}{k\,(k-1)}\,\frac{1}{a^{k/2-1}}\left(1 - \frac{(2k-1)(k-2)}{48\,a^2} + O(a^{-4})\right), \quad k \geq 3\ . \tag{2.10}
$$

The coefficients a_k are related to b_k and given by

$$
\begin{aligned}
a_3 &= b_3 = \frac{1}{6\,a^{1/2}}\ 1 - \frac{5}{48\,a^2} + O(a^{-4}) \quad , \\
a_4 &= b_4 = -\frac{1}{12\,a}\left(1 - \frac{7}{24\,a^2} + O(a^{-4})\right) , \\
a_5 &= b_5 = \frac{1}{20\,a^{3/2}}\left(1 - \frac{9}{16\,a^2} + O(a^{-4})\right) , \\
a_6 &= b_6 + \frac{1}{2}\,b_3^2 = \frac{1}{72\,a}\left(1 - \frac{12}{5\,a} - \frac{5}{24\,a^2} + O(a^{-3})\right) , \\
a_7 &= b_7 + b_3 b_4 = -\frac{1}{72\,a^{3/2}}\left(1 - \frac{12}{7\,a} + O(a^{-2})\right) \quad , \\
a_8 &= b_8 + \frac{1}{2}\,b_4^2 + b_3 b_5 = \frac{17}{1440\,a^2}\left(1 + O(a^{-1})\right) , \\
a_9 &= b_9 + \frac{1}{6}\,b_3^3 + b_3 b_6 + b_4 b_5 = \frac{1}{1296\,a^{3/2}}\left(1 - \frac{63}{5\,a} + O(a^{-2})\right)
\end{aligned}
\tag{2.11}
$$

and

$$
a_{3\ell-4,\ 3\ell-2,\ 3\ell} = O(a^{-\ell/2}) \quad , \quad \ell \geq 3 \quad . \tag{2.12}
$$

Equations (2.2), (2.3), and (2.9) - (2.12) give us an idea of how the limiting normal distribution is approached as the average value a increases.

The normalized cross section P_n has the maximum value $1/\sqrt{2\pi}\ \beta$ at $n = m$, which is called the mode or modal multiplicity.[10] The width parameter γ and β have the same limit $\sqrt{a}$, but γ approaches the limit faster than β does.

III. MOMENTS AND CUMULANTS

A statistical system with correlations is conveniently discussed in terms of various moments. They are defined through the characteristic function, c.f.,

$$\phi(t) = \sum_{n=0}^{\infty} e^{int} P_n \tag{3.1}$$

$$= \sum_{k=0}^{\infty} \frac{\mu_k (it)^k}{k!} = \exp\left[\sum_{k=1}^{\infty} \frac{\kappa_k (it)^k}{k!}\right]$$

$$= 1 + \sum_{k=1}^{\infty} \frac{F_k (e^{it}-1)^k}{k!} = \exp\left[\sum_{k=1}^{\infty} \frac{f_k (e^{it}-1)^k}{k!}\right] \tag{3.2}$$

$$= e^{i\mu_1 t}\left[1 + \sum_{k=2}^{\infty} \frac{d_k (it)^k}{k!}\right],$$

where $\mu_k = \overline{n^k}$, κ_k, $F_k = \overline{n(n-1)\cdots(n-k+1)}$, f_k and $d_k = \overline{(n-\bar{n})^k}$ are moments, cumulants,[9] factorial moments, correlation moments,[11] and dispersion moments,[12] respectively.

All these moments are related to each other. Some useful relations are given below:

$$\begin{aligned}
\kappa_1 &= f_1 = \bar{n} = \mu_1 \\
\kappa_2 &= f_1 + f_2 = d_2 \\
\kappa_3 &= f_1 + 3f_2 + f_3 = d_3 \\
\kappa_4 &= f_1 + 7f_2 + 6f_3 + f_4 = d_4 - 3d_2^2 \\
\kappa_5 &= f_1 + 15f_2 + 25f_3 + 10f_4 + f_5 = d_5 - 10d_3 d_2
\end{aligned} \tag{3.3}$$

and

$$\kappa_k = \sum_{\ell=1}^{k} c_k^\ell f_\ell , \tag{3.4}$$

where

$$c_k^\ell = \frac{1}{\ell!} \sum_{\substack{k_1+k_2+\cdots+k_\ell=k \\ k_i \geq 1}} \frac{k!}{k_1!\,k_2!\cdots k_\ell!}$$

$$= \begin{cases} \dfrac{1}{(\ell-1)!} \displaystyle\sum_{r=1}^{\ell} \binom{\ell-1}{r-1} (-1)^{\ell-r}\, r^{k-1} , & k \geq \ell \\ 0 & \text{otherwise .} \end{cases} \tag{3.5}$$

It follows from Eq. (3.3) that the first two cumulants are positive definite. The correlation moments f_k are cumulants of the factorial moments F_k, and therefore the relationship between F_k and f_k is identical to that between μ_k and κ_k. The latter is given in Ref. 9.

IV. ASYMPTOTIC EXPANSION IN TEMPERATE CORRELATION MODELS

Inverting Eq. (3.1), we obtain

$$P_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-int} \phi(t)\, dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left[-int + \sum_{k=1}^{\infty} \frac{\kappa_k (it)^k}{k!}\right] dt$$

$$= \exp\left(\sum_{k=3}^{\infty} \frac{\kappa_k (-D)^k}{k!}\right) \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left(-int - \frac{\kappa_2 t^2}{2}\right) dt, \qquad (4.1)$$

where the abbreviation

$$D = \frac{\partial}{\partial n} \qquad (4.2)$$

is used. Obviously, such a formal manipulation is not permissible for an arbitrary distribution P_n or a c.f. $\phi(t)$. Very roughly speaking, it should be allowed for a distribution which is sufficiently smooth and vanishes sufficiently fast as $n \longrightarrow \infty$. The latter condition is also necessary for all moments or cumulants to exist. In any event, we retrict ourselves to a class of distributions which permit this manipulation. For practical application in physics, this restriction does not seem a serious hazard. In particular, in the case of multiplicity distributions, conservation of energy requires a cut-off of the distribution for $n > N \propto \sqrt{s}$.

Now the integral in the last term of Eq. (4.1) is computed as follows:

$$\chi(n) \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left[-i(n-\kappa_1)t - \frac{\kappa_2 t^2}{2}\right] dt$$

$$= \exp\left[-\frac{(n-\kappa_1)^2}{2\kappa_2}\right]\frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left[-\frac{\kappa_2}{2}\left(t + \frac{i(n-\kappa_1)}{\kappa_2}\right)^2\right] dt$$

$$= \frac{1}{\sqrt{2\pi\kappa_2}} \exp\left[-\frac{(n-\kappa_1)^2}{2\kappa_2}\right] \cdot \frac{1}{\sqrt{\pi}} \int_{-\sqrt{\frac{\kappa_2}{2}}\left(\pi - \frac{i(n-\kappa_1)}{\kappa_2}\right)}^{\sqrt{\frac{\kappa_2}{2}}\left(\pi + \frac{i(n-\kappa_1)}{\kappa_2}\right)} e^{-t^2} dt$$

$$= \frac{\exp\left[-\frac{(n-\kappa_1)^2}{2\kappa_2}\right]}{\sqrt{2\pi\kappa_2}} \operatorname{Re}\left\{\operatorname{Erf}\left(\sqrt{\frac{\kappa_2}{2}}\left(\pi + \frac{i(n-\kappa_1)}{\kappa_2}\right)\right)\right\} , \qquad (4.3)$$

where the error function is defined by

$$\operatorname{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt \xrightarrow[x \to \infty]{} 1 - \frac{1}{\sqrt{\pi}} \frac{e^{-x^2}}{x}\left(1 + O(x^{-2})\right) . \qquad (4.4)$$

Thus we have the asymptotic form for $\chi(n)$:

$$\chi(n) \xrightarrow[\kappa_2 \to \infty]{} \frac{\exp\left[-\frac{(n-\kappa_1)^2}{2\kappa_2}\right]}{\sqrt{2\pi\kappa_2}}\left[1 - \sqrt{\frac{2}{\pi}} \operatorname{Re}\frac{\exp\left\{-\frac{\kappa_2}{2}\left(\pi^2 - \left(\frac{n-\kappa_1}{\kappa_2}\right)^2\right) - i\pi(n-\kappa_1)\right\}}{\sqrt{\kappa_2}\left\{\pi + \frac{i(n-\kappa_1)}{\kappa_2}\right\}} \times \left(1 + O\left(\kappa_2^{-1}\right)\right)\right] . \qquad (4.5)$$

Notice that the second term in the brackets of Eq. (4.5) vanishes exponentially in the limit $\kappa_2 \longrightarrow \infty$ as long as the condition

$$\left| \frac{n - \kappa_1}{\kappa_2} \right| < \pi \tag{4.6}$$

is satisfied.

From Eq. (4.1), (4.3), and (4.5), it then follows that

$$P_n = \exp\left[\sum_{k=3}^{\infty} \frac{\kappa_k (-D)^k}{k!}\right] \frac{1}{\sqrt{2\pi\kappa_2}} \exp\left[-\frac{(n-\kappa_1)^2}{2\kappa_2}\right] \times$$

$$\times \left[1 + O\left\{\frac{\exp\left[-\frac{\kappa_2}{2}\left(\pi^2 - \left(\frac{n-\kappa_1}{\kappa_2}\right)^2\right)\right]}{\sqrt{\kappa_2}}\right\}\right] \tag{4.7}$$

The first term of Eq. (4.7) is of the form used by Haldane in his analysis of the mode and median of a nearly normal distribution with given cumulants,[13] and is an exact formula for a continuous distribution in the interval $(-\infty, \infty)$. In other words, the problem of obtaining the asymptotic distribution function in terms of cumulants for the case of discrete distribution in the range $(0, \infty)$ is identical to that for the continuous distribution in the range $(-\infty, \infty)$, apart from the terms that vanish exponentially, for the range of n which satisfies the condition (6).

If, moreover, we assume that

$$\gamma_{k-2} \equiv \frac{\kappa_k}{\kappa_2^{k/2}} = O(\epsilon^{k-2}) , \qquad k \geq 3 , \tag{B}$$

with ϵ a small number, we can derive an asymptotic expansion for Eq. (4.7), following the method of Haldane.[13] We may refer to the assumption (B) as temperate correlation models. We further divide the models into two cases: If

$$\frac{\kappa_k}{\kappa_2} \quad \text{are bounded,} \quad (k \geq 3) , \tag{A}$$

so that

$$\epsilon \approx \frac{1}{\sqrt{\kappa_2}} \longrightarrow 0 \quad \text{as} \quad \kappa_2 \rightarrow \infty , \tag{4.8}$$

we may call them weak correlation models, while those which satisfy the condition (B) with a small but finite ϵ may be called moderate correlation models.

A special case of the former is the so-called short correlation models,[6-8] the name which originates from the behavior of correlations in rapidity variables. In such models, energy dependence of cumulants or correlation moments are

$$\kappa_k \approx f_k = O(\ln s) . \tag{4.9}$$

On the other hand, long correlation models lead to

$$\kappa_k \approx f_k = O\left((\ln s)^k\right) , \tag{4.10}$$

i.e.,

$$\gamma_{k-2} \equiv \frac{\kappa_k}{\kappa_2^{k/2}} \quad \text{are bounded} . \tag{4.11}$$

The condition (B) further requires that γ_k be successively smaller as k increases.

Assuming the condition (A) or (B), we use $O(\epsilon)$ and $O\left(\kappa_2^{-1/2}\right)$ synonymously, unless otherwise stated explicitly, since the asymptotic expansion formulas are the same for both the cases. The rates of convergence are different, however.

Then, using the definition of the Hermite polynomials,

$$H_r(x) = (-1)^r e^{x^2/2} \left(\frac{d}{dx}\right)^r e^{-x^2/2} , \tag{4.12}$$

we obtain[13]

$$P_n = \frac{\exp\left(-x^2/2\right)}{\sqrt{2\pi\kappa_2}} \left[1 + \frac{\gamma_1}{6} H_3(x) + \frac{\gamma_2}{24} H_4(x) + \frac{\gamma_3}{120} H_5(x) + \right.$$

$$\left. + \frac{\gamma_1^2}{72} H_6(x) + \frac{\gamma_1\gamma_2}{144} H_7(x) + \frac{\gamma_1^3}{1296} H_9(x) + O\left(\kappa_2^{-2}\right)\right], \tag{4.13}$$

where γ_k is defined in Eq. (4.11) and

$$x = \frac{n - \kappa_1}{\sqrt{\kappa_2}} . \tag{4.14}$$

Equation (4.13) enables us to derive an asymptotic form which is similar to Eq. (2.2) and (2.3). Or, alternatively, we may use the expression

$$P_n = \exp\left[-(\kappa_1 - m) D + \frac{\kappa_2 - \gamma^2}{2} D^2 + \sum_{k=3}^{\infty} \frac{\kappa_k (-D)^k}{k!}\right] \times$$

$$\times \frac{1}{\sqrt{2\pi}\,\gamma} \exp\left(-\frac{(n-m)^2}{2\gamma^2}\right) \left[1 + O\left(\frac{\exp\left\{-\frac{\gamma^2}{2}\left(\pi^2 - \left(\frac{n-m}{\gamma^2}\right)^2\right)\right\}}{\gamma}\right)\right], \tag{4.15}$$

which can be easily understood from the way formula (4.7) was derived, or from the identities,

$$\exp(aD)\, f(n) = f(n+a) \tag{4.16}$$

and

$$\exp\left(\frac{b}{2}D^2\right)\exp -\left\{\frac{(n-d)^2}{2c}\right\} = \sqrt{\frac{c}{b+c}}\exp\left\{-\frac{(n-d)^2}{2(b+c)}\right\}, \quad c>0, \quad b+c>0\,. \qquad (4.17)$$

[Equation (4.16) is a formal expression of the Taylor expansion and Eq. (4.17) can be proved by taking the Fourier transform of both sides.] In Eq. (4.15), m and γ^2 can be arbitrary, but later we will identify them as the mode and the correct width.

Using either form, (4.13) or (4.15), we derive the asymptotic form

$$P_n = \frac{1}{\sqrt{2\pi}\,\beta}\exp\left[-\frac{(n-m)^2}{2\gamma^2}\right]\left\{1+\sum_{k=3}^{\infty} a_k\left(\frac{n-m}{\gamma}\right)^k\right\} \qquad (4.18)$$

$$= \frac{1}{\sqrt{2\pi}\,\beta}\exp\left[-\frac{1}{2}\left(\frac{n-m}{\gamma}\right)^2+\sum_{k=3} b_k\left(\frac{n-m}{\gamma}\right)^k\right], \qquad (4.19)$$

where the parameters are computed in Appendix B;

$$m = \kappa_1 - \frac{\kappa_3}{2\kappa_2} + \left(\frac{1}{8}\frac{\kappa_5}{\kappa_2^2} - \frac{5}{12}\frac{\kappa_3\kappa_4}{\kappa_2^3} + \frac{1}{4}\frac{\kappa_3^3}{\kappa_2^4}\right) + O\left(\kappa_2^{-2}\right),$$

$$\beta = \sqrt{\kappa_2}\left(1 - \frac{1}{8}\frac{\kappa_4}{\kappa_2^2} + \frac{1}{12}\frac{\kappa_3^2}{\kappa_2^3} + O\left(\kappa_2^{-2}\right)\right), \qquad (4.20)$$

$$\gamma = \sqrt{\kappa_2}\left(1 - \frac{1}{4}\frac{\kappa_4}{\kappa_2^2} + \frac{1}{4}\frac{\kappa_3^2}{\kappa_2^3} + O\left(\kappa_2^{-2}\right)\right),$$

$$a_3 = b_3 = \frac{1}{6}\frac{\kappa_3}{\kappa_2^{3/2}} + \left(\frac{3}{8}\frac{\kappa_3\kappa_4}{\kappa_2^{7/2}} - \frac{1}{12}\frac{\kappa_5}{\kappa_2^{5/2}} - \frac{7}{24}\frac{\kappa_3^3}{\kappa_2^{9/2}}\right) + O\left(\kappa_2^{-5/2}\right),$$

$$a_4 = b_4 = \frac{1}{24}\frac{\kappa_4}{\kappa_2^2} - \frac{1}{8}\frac{\kappa_3^2}{\kappa_2^3} + O\left(\kappa_2^{-2}\right) ,$$

$$a_5 = b_5 = \frac{1}{120}\frac{\kappa_5}{\kappa_2^{5/2}} - \frac{1}{12}\frac{\kappa_3\kappa_4}{\kappa_2^{7/2}} + \frac{1}{8}\frac{\kappa_3^3}{\kappa_2^{9/2}} + O\left(\kappa_2^{-2}\right),$$

$$a_6 = \frac{1}{2} b_3^2 + O\left(\kappa_2^{-2}\right) = \frac{1}{72}\frac{\kappa_3^2}{\kappa_2^3} + O\left(\kappa_2^{-2}\right) ,$$

$$a_7 = b_3 b_4 + O\left(\kappa_2^{-5/2}\right) = \frac{1}{144}\frac{\kappa_3\kappa_4}{\kappa_2^{7/2}} - \frac{1}{48}\frac{\kappa_3^3}{\kappa_2^{9/2}} + O\left(\kappa_2^{-5/2}\right) ,$$

$$a_8 = O\left(\kappa_2^{-2}\right) ,$$

and

$$a_9 = \frac{1}{3!} b_3^3 + O\left(\kappa_2^{-5/2}\right) = \frac{1}{1296}\frac{\kappa_3^3}{\kappa_2^{9/2}} + O\left(\kappa_2^{-5/2}\right) . \tag{4.21}$$

The order of magnitude of the coefficients a_k and b_k is represented by

$$a_{3\ell-4,\ 3\ell-2,\ 3\ell} = O\left(\kappa_2^{-\ell/2}\right) , \qquad \ell \geq 3 , \tag{4.22}$$

and

$$b_k = O\left(\kappa_2^{-(k/2-1)}\right) , \qquad k \geq 3 , \tag{4.23}$$

which are analogues of Eq. (2.12) and (2.10).

Expressions (4.20) - (4.23) contain the asymptotic forms (2.9) - (2.12) for the Poisson distribution as a special case, i.e., the former reduces to the latter if one puts $\kappa_k = a$. We notice that the speed of convergence to the limit for the parameter γ is not necessarily faster than that for β unless $\kappa_2\kappa_4 \approx \kappa_3^2$, in contrast to the case of the Poisson distribution.

Finally, we present a simple example which does not satisfy the conditon (A) or (B). Consider a distribution (continuous, for simplicity),

$$P_n = \frac{1}{2\bar{n}}, \qquad \text{for } 0 < n < 2\bar{n}$$
$$= 0, \qquad \text{for } n > 2\bar{n} \quad . \tag{4.24}$$

The c.f. is expressed as

$$\phi(t) = \frac{1}{2\bar{n}} \int_0^{2\bar{n}} e^{int}\, dn = e^{i\bar{n}t} \frac{\sin \bar{n}t}{\bar{n}t}$$

$$= \sum_{k=0}^{\infty} \frac{(2\bar{n})^k}{(k+1)!} (it)^k , \tag{4.25}$$

which gives the moments[14]

$$\mu_k = \frac{(2\bar{n})^k}{k+1} \quad . \tag{4.26}$$

In order to find the cumulants, take the logarithm of Eq. (4.25),

$$\ln \phi(t) = i\bar{n}t + \ln \frac{\sin \bar{n}t}{\bar{n}t}$$

$$= i\bar{n}t + \frac{\bar{n}^2}{3} \frac{(it)^2}{2} + \sum_{k=2}^{\infty} \frac{(2\bar{n})^{2k} B_{2k}}{2k} \frac{(it)^{2k}}{(2k)!} , \text{ for } \bar{n}t < \pi , \tag{4.27}$$

where B_{2k} is the Bernoulli number[15]

$$B_{2k} = (-1)^{k-1} \frac{2(2k)!}{(2\pi)^{2k}} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \quad . \tag{4.28}$$

It is easy to see that the cumulants are

$$\kappa_1 = \bar{n}$$

$$\kappa_2 = \frac{\bar{n}^2}{3}$$

$$\kappa_{2k} = \frac{(2\bar{n})^{2k}\, B_{2k}}{2k} \tag{4.29}$$

and

$$\kappa_{2k+1} = 0 , \qquad k \geq 1 ,$$

and therefore we have

$$\left|\frac{\kappa_{2k}}{\kappa_2^k}\right| = \frac{(2\sqrt{3})^{2k}\,|B_{2k}|}{2k} > 2\left(\frac{\sqrt{3}}{\pi}\right)^{2k} (2k-1)! \longrightarrow \infty ,$$

$$\text{as } k \longrightarrow \infty , \tag{4.30}$$

which violates the condition (B). In this example, absence of "smoothness" in the distribution function is related to the violent behavior of cumulants as shown in Eq. (4.30).

V. ASYMPTOTIC LIMIT AND SCALING

The assumption (A) or (B) corresponds to very different physical models, although we can use the similar asymptotic expansion, as was mentioned in the preceding section. The difference of both the models lies in the behavior of the parameters as functions of energy.

(a) Weak correlation models

Let us assume that

$$\kappa_1 = \bar{n} = O(\ln s) \tag{5.1}$$

and consider a little more general case than Eq. (4.9), e.g.,

$$\kappa_k \approx f_k = O\left((\ell n\ s)^p\right), \qquad k \geq 2\ , \tag{5.2}$$

where p is a positive constant.

(i) The asymptotic limit

The asymptotic expansion (4.18) will approach the limiting Gaussian form

$$P_n = \frac{\exp\left[-\frac{(n-m)^2}{2\gamma^2}\right]}{\sqrt{2\pi}\ \beta}\ , \tag{5.3}$$

where

$$m \qquad \bar{n} - \frac{1}{2}\frac{\kappa_3}{\kappa_2} + O\left(\kappa_2^{-1}\right) \tag{5.4}$$

and

$$\beta = \frac{1}{\sqrt{2\pi}}\ \frac{\sigma_{inel}}{\sigma_m} \longrightarrow \sqrt{\kappa_2} \tag{5.5}$$

$$\gamma \longrightarrow \sqrt{\kappa_2} \tag{5.6}$$

However, the present energy is not sufficiently high to realize such a limit. This may be understood from the fact that the cross sections for small multiplicities are still not small and therefore the condition for using the simplest asymptotic limit, Eq. (5.3) - (5.6), is not satisfied. As a matter of fact, the normal distribution (5.3) with condition $\beta = \gamma$ implies that the distribution is normalized (automatically) for the range $(-\infty, \infty)$ but not for the range $(0, \infty)$. We may notice, however, that the parameters β and γ approach their limiting value with different speeds, as is seen from Eq. (4.20). A simple modification, keeping the normal distribution, would be to impose the normalization condition,

$$\beta = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{(n-m)^2}{2\gamma^2}} dn$$

$$= m g\left(\frac{\gamma}{m}\right) , \tag{5.7}$$

where

$$g(x) = \frac{x}{2}\left(1 + \mathrm{Erf}\left(\frac{1}{\sqrt{2}\, x}\right)\right) . \tag{5.8}$$

It is easy to see that the asymptotic form of the function g(x) is

$$g(x) \xrightarrow[x \to 0]{} x - \frac{x^2}{\sqrt{2\pi}} e^{-1/2x^2}\left(1 + O(x^2)\right) ,$$

$$\xrightarrow[x \to \infty]{} \frac{x}{2} + \frac{1}{\sqrt{2\pi}} + O(x^{-2}) , \tag{5.9}$$

and the inequality

$$\frac{1}{2} < \frac{\beta}{\gamma} < 1 \tag{5.10}$$

is satisfied.

The analysis based on Eq. (5.3) and (5.7) was carried out in Ref. 2 - 4, and shows that a reasonable agreement with experiment is obtained as long as the two-prong events are neglected.

(ii) Correction to the normal form

The question of how the two-prong events should be treated is a difficult one. If the total two-prong events are to be included, the Gaussian limit would not fit the experimental data. This is because the elastic cross section at high energy is mostly of diffractive nature and is too large to be explained by a Gaussian distribution or any other simple distribution which is supposed to cope with inelastic events. It would be more natural to use a model consisting of two components, one for the diffractive and the other for the non-diffractive (inelastic) component, as was elaborated by Quigg and Jackson.[16] The difficulty is, however, that we do not have a theoretical idea which enables us to make a clear-cut separation of the elastic amplitude into the diffractive and nondiffractive components.

The simplest possible assumption is to identify the inelastic two-prong cross section with the term $P_{n_-=0}$. This would imply that the nondiffractive component in the elastic amplitude is quite small at high energy. This is compatible with a remarkable constancy of the elastic cross section in the energy range 50 - 300 GeV (see Table 1). Adopting this assumption, we see that comparison with experiment rules out the asymptotic form, Eq. (5.3) and (5.7); the cross section of the two-pront events is too small to be fitted by a Gaussian distribution. In other words, an asymmetry around the modal point becomes evident and a correction to the normal form is definitely needed. In particular, the a_3 or b_3 term, which is the dominant one in the brackets of Eq. (4.18) or (4.19), should be included. Notice that $a_3 = b_3 \propto 1/\kappa_2^{1/2}$, while the coefficients a_k or b_k of all the other terms are at most of the order $1/\kappa_2$, according to Eq. (4.21) - (4.23). In fact, we would expect to have a positive value for a_3, in order to explain a lower value for the two-prong inelastic cross section.

We are thus led to use a modified asymptotic form

$$P_n = \frac{1}{\sqrt{2\pi}\,\beta}\, e^{-\frac{1}{2}\left(\frac{n-m}{\gamma}\right)^2} \left\{1 + a_3\left(\frac{n-m}{\gamma}\right)^3\right\} \quad , \tag{5.11}$$

or

$$P_n = \frac{1}{\sqrt{2\pi}\,\beta} \exp\left[-\frac{1}{2}\left(\frac{n-m}{\gamma}\right)^2 + a_3\left(\frac{n-m}{\gamma}\right)^3\right] \quad , \tag{5.12}$$

instead of Eq. (5.3). Around the modal point, both the formulae, Eq. (5.11) and (5.12), give roughly the same prediction, while away from the modal point, they may differ from each other. That will be reflected in the determination of a_3 when experimental data are fitted with these formulae. Which of these two should be used is a question of efficiency to reproduce the data, since they are equivalent if one takes into consideration the infinite terms.

There are some advantages to using formula (5.12): (a) It is clearly positive definite, and (b) convergence may be more efficient, as may be indicated from the comparison of Eq. (4.22) and (4.23), should a few more terms be included. Nevertheless, we will use formula (5.11) in our analysis of the experimental data for the following reasons: (a) This is the way the formula has been derived (see Appendix B), (b) a_3 is small and the accuracy of the present experiment is not sufficient to select either form, and (c) it is easy to handle the normalization condition which reads

$$\frac{\beta}{m} = g\left(\frac{\gamma}{m}\right) + \frac{a_3}{\sqrt{2\pi}} \frac{\gamma}{m}\left(2 + \frac{m^2}{\gamma^2}\right) e^{-\frac{1}{2}\frac{m^2}{\gamma^2}} \quad . \tag{5.13}$$

(iii) The asymptotic limit of the ratio γ/m

Let us define the quantity

$$\lim_{s \to \infty} \frac{\gamma}{m} = \lim_{s \to \infty} \frac{\sqrt{\kappa_2}}{\kappa_1} = d \quad . \tag{5.14}$$

If we use the normalization equation (5.7) or (5.13), we would be led to conclude that

$$d = 0 , \tag{5.15}$$

since at infinite energy, we have $\beta = \gamma$ and $a_3 = 0$, and the equation, $x = g(x)$, has the unique solution, $x = 0$. Equation (5.15) restricts the value of the exponent p in Eq. (5.2) to

$$0 < p < 2 . \tag{5.16}$$

This is a somewhat surprising result, in view of the fact that our asymptotic expansion should satisfy the normalization condition automatically (it is built in) and we have used neither such a condition nor Eq. (5.16) in its derivation.

It is possible that because of several limiting processes involved and approximations made in the discussion of the preceding section, we may be forced to a false conclusion. In fact, the precise normalization condition is a discrete sum[2] of a finite number of terms (due to energy conservation). It is also worth noting that the contact of the function $y = g(x)$ and $y = x$ is of infinite order at the origin $x = 0$, and Eq. (5.7) and the asymptotic relation $\beta \longrightarrow \gamma$ are asymptotically compatible as long as

$$d \lesssim 1 . \tag{5.17}$$

If d is nonvanishing, we have a scaling of the distribution function, as will be discussed in the next subsection.

(b) Moderate correlation models

(i) The asymptotic limit

In this case, the asymptotic expansions which are to be compared with experiment read

$$P_n = \frac{1}{\sqrt{2\pi}\,\beta}\, e^{-\frac{1}{2}\frac{(n-m)^2}{\gamma^2}} \left\{ 1 + a_3\left(\frac{n-m}{\gamma}\right)^3 + O(\epsilon^2) \right\} \tag{5.18}$$

$$m = \kappa_1 - \frac{\kappa_3}{2\kappa_2}\left(1 + O(\epsilon^2)\right) \tag{5.19}$$

$$\beta = \sqrt{\kappa_2}\left(1 + O(\epsilon^2)\right) \tag{5.20}$$

$$\gamma = \sqrt{\kappa_2}\left(1 + O(\epsilon^2)\right) \tag{5.21}$$

$$a_3 = \frac{\kappa_3}{6\,\kappa_2^{3/2}}\left(1 + O(\epsilon^2)\right) \quad , \text{ etc.} \tag{5.22}$$

These are the same formulae as in weak correlation models, but have a very different meaning. (1) All the asymptotic relations (5.4) - (5.6) are valid only approximately even at infinite energy. (2) In particular, a_3 is small (order ϵ) but does not vanish as $s \longrightarrow \infty$, and most importantly (3)

$$\bar{n} - m = \frac{\kappa_3}{2\,\kappa_2} = O(\epsilon)\sqrt{\kappa_2} \longrightarrow \infty . \tag{5.23}$$

In weak correlation models, the quantity corresponding to Eq. (5.23) approaches a constant. Although we use the same formula for analyzing experimental data, the behavior of various parameters decides which of the models is the correct one, as was pointed out earlier. Incidentally, we remark that Eq. (5.17) should be valid as long as ϵ is small.

(ii) Scaling of the distribution function

In moderate correlation models, the limit

$$\lim_{s \to \infty} \frac{\gamma}{m} = \lim_{s \to \infty} \frac{\sqrt{\kappa_2}}{\kappa_1 - \frac{\kappa_3}{2\kappa_2}} \left(1 + O(\epsilon^2)\right) = d \tag{5.24}$$

need not vanish. If that is the case, we obtain a scaling law

$$mP_n = \frac{1}{\sqrt{2\pi}\, b}\, e^{-\frac{1}{2d^2}\left(\frac{n}{m} - 1\right)^2} \left[1 + \frac{a_3}{d^3}\left(\frac{n}{m} - 1\right)^3 + O(\epsilon^2)\right] \tag{5.25}$$

or

$$= \frac{1}{\sqrt{2\pi}\, b} \exp\left[-\frac{1}{2d^2}\left(\frac{n}{m} - 1\right)^2 + \frac{a_3}{d^3}\left(\frac{n}{m} - 1\right)^3 + O(\epsilon^2)\right] \tag{5.26}$$

where

$$b = \lim_{s \to \infty} \frac{\beta}{m} = d\left(1 + O(\epsilon^2)\right) . \tag{5.27}$$

The energy dependence of Eq. (5.25) and (5.26) appears only through the modal multiplicity. This is similar to the KNO scaling[17] in that the scaling law is given by

$$\bar{n}\, P_n = \psi\, (n/\bar{n}) \tag{5.28}$$

Our scaling law is, instead,

$$m\, P_n = \varphi\, (n/m) , \tag{5.29}$$

where the scaling function φ is approximately Gaussian.

Both of the scaling laws coincide at infinite energy, provided that

$$\lim_{s \to \infty} \frac{m}{\bar{n}} = \lim_{s \to \infty} \left(1 - \frac{\kappa_3}{2\kappa_1\kappa_2}\left(1 + O(\epsilon^2)\right)\right) \tag{5.30}$$

is a finite constant. Equation (4.10) is a sufficient condition to realize such a case. Equation (5.26) is then equivalent to the formula which was used by Slattery[18] in his analysis of the experimental data. Olesen, on the other hand, used the Gaussian scaling function[19] in a similar analysis.

(c) The asymptotic relation at the mode and the Weisberger relation

Equation (5.5) or its original form in Eq. (4.20) may be written in the following form:

$$\sqrt{\kappa_2}\, P_m \equiv \sqrt{\kappa_2}\, \frac{\sigma_m}{\sigma_{inel}} = \frac{1}{\sqrt{2\pi}} \left(1 + \frac{1}{8}\frac{\kappa_4}{\kappa_2^2} - \frac{1}{12}\frac{\kappa_3^2}{\kappa_2^3} + O(\epsilon^4)\right) = \frac{\sqrt{\kappa_2}}{\sqrt{2\pi}\,\beta}. \tag{5.31}$$

On the other hand, using the saddle point method, Weisberger obtained an asymptotic relation[20]

$$\lim_{s \to \infty} \sqrt{\kappa_2}\, P_{\bar{n}} = \frac{1}{\sqrt{2\pi}} \tag{5.32}$$

for the case

$$\kappa_k \propto \ln s \quad . \tag{5.33}$$

Equation (4.20) permits us to calculate a correction term to Eq. (5.32),

$$\sqrt{\kappa_2}\, P_{\bar{n}} = \frac{1}{\sqrt{2\pi}} \left(1 + \frac{1}{8}\frac{\kappa_4}{\kappa_2^2} - \frac{5}{24}\frac{\kappa_3^2}{\kappa_2^3} + O(\epsilon^4)\right) \quad . \tag{5.34}$$

Using the empirical values of cumulants of the multiplicity distribution for the 303-GeV pp collision (Table 2 and $\kappa_4 = 6.0 \pm 4.8$ from Ref. 1), we make an estimate of correction in Eq. (5.31) and (5.34),

$$\sqrt{\kappa_2}\, P_m = \frac{1}{\sqrt{2\pi}} (1 - 0.005 \pm 0.03) \tag{5.31'}$$

and

$$\sqrt{\kappa_2}\, P_n = \frac{1}{\sqrt{2\pi}} (1 - 0.06 \pm 0.05) . \tag{5.34'}$$

The asymptotic relation (5.31) is equivalent to the Weisberger relation (5.32) in weak correlation models, while the latter needs correction terms to be added in moderate correlation models. Equations (5.31') and (5.34') may indicate that the convergence to or the approximation of the asymptotic relation at the mode, Eq. (5.5), is better than that of the Weisberger relation.

VI. COMPARISON WITH EXPERIMENT

We analyze the experimental data for pp collision with 50 - 300 GeV/c laboratory momentum based on the formula given by Eq. (5.11) or (5.18) with or without the constraint

$$P_{n_- = 0} = \text{the 2-prong inelastic cross section} \tag{6.1}$$

We consider the negative charge multiplicity distribution in order to take care of charge conservation, ($n_- = n_{ch}/2 - 1$).

The χ^2-fit of the data is shown in Fig. 1 and the parameters thus determined are listed in Table 1. Also given in Table 2 are the values of cumulants obtained from the experimental data.[1] Figures 2 and 3 and Table 3 are presented in order to show the energy dependence of some of the parameters and the validity of the asymptotic relations among them. Case I (II) of Tables 1 and 3 corresponds to the analysis with (without) the constraint (6.1) and its best fit of the experimental data is represented by solid (dashed) curves in Fig. 1. Incidentally, we did not use the normalization condition (5.13), since it must be approximately satisfied by the χ^2-fitted solution, anyway.

Summarizing the result of the analysis, it may be said that the asymptotic form (5.11) well represents the experimental data up to 300 GeV. In particular, should the 2-prong inelastic cross section be included in the analysis, the necessity of the a_3 term is evident, as was anticipated. However, an improvement of the χ^2/N ratio for the case II at 69 and 303 GeV might suggest a possibility that the constraint (6.1) is too stringent. Some portion of the elastic amplitude may be identified as nondiffractive and be added to the inelastic cross section, although its magnitude is unknown. An alternative way of decreasing the χ^2/N value is to introduce a few more correction terms. Should the accuracy be improved in future experiments, this would be a useful approach. We may point out also that there is some irregularity in the experimental data at 303 GeV which contributes to a high χ^2-value.

If Case II is preferred, then the existence of the a_3 term becomes inconclusive. Nevertheless, it should be pointed out that even in such a case, a small value for a_3 changes the value of the parameter γ significantly. Compare the values of γ in Table 1 and those given in Ref. 4. Let me mention also that the a_3 term dominates the first term of Eq. (5.11) in the prediction of high multiplicity events. Moreover, the values of γ obtained in this article are more appealing than those of Ref. 4 in the sense that (1) they are close to those determined locally around the modal point and (2) they better satisfy the expected asymptotic relations. The former point should be taken seriously since the asymptotic expansion (4.18) or (4.19) is the best approximation around the mode. The latter point will be discussed in the next section.

VII. DISCUSSION

We discuss further aspects of our analysis.

(1) Asymptotic relations

Both the asymptotic formulae

$$\frac{\beta}{\sqrt{\kappa_2}} = 1 + O(\epsilon^2) \tag{7.1}$$

and

$$\frac{\beta}{\gamma} = 1 + O(\epsilon^2) \tag{7.2}$$

are in reasonably good accord with those in Table 3. In particular, the convergence of Eq. (7.1) seems much faster, while Eq. (7.2) is satisfied in Case I but the departure from it is somewhat larger in Case II.

The energy dependence of the parameters β, γ, and $\sqrt{\kappa_2}$, shown in Fig. 2, indicates a linear increase in $\ln s$, with a slope of approximately 1/2, i.e.,

$$\beta,\ \gamma,\ \sqrt{\kappa_2} = \frac{1}{2}\ln s + \text{constant} . \tag{7.3}$$

The asymptotic relation

$$\lim_{s \to \infty} (\bar{n} - m) = \frac{2\kappa_3}{\kappa_2}\left(1 + O(\epsilon^2)\right) \tag{7.4}$$

is also consistent with the values given in Table 3 or Fig. 2, although larger errors which originate from those of κ_3 do not permit us to draw a definite conclusion.

Figure 3 and Table 3a show the validity of the asymptotic equality

$$a_3 = \frac{\kappa_3}{6\kappa_2^{3/2}}\left(1+O(\epsilon^2)\right), \quad \text{as } s \longrightarrow \infty . \tag{7.5}$$

We may notice that the quantity

$$\frac{\kappa_3}{6\kappa_2^{3/2}}$$

is roughly constant over the energy range 50 - 300 GeV. If this constancy persists at higher energy, we will be forced to choose moderate correlation models over weak correlation models and will have scaling with an approximately Gaussian scaling function. This is a view consistent with the analysis of Slattery.[18]

If that is the case, what is the magnitude of the expansion parameter ϵ? At 303 GeV, we have

$$\epsilon = \frac{\gamma_2}{\gamma_1} = \frac{\kappa_4}{\sqrt{\kappa_2}\ \kappa_3} = 0.39 \pm 0.38 \quad . \tag{7.6}$$

Unfortunately, this does not tell us much because of the large error which is due to that of κ_4. We should point out, however, that the convergence is better than what Eq. (7.6) might indicate, thanks to numerical factors. In order to see this, the first few coefficients of the asymptotic expansion, Eq. (4.18) and (4.19), are calculated using Eq. (4.21).

$$\begin{aligned}
a_3 = b_3 &= \frac{1}{6}\frac{\kappa_3}{\kappa_2^{3/2}} = 0.11 \pm 0.03 \\
a_4 = b_4 &= -0.047 \pm 0.028 \\
a_5 = b_5 &= 0.024 \pm 0.030 \\
a_6 &= 0.006 \pm 0.003 \\
a_7 &= -0.005 \pm 0.005 \\
a_9 &= 0.0002 \pm 0.0001 ,
\end{aligned} \tag{7.7}$$

where we have neglected the term which contains κ_5.

(2) The analysis with Eq. (5.12)

In order to see a difference between using Eq. (5.11) and (5.12) in our analysis, we made the χ^2-fit of the 303-GeV data with the assumption of Eq. (5.12). The parameters thus obtained are given in the last column of Table 1 (Case III), while the best fit curve for the distribution function is almost identical to the solid curve of Fig. 1e. As is seen also from Table 1, the fit is not significantly different from that of Eq. (5.11), except the value of a_3.

In conclusion, asymptotic multiplicity distributions seem to approach an approximately Gaussian form and suggest that correlations among the produced particles at high energy are not strong.

APPENDIX A

ASYMPTOTIC EXPANSION FOR THE POISSON DISTRIBUTION

From Eq. (2.1) and (2.4), it follows that

$$\ln \frac{a}{n} - \frac{1}{2n} + \frac{1}{12n^2} + O(n^{-4}) \; , \; 0, \quad \text{for } n = m. \tag{A.1}$$

Substituting the solution of the form

$$m = a + \alpha_1 + \frac{\alpha_2}{a} + \frac{\alpha_3}{a^2} + \ldots \tag{A.2}$$

in Eq. (A.1), we obtain

$$\left(\alpha_1 + \frac{1}{2}\right)\frac{1}{a} + \left(\alpha_2 - \frac{\alpha_1^2}{2} - \frac{\alpha_1}{2} - \frac{1}{12}\right)\frac{1}{a^2} +$$

$$+ \left(\alpha_3 - \alpha_1\alpha_2 + \frac{\alpha_1^3}{3} - \frac{\alpha_2}{2} + \frac{\alpha_1^2}{2} + \frac{\alpha_1}{6}\right)\frac{1}{a^3} + O(a^{-4}) = 0 \; , \tag{A.3}$$

which gives

$$\alpha_1 = -\frac{1}{2} \; , \qquad \alpha_2 = -\frac{1}{24} \; , \qquad \text{and } \alpha_3 = 0 \; . \tag{A.4}$$

Using Eq. (A.2), (A.4), (2.6), and (2.7), we get the expressions

$$\frac{1}{\gamma^2} = \frac{1}{m} - \frac{1}{2m^2} + \frac{1}{6m^3} = \frac{1}{a}\left(1 - \frac{1}{24a^2} + O(a^{-4})\right) \tag{A.5}$$

and

$$\frac{1}{\beta} = \exp\left[m \ln a - \left(m + \frac{1}{2}\right)\ln m + m - a - \frac{1}{12m} + O(m^{-3})\right]$$

$$= \frac{1}{\sqrt{a}} \exp\left[\frac{1}{24a} + O(a^{-3})\right] \quad . \tag{A.6}$$

The coefficients b_k in Eq. (2.3) can be computed using Eq. (2.8):

$$b_k = \frac{\gamma^k}{k} \left[(-1)^{k-1} \frac{(k-2)!}{m^{k-1}} + \frac{(-1)^k (k-1)}{2\,m^k} + \frac{(-1)^{k+1} k}{12\,m^{k+1}} + O(m^{-k-3}) \right]$$

$$= \frac{(-1)^{k-1}}{k(k-1)} \frac{1}{a^{k/2-1}} \left[1 + \frac{k}{48\,a^2} + O(a^{-4}) \right] \left[1 - \frac{(k-1)^2}{24\,a^2} + O(a^{-4}) \right]$$

$$= \frac{(-1)^{k-1}}{k(k-1)} \frac{1}{a^{k/2-1}} \left[1 - \frac{(2k-1)(k-2)}{48\,a^2} + O(a^{-4}) \right] . \qquad \text{(A.7)}$$

It is easy to see that a_k are given by Eq. (2.11). In order to obtain the order of magnitude for a_k, we observe that the dominant contribution in a_k is given by the term $b_{k-3} b_3$ or $b_{k-3\ell} b_3^{\ell}$, since b_3 is the largest of all the coefficients. Then, by induction, we can prove Eq. (2.12).

APPENDIX B

DERIVATION OF THE ASYMPTOTIC FORM, EQ. (4.20 - 23)

Defining the quantities,

$$y = \frac{n - m}{\gamma} \quad ,$$

$$D_y = \frac{\partial}{\partial y} \quad ,$$

$$\frac{\kappa_1 - m}{\gamma} = \lambda_1 \quad ,$$

$$\frac{\kappa_2 - \gamma^2}{\gamma^2} = \lambda_2 \quad , \tag{B.1}$$

and

$$\frac{\kappa_k}{\gamma^k} = \lambda_k \, , \qquad k \geq 3$$

we may express Eq. (4.15) as[21]

$$P_n = \exp\left[\sum_{k=1}^{\infty} \frac{\lambda_k(-D_y)^k}{k!}\right] \frac{1}{\sqrt{2\pi}\;\gamma}\, e^{-\frac{y^2}{2}} \quad , \tag{B.2}$$

where the extra term which vanishes exponentially as $\kappa_2 \longrightarrow \infty$ was dropped. In Eq. (B.2), m and γ should be determined in such a way that Eq. (B.2) coincides with Eq. (4.19). Anticipating that

$$\lambda_1 = O\left(\frac{1}{\gamma}\right) = O\left(\frac{1}{\kappa_2^{-1/2}}\right),$$

$$\lambda_2 = O\left(\frac{1}{\gamma^2}\right) = O\left(\frac{1}{\kappa_2^{-1}}\right) \, , \tag{B.3}$$

and

$$\lambda_k = O\left(\gamma^{-k+2}\right) = O\left(\kappa_2^{-\frac{k}{2}+1}\right) , \qquad k \geq 3 ,$$

Eq. (B.2) can be expanded as[22]

$$P_n = \frac{1}{\sqrt{2\pi}\,\gamma}\, e^{-\frac{y^2}{2}} \Big[1 + \lambda_1 H_1(y) + \frac{1}{2!}\left(\lambda_1^2 + \lambda_2\right) H_2(y) +$$

$$+ \frac{1}{3!}\left(\lambda_3 + 3\lambda_2\lambda_1 + \lambda_1^3\right) H_3(y) + \frac{1}{4!}\left(\lambda_4 + 4\lambda_3\lambda_1\right) H_4(y) +$$

$$+ \frac{1}{5!}\left(\lambda_5 + 5\lambda_4\lambda_1 + 10\lambda_3\lambda_2 + 10\lambda_3\lambda_1^2\right) H_5(y) + \frac{10}{6!}\lambda_3^2 H_6(y) +$$

$$+ \frac{1}{7!}\left(35\lambda_4\lambda_3 + 70\lambda_3^2\lambda_1\right) H_7(y) +$$

$$+ \frac{280}{9!}\lambda_3^3 H_9(y) + O(\gamma^{-4}) \Big] \qquad \text{(B.4)}$$

The assumption $\lambda_2 = O(\gamma^{-2})$, instead of $O(\gamma^{-1})$, is justified a posteriori, and is expected also from the solution for the Poisson distribution. (Otherwise, we would have to keep a few more terms in Eq. (B.4). Explicit calculation, then, shows that the assumption is correct.) In order to simplify the algebra, we use this assumption from the outset.

The explicit forms of the relevant Hermite polynomials are given below:

$$H_1(y) = y$$

$$H_2(y) = y^2 - 1$$

$$H_3(y) = y^3 - 3y$$

$$H_4(y) = y^4 - 6y^2 + 3 \qquad \text{(B.5)}$$

$$H_5(y) = y^5 - 10y^3 + 15y$$
$$H_6(y) = y^6 - 15y^4 + 45y^2 - 15$$
$$H_7(y) = y^7 - 21y^5 + 105y^3 - 105y$$
$$H_8(y) = y^8 - 28y^6 + 210y^4 - 420y^2 + 105$$
$$H_9(y) = y^9 - 36y^7 + 378y^5 - 1260y^3 + 945y \qquad \text{(B.5)}$$

The condition that the terms linear and quadratic in y are missing in the parentheses of Eq. (B.4) leads to two equations:

$$\lambda_1 - \frac{1}{2!}\left(\lambda_3 + 3\lambda_2\lambda_1 + \lambda_1^3\right) + \frac{3}{4!}\left(\lambda_5 + 5\lambda_4\lambda_1 + 10\lambda_3\lambda_2 + 10\lambda_3\lambda_1^2\right) -$$
$$- \frac{105}{7!}\left(35\lambda_4\lambda_3 + 70\lambda_3^2\lambda_1\right) + \frac{280 \times 945}{9!}\lambda_3^3 = 0 \qquad \text{(B.6)}$$

and

$$\frac{1}{2}\left(\lambda_2 + \lambda_1^2\right) - \frac{6}{4!}\left(\lambda_4 + 4\lambda_3\lambda_1\right) + \frac{450}{6!}\lambda_3^2 = 0 \; . \qquad \text{(B.7)}$$

The solutions for λ_1 and λ_2 which are of the form

$$\lambda_1 = a\lambda_3 + b\lambda_4 + c\lambda_5 + d\lambda_3^2 + e\lambda_3\lambda_4 + f\lambda_3^3 + O(\gamma^{-4}) \qquad \text{(B.8)}$$

$$\lambda_2 = b'\lambda_4 + c'\lambda_5 + d'\lambda_3^2 + e'\lambda_3\lambda_4 + f'\lambda_3^3 + O(\gamma^{-4}) \qquad \text{(B.9)}$$

are sought by substituting them in Eq. (B.6) and (B.7). We thus obtain

$$a = \frac{1}{2}, \quad b = 0, \quad c = -\frac{1}{8}, \quad d = 0, \quad e = \frac{1}{6}, \quad f = 0,$$
$$b' = \frac{1}{2}, \quad c' = 0, \quad d' = -\frac{1}{2}, \quad e' = f' = 0 \qquad \text{(B.10)}$$

i.e.,

$$\lambda_1 = \frac{\kappa_1 - m}{\gamma} = \frac{1}{2}\lambda_3 - \frac{1}{8}\lambda_5 + \frac{1}{6}\lambda_3\lambda_4 + O(\gamma^{-4}) \tag{B.11}$$

$$\lambda_2 = \frac{\kappa_2 - \gamma^2}{\gamma^2} = \frac{1}{2}\lambda_4 - \frac{1}{2}\lambda_3^2 + O(\gamma^{-4})$$

$$= \frac{1}{2}\frac{\kappa_4}{\gamma^4} - \frac{1}{2}\frac{\kappa_3^2}{\gamma^6} + O(\gamma^{-4}) \; . \tag{B.12}$$

These equations can be solved easily; we get

$$\gamma^2 = \kappa_2 - \frac{1}{2}\frac{\kappa_4}{\kappa_2} + \frac{1}{2}\frac{\kappa_3^2}{\kappa_2^2} + O\left(\kappa_2^{-2}\right) \tag{B.13}$$

and

$$m = \kappa_1 - \frac{1}{2}\frac{\kappa_3}{\gamma^2} + \frac{1}{8}\frac{\kappa_5}{\gamma^4} - \frac{1}{6}\frac{\kappa_3\kappa_4}{\gamma^6} + O(\gamma^{-4})$$

$$= \kappa_1 - \frac{1}{2}\frac{\kappa_3}{\kappa_2} + \frac{1}{8}\frac{\kappa_5}{\kappa_2^2} - \frac{5}{12}\frac{\kappa_3\kappa_4}{\kappa_2^3} + \frac{1}{4}\frac{\kappa_3^3}{\kappa_2^4} + O\left(\kappa_2^{-2}\right). \tag{B.14}$$

From the value for $n = m$, $(y = 0)$, the parameter β is determined as follows:

$$\frac{1}{\beta} = \frac{1}{\gamma}\left[1 - \frac{1}{2}\left(\lambda_1^2 + \lambda_2\right) + \frac{1}{8}\left(\lambda_4 + 4\lambda_3\lambda_1\right) - \frac{150}{6}\lambda_3^2 + O(\gamma^{-4})\right]$$

$$= \frac{1}{\gamma}\left[1 + \frac{1}{6}\lambda_3^2 - \frac{1}{8}\lambda_4\right]$$

$$= \frac{1}{\gamma}\left[1 - \frac{1}{8}\frac{\kappa_4}{\kappa_2^2} + \frac{1}{6}\frac{\kappa_3^2}{\kappa_2^3}\right]$$

i.e.,

$$\beta = \sqrt{\kappa_2}\left[1 - \frac{1}{8}\frac{\kappa_4}{\kappa_2^2} + \frac{1}{12}\frac{\kappa_3^2}{\kappa_2^3} + O\left(\kappa_2^{-2}\right)\right] \tag{B.15}$$

The parameters a_k are given by the coefficients of y^k; explicitly written,

$$a_3\left(1 - \frac{1}{8}\frac{\kappa_4}{\kappa_2^2} + \frac{1}{6}\frac{\kappa_3^2}{\kappa_2^3}\right) = \frac{1}{6}\left(\lambda_3 + 3\lambda_2\lambda_1 + \lambda_1^3\right) - \frac{1}{12}\left(\lambda_5 + 5\lambda_4\lambda_3 + 10\lambda_3\lambda_2 + 10\lambda_3\lambda_1^2\right) +$$

$$+ \frac{1}{48}\left(35\lambda_4\lambda_3 + 70\lambda_3^2\lambda_1\right) - \frac{35}{36}\lambda_3^3$$

$$= \frac{1}{6}\lambda_3 - \frac{1}{12}\lambda_5 + \frac{11}{48}\lambda_3\lambda_4 - \frac{5}{36}\lambda_3^3$$

$$= \frac{1}{6}\frac{\kappa_3}{\kappa_2^{3/2}} + \frac{17}{48}\frac{\kappa_3\kappa_4}{\kappa_2^{7/2}} - \frac{1}{12}\frac{\kappa_5}{\kappa_2^{5/2}} - \frac{19}{72}\frac{\kappa_3^3}{\kappa_2^{9/2}} + O\left(\kappa_2^{-5/2}\right),$$

$$a_4 = \frac{1}{4!}\left(\lambda_4 + 4\lambda_3\lambda_1\right) - \frac{150}{6!}\lambda_3^2$$

$$= \frac{1}{24}\lambda_4 - \frac{1}{8}\lambda_3^2$$

$$= \frac{1}{24}\frac{\kappa_4}{\kappa_2^2} - \frac{1}{8}\frac{\kappa_3^2}{\kappa_2^3} + O\left(\kappa_2^{-2}\right),$$

$$a_5 = \frac{1}{5!}\left(\lambda_5 + 5\lambda_4\lambda_1 + 10\lambda_3\lambda_2 + 10\lambda_3\lambda_1^2\right) - \frac{21}{7!}\left(35\lambda_4\lambda_3 + 70\lambda_3\lambda_1^2\right) + \frac{280 \times 378}{9!}\lambda_3^3$$

$$= \frac{1}{120}\lambda_5 - \frac{1}{12}\lambda_4\lambda_3 + \frac{1}{8}\lambda_3^3$$

$$= \frac{1}{120}\frac{\kappa_5}{\kappa_2^{5/2}} - \frac{1}{12}\frac{\kappa_3\kappa_4}{\kappa_2^{7/2}} + \frac{1}{8}\frac{\kappa_3^3}{\kappa_2^{9/2}} + O\left(\kappa_2^{-5/2}\right),$$

$$a_6 = \frac{10}{6!}\gamma_3^2 = \frac{1}{72}\frac{\kappa_3^2}{\kappa_2^3} + O\left(\kappa_2^{-2}\right), \qquad \text{(B.16)}$$

$$a_7 = \frac{1}{24}\left(\lambda_4\lambda_3 + 2\lambda_3^2\lambda_1\right) - \frac{1}{36}\lambda_3^3$$

$$= \frac{1}{144}\lambda_4\lambda_3 - \frac{1}{48}\lambda_3^3$$

$$= \frac{1}{144}\frac{\kappa_3\kappa_4}{\kappa_2^{7/2}} - \frac{1}{48}\frac{\kappa_3^3}{\kappa_2^{9/2}} + O\left(\kappa_2^{-5/2}\right) ,$$

$$a_8 = O\left(\kappa_2^{-2}\right),$$

$$a_9 = \frac{1}{1296}\lambda_3^3 = \frac{1}{1296}\frac{\kappa_3^3}{\kappa_2^{9/2}} + O\left(\kappa_2^{-5/2}\right),$$

which lead to the expression of Eq. (4.21). The dominant term in the parameter a_k $(k \geq 6)$ is given by $\lambda_{k-3}\lambda_3$ and its general form is shown, by induction, to be

$$a_{3\ell-4,\ 3\ell-2,\ 3\ell} = O\left(\kappa_2^{-\ell/2}\right) . \tag{B.17}$$

The coefficients b_k in Eq. (4.19) are related to a_k through the relations in (2.11), and therefore must behave like

$$b_k = O\left(\kappa_2^{-(k/2-1)}\right) , \tag{B.18}$$

at most, otherwise it would upset Eq. (B.17), according to an argument similar to that given at the end of Appendix A.

(q.e.d.)

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Table Captions

Table 1 The values for the parameters. Column I(II) corresponds to the χ^2 fit by Eq. (5.11) with (without) the 2 prong inelastic cross section included. The last Column III for the 303 GeV experiment is obtained under the assumption Eq. (5.12).

Table 2 Experimental values for cumulants.

Table 3 The ratio of various parameters

a. Case I

b. Case II

TABLE I

	50		69		102	
	I	II	I	II	I	II
m_-	1.26 ± 0.08	1.13 ± 0.10	1.34 ± 0.01	1.18 ± 0.02	1.46 ± 0.08	1.20 ± 0.12
σ_{m_-}	9.32 ± 0.30	9.39 ± 0.31	8.52 ± 0.07	8.71 ± 0.09	8.05 ± 0.27	8.13 ± 0.25
γ	1.47 ± 0.14	1.62 ± 0.11	1.61 ± 0.01	1.75 ± 0.01	1.70 ± 0.04	2.18 ± 0.13
a_3	0.036 ± 0.29	0.004 ± 0.26	0.078 ± 0.008	0.050 ± 0.005	0.117 ± 0.019	0.001 ± 0.60
χ^2	4.0	3.4	13.0	3.0	2.9	1.5
N(degree of freedom)	4	3	5	4	5	4
β	1.34 ± 0.04	1.38 ± 0.05	1.50 ± 0.01	1.52 ± 0.02	1.63 ± 0.05	1.71 ± 0.05
$\Sigma\sigma_n$(mb)	31.3	32.5	32.1	33.3	32.8	34.9
σ_{inel}(exp)	31.12 ± 1.33		31.32 ± 0.52		32.8 ± 1.1	
σ_{el}(exp)	6.9 ± 0.2		7.0 ± 0.2		6.9 ± 1.0	

continued

TABLE I (continued)

	205		303		
	I	II	I	II	III
m_-	2.06 ± 0.06	2.06 ± 0.05	2.51 ± 0.08	2.43 ± 0.07	2.63 ± 0.10
σ_{m_-}	6.58 ± 0.14	6.58 ± 0.10	5.75 ± 0.15	5.67 ± 0.16	5.79 ± 0.16
γ	2.05 ± 0.03	2.06 ± 0.03	2.32 ± 0.05	2.66 ± 0.21	2.35 ± 0.09
a_3	0.104 ± 0.012	0.100 ± 0.012	0.097 ± 0.015	0.029 ± 0.67	0.047 ± 0.88
χ^2	8.3	8.3	17.6	11.5	17.0
N(degree of freedom)	7	6	9	8	9
β	1.98 ± 0.04	1.98 ± 0.03	2.25 ± 0.06	2.35 ± 0.07	2.30 ± 0.06
$\Sigma\sigma_n$(mb)	32.6	32.7	32.4	33.4	32.5
σ_{inel}(exp)	32.7 ± 1.2			31.8 ± 1.0	
σ_{el}(exp)	6.8 ± 0.3			7.2 ± 0.4	

TABLE 2

	50	69	102	205	303
$\kappa_1 = f_1 = \bar{n}_-$	1.66 ± 0.07	1.95 ± 0.04	2.17 ± 0.07	2.82 ± 0.08	3.43 ± 0.08
$\kappa_2 = f_1 + f_2$	1.67 ± 0.11	2.09 ± 0.07	2.56 ± 0.12	3.77 ± 0.22	4.79 ± 0.26
$\kappa_3 = f_1 + 3f_2 + f_3$	1.39 ± 0.32	2.06 ± 0.21	2.70 ± 0.56	4.46 ± 0.88	7.11 ± 1.06
$\sqrt{\kappa_2}$	1.29 ± 0.04	1.45 ± 0.02	1.60 ± 0.04	1.94 ± 0.06	2.19 ± 0.06
$\kappa_3/2\kappa_2$	0.42 ± 0.12	0.49 ± 0.07	0.53 ± 0.13	0.59 ± 0.15	0.74 ± 0.15
$\kappa_3/6\kappa_2^{3/2}$	0.107 ± 0.028	0.114 ± 0.017	0.110 ± 0.031	0.102 ± 0.029	0.113 ± 0.026

TABLE 3a

	50	69	102	205	303
$\beta / \sqrt{\kappa_2}$	1.04 ± 0.07	1.04 ± 0.02	1.03 ± 0.07	1.02 ± 0.06	1.03 ± 0.06
β / γ	0.91 ± 0.12	0.93 ± 0.01	0.96 ± 0.05	0.97 ± 0.03	0.97 ± 0.05
β / m	1.06 ± 0.10	1.12 ± 0.02	1.12 ± 0.10	0.96 ± 0.05	0.90 ± 0.05
γ / m	1.17 ± 0.19	1.20 ± 0.02	1.16 ± 0.09	1.00 ± 0.04	0.92 ± 0.05
$(\bar{n}-m) / \frac{\kappa_3}{2\kappa_2}$	0.95 ± 0.53	1.24 ± 0.24	1.34 ± 0.52	1.29 ± 0.50	1.24 ± 0.43
$a_3 / \frac{\kappa_3}{6\,\kappa_2^{3/2}}$	0.34 ± 2.8	0.68 ± 0.17	1.06 ± 0.47	1.02 ± 0.41	0.86 ± 0.33

TABLE 3b

	50	69	102	205	303
$\beta/\sqrt{\kappa_2}$	1.07 ± 0.07	1.05 ± 0.03	1.08 ± 0.07	1.02 ± 0.05	1.07 ± 0.07
β/γ	0.85 ± 0.09	0.87 ± 0.02	0.78 ± 0.07	0.96 ± 0.03	0.88 ± 0.10
β/m	1.22 ± 0.10	1.29 ± 0.04	1.43 ± 0.19	0.96 ± 0.05	0.97 ± 0.09
γ/m	1.43 ± 0.22	1.48 ± 0.04	1.82 ± 0.29	1.00 ± 0.04	1.09 ± 0.06
$(\bar{n}-m)/\frac{\kappa_3}{2\kappa_2}$	1.26 ± 0.67	1.57 ± 0.31	1.83 ± 0.71	1.29 ± 0.48	1.35 ± 0.44

Figure Captions

Figure 1 The negative charge multiplicity distribution in the pp collision. The solid line and the dashed line represent the χ^2 fit with and without the 2 prong inelastic events, respectively.

a. 50 GeV

b. 69 GeV

c. 102 GeV

d. 205 GeV (The dashed line coincides with the solid line.)

e. 303 GeV

Figure 2 Energy dependence of the parameters in Case I. The asymptotic equalities read $\beta = \gamma = \sqrt{\kappa_2}$ and $\bar{n} - m = \kappa_3/2\kappa_2$. The solid line is an eye-fitted linear curve with gradient 1/2.

Figure 3 Test of the asymptotic equality $a_3 = \kappa_3/6\kappa_2^{3/2}$. (Case I)

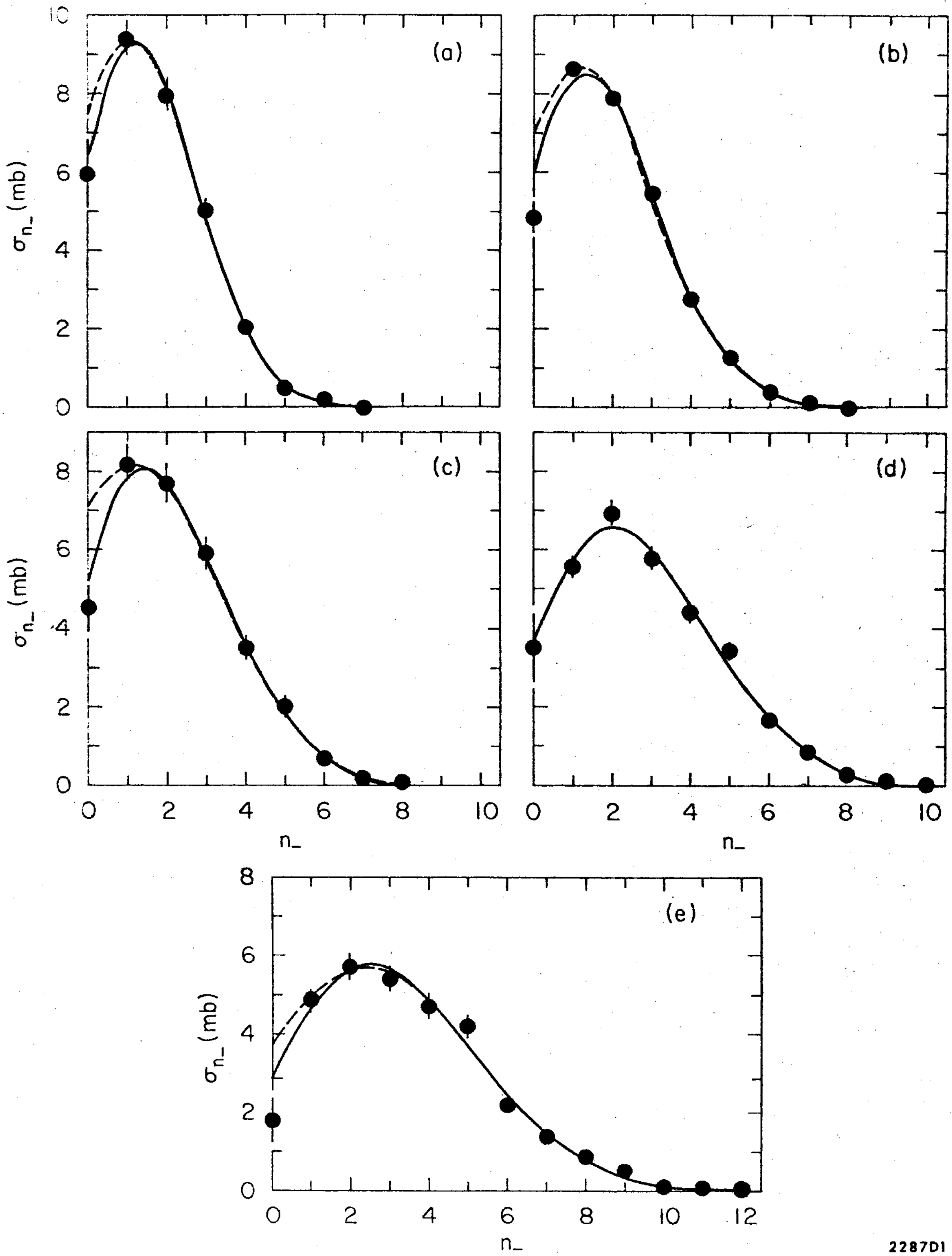
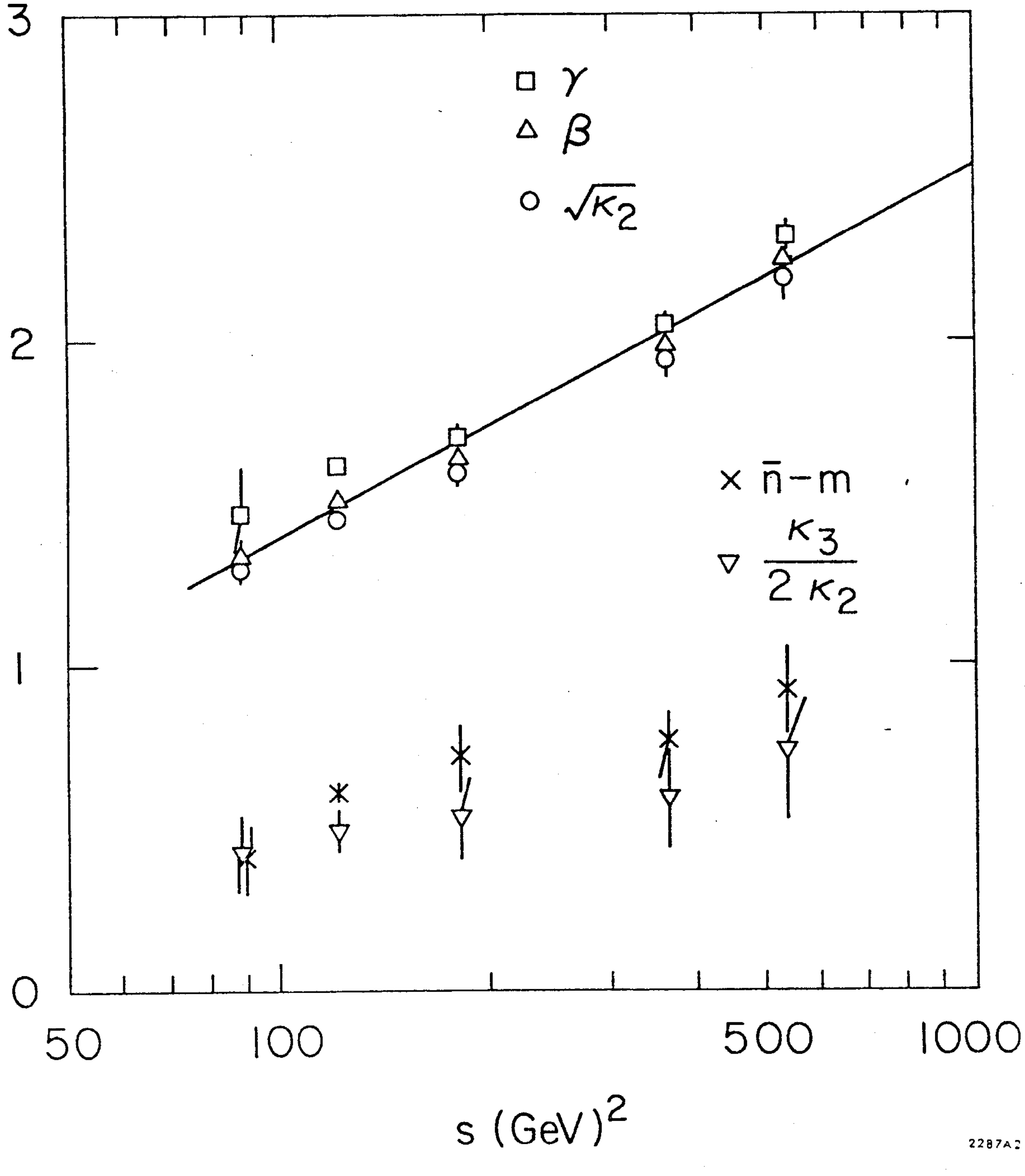
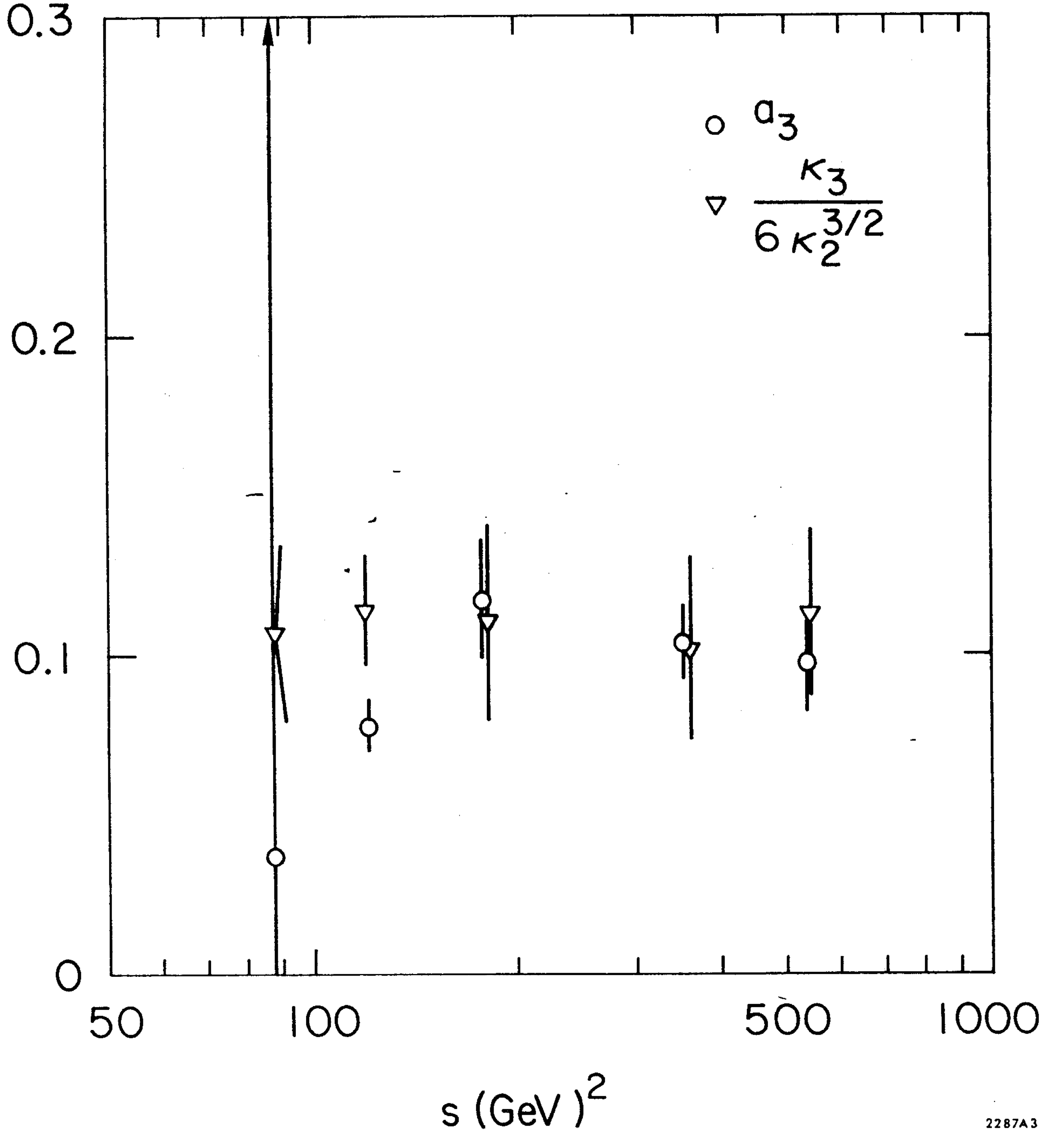


Fig. 1



3
2
1
0
□ γ
△ β
○ $\sqrt{\kappa_2}$
× $\bar{n}-m$
▽ $\frac{\kappa_3}{2\kappa_2}$
50
100
500
1000
s (GeV)^2
2287A2

Fig. 2



0.3
0.2
0.1
0
50
100
500
1000
s (GeV)^2
a_3
$\frac{\kappa_3}{6\,\kappa_2^{3/2}}$
2287A3

Fig. 3