MODE AND SCALING IN CHARGED MULTIPLICITY DISTRIBUTIONS[*]

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ABSTRACT

A quasi-normal expansion is used to examine a possibility for scaling of charged multiplicity distributions in pp collisions.

It has been pointed out by several authors that the charged multiplicity cross section σ_n in pp collisions[1] with incident energies 50–300 GeV are well represented by normal[2-5] or approximately normal[6,7] distributions. The KNO scaling[8]

$$P_n \equiv \frac{\sigma_n}{\sigma_{inel}} = \frac{1}{\bar{n}} \, \psi\left(\frac{n}{\bar{n}}\right) \tag{1}$$

also seems not inconsistent with experiment at the present energy, where ψ is approximately Gaussian.[4,6]

A mathematical basis which leads us to obtain a quasi-normal distribution was discussed in references 7, 9, and 10. It is an analogue of the central limit theorem and can be stated in the following way: the asymptotic expansion at the mode[11] m,

$$P_n = \frac{1}{\sqrt{2\pi}\,\beta} \exp\left[-\frac{1}{2}\left(\frac{n-m}{\gamma}\right)^2\right] \left\{1 + \sum_{k=3}^{\infty} a_k\left(\frac{n-m}{\gamma}\right)^k\right\} , \tag{2a}$$

$$= \frac{1}{\sqrt{2\pi}\,\beta} \exp\left\{-\frac{1}{2}\left(\frac{n-m}{\gamma}\right)^2 + \sum_{k=3}^{\infty} b_k\left(\frac{n-m}{\gamma}\right)^k\right\} , \tag{2b}$$

is valid provided that

$$\kappa_2 \to \infty \quad \text{as} \quad s \to \infty \tag{3}$$

and that the condition

$$\left| \frac{n-m}{\gamma^2} \right| < \pi \tag{4}$$

is satisfied. The parameters β, m, γ, a_k and b_k can be expressed in terms of moments, deviants[10] or cumulants[12,13]. If correlations of the produced particles are temperate[7,9] in the sense that higher cumulants κ_k satisfy the condition

$$\kappa_k / \kappa_2^{k/2} = O(\epsilon^{k-2}) , \quad k \geq 3 , \tag{5}$$

with

$$\epsilon \ll 1 \ , \tag{6}$$

then we have

$$a_{3\ell-4,\,3\ell-2,\,3\ell} = O(\epsilon^{\ell}) \ , \quad \ell \geq 1 \ , \tag{7}$$

and

$$b_k = O(\epsilon^{k-2}) \ , \quad k \geq 3 \ . \tag{8}$$

Therefore, only a few terms in expansion (4) are important.

If, moreover, the limits[7]

$$\lim_{s \to \infty} \frac{\beta}{m} = \lim_{s \to \infty} \frac{\sqrt{\kappa_2}}{\kappa_1 - \frac{1}{2}\frac{\kappa_3}{\kappa_2}} \left(1 + O(\epsilon^2) \right) = b \ , \tag{9}$$

$$\lim_{s \to \infty} \frac{\gamma}{m} = \lim_{s \to \infty} \frac{\sqrt{\kappa_2}}{\kappa_1 - \frac{1}{2}\frac{\kappa_3}{\kappa_2}} \left(1 + O(\epsilon^2) \right) = d \ , \tag{10}$$

are nonvanishing, Eq. (2) reduces to

$$m\frac{\sigma_n}{\sigma_{\text{inel}}} = \frac{1}{\sqrt{2\pi}\,b} \exp\left[-\frac{1}{2d^2}\left(\frac{n}{m}-1\right)^2 \right] \left\{ 1 + \frac{a_3}{d^3}\left(\frac{n}{m}-1\right)^3 + \ldots \right\} \tag{11a}$$

$$= \frac{1}{\sqrt{2\pi}\,b} \exp\left\{ -\frac{1}{2d^2}\left(\frac{n}{m}-1\right)^2 + \frac{a_3}{d^3}\left(\frac{n}{m}-1\right)^3 + \ldots \right\} \tag{11b}$$

This is a scaling in terms of the variable n/m, while the KNO scaling is expressed in terms of the variable $n/\bar{n}$ (Eq. (1)). Both scaling laws coincide at infinite energy provided that the limit

$$\lim_{s \to \infty} \frac{m}{\bar{n}} = \lim_{s \to \infty} \left\{ 1 - \frac{\kappa_3}{2\kappa_1\kappa_2}\left(1 + O(\epsilon^2) \right) \right\} \tag{12}$$

exists.[14] Certainly that is the case if

$$\kappa_k \propto (\ell n\, s)^k \, . \tag{14}$$

The advantages of using Eq. (11) over the KNO scaling (1) are that (i) the approximation is best around the mode, and (ii) the scaling function is proved to be quasi-normal.

From this point of view, we analyze the pp collision experimental data based on Eq. (11). The results are shown in Figure 1 a, b, and in columns A and B of Table 1. A few remarks are given below.

(1) With the accuracy and the energy range of the present experiment, we do not see an essential difference between the two forms of Eq. (11). Any improvement of the accuracy or increase of the energy of the experiment might differentiate the two forms with respect to the effectiveness of representing the experimental data. Although the two forms of Eq. (11) are mathematically equivalent when one takes an infinite number of terms, there is a difference in practice when only a finite number of terms are considered.

(2) The necessity of the a_3 term is clearly exhibited. We repeated the analysis neglecting the errors due to σ_{inel}, since their inclusion forces us to give more weight to the events of high multiplicities. The results are shown in columns A' and B' of Table 1, in which the real χ^2 value should be obtained from the listed value divided by a factor 1.5 ~ 2. In this way, the relative weight of the experimental data around the mode is increased, which is a reasonable procedure because of the nature of expansion (11). We do not see a big difference in the results, however.

In conclusion, the scaling based on Eq. (11) is consistent with the present experimental data.

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References and Footnotes

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11. Mode is the value of the multiplicity for which the distribution function is maximum.

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13.   Cumulants are defined by

$$\phi(t) = \sum_{n=0}^{\infty} e^{int} P_n = \exp\left[\sum_{k=1}^{\infty} \frac{\kappa_k (it)^k}{k!}\right]$$

e.g. $\kappa_1 = \overline{n}$, $\kappa_2 = \overline{(n-\overline{n})^2}$, $\kappa_3 = \overline{(n-\overline{n})^3}$,   etc.

14.   In this case, Eq. (11b) gives an explanation of the scaling function used in

reference 6.

Figure Captions

Figure 1   Scaling of negative charge multiplicity distributions in pp collisions.

a.   With scaling function, Eq. (11a).

b.   With scaling function, Eq. (11b).

Table Captions

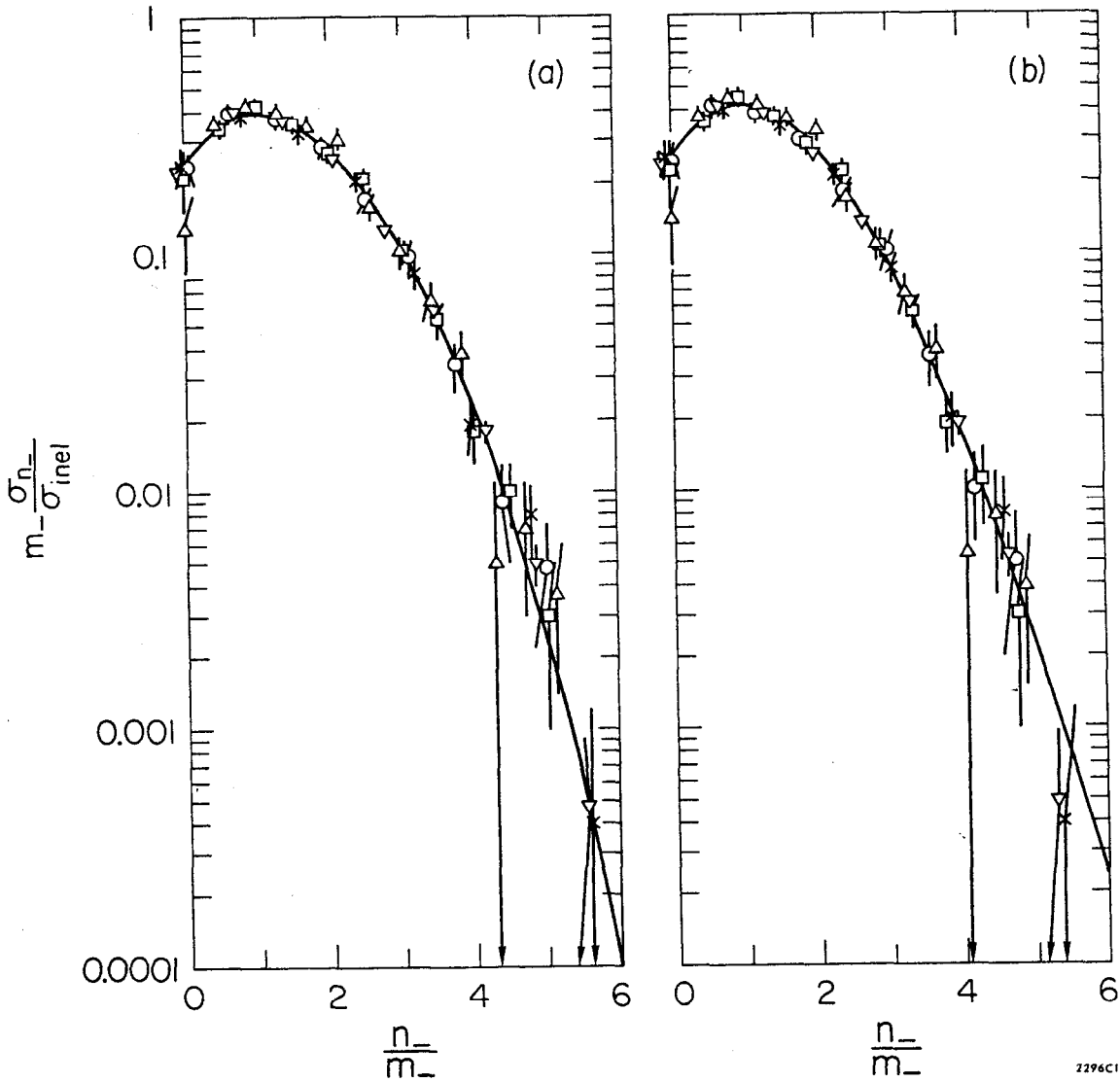Table I   Values of the parameters for the best fit [N (degree of freedom) = 42].

A.   With scaling function, Eq. (11a).

B.   With scaling function, Eq. (11b).

The prime indicates the case where the errors due to σ_{inel} are

neglected. The number in the parenthesis that follows m represents

energy in the unit GeV.

TABLE I

	A	A'	B	B'
$\beta/m = b$	1.02 ± 0.01	1.03 ± 0.01	0.97 ± 0.02	0.98 ± 0.01
$\gamma/m = d$	1.10 ± 0.01	1.11 ± 0.01	1.05 ± 0.02	1.08 ± 0.01
a_3	0.071 ± 0.006	0.081 ± 0.005	0.036 ± 0.072	0.037 ± 0.002
m(50)	1.25 ± 0.03	1.23 ± 0.01	1.31 ± 0.03	1.29 ± 0.01
m(69)	1.44 ± 0.01	1.41 ± 0.01	1.51 ± 0.02	1.48 ± 0.01
m(102)	1.59 ± 0.04	1.56 ± 0.03	1.68 ± 0.04	1.63 ± 0.02
m(205)	1.99 ± 0.03	1.96 ± 0.01	2.10 ± 0.04	2.06 ± 0.03
m(303)	2.33 ± 0.05	2.28 ± 0.03	2.46 ± 0.05	2.39 ± 0.03
χ^2	35.6	60.4	35.0	62.4

Fig. 1