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THEOREMS AND BOUNDS RELATING TO THE TWO PHOTON DECAY OF THE PION*

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ABSTRACT

An exact bound on the $\pi^o \rightarrow 2\gamma$ vertex function is derived. From it we are able to estimate bounds to the corrections to the vector dominance model, the anomalous constant S, and the asymptotic behavior of the form factor. A theorem is derived which states that the coefficient (K) of the axial current in the short distance expansion of two electromagnetic currents is finite if $\sigma_{e^+e^-} \rightarrow 1/S$, but vanishes if $\sigma_{e^+e^-}$ falls faster.

Interest in the two photon decay of the π^o has intensified over the past few years as the focus of attention has shifted from the old vector meson dominance (VD) model[1] to the more fashionable ideas of anomalies and light cone operator product expansions.[2,3,4] For example, last year Crewther[2] showed that the anomalous constant S (whose size governs the two photon decay of a massless pion) can be related to R, the asymptotic ratio of $\sigma_{e^+e^-}$ to that for Bhabha scattering, and K, the coefficient of the axial vector current, A^μ, in the short distance expansion of the product of the two electromagnetic currents (j_μ); ($4S \cong KR$).[5] The power of this kind of analysis lies, of course, in the remarkable feature that

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one can apparently glean information concerning the charge structure of the basic constituents of the electromagnetic current from a low energy decay process. For instance, in the usual quark model $K = 1$ and $R = 2/3$ so $S = 1/6$ which differs for the empirical result by roughly a factor of 3. This has led Gell-Mann[4] to extend the old quark model from the original three basic units to nine in such a way that R is now 2 and $S = 1/3$ which is more in accord with experiment (K is still unity). In making such a comparison one is, of course, assuming a smooth continuation in the pion mass and this assumption has recently been questioned by some authors.[4] For example, Preparata[4] has constructed a VD model whose continuum contributions are "controlled by the light cone" but which vanishes in the massless pion limit where only the anomaly survives. For the physical pion, the anomaly contributes only a small percentage of the amplitude and seems to be intimately connected to the corrections to the VD model.

It is the purpose of the present paper to derive an exact inequality for the $\pi^0\gamma\gamma$ vertex function G from which several interesting results pertinent to these models can be derived. The inequality is a fairly straightforward one which follows from the Schwartz inequality and bounds G in terms of R and W_1, one of the structure functions describing inelastic electron scattering from pions. These results can be summarized as follows: (a) an upper bound for the corrections to the VD model (which turns out to be relatively weak); (b) an upper bound on S, which depends somewhat on the extrapolation to zero pion mass; (c) a rigorous bound on the behavior of G as a function of the mass of one of the photons (the other being held fixed), see Eq. (8) below; (d) a bound on K which is perilously close to 1; and (e) a theorem which, roughly speaking, states that K is finite if R vanishes (i.e. when Z_3 for the photon is finite).

The details of these assertions will be discussed below. First, to the derivation of the basic inequality. We begin by defining the form factor $G(q^2, k^2)$ via the equation

$$G(q^2,k^2)\,\epsilon_{\mu\nu\rho\sigma}p^\rho q^\sigma \equiv i\int d^4x\, e^{-iq\cdot x} < 0 \mid T[j_\mu(x) j_\nu(0)] \mid \pi^o(p) > . \tag{1}$$

Here, q and k are the photon 4-momenta and p that of the pion (of mass μ). When $q^2 < 4\mu^2$ the imaginary part of G can be bounded using the Schwartz inequality; in the frame where $\underset{\sim}{p} = \underset{\sim}{0}$ and $\hat{\underset{\sim}{q}} \equiv \hat{\underset{\sim}{z}}$ we have

$$\begin{aligned} \mid \mathrm{Im}\, G(q^2,k^2) \mid^2 \leq \left(\frac{1}{4\mu^2 q^2}\right) \sum_N \mid < 0 \mid j_x(0) \mid N > \mid^2 \cdot (2\pi)^4 \delta^{(4)}(p_N - k) \\ \times \sum_N \mid < N \mid j_y(0) \mid \pi(p) > \mid^2 (2\pi)^4 \delta^{(4)}(p_N - p - q) \ . \end{aligned} \tag{2}$$

Both sums in (2) define well-known invariant functions: the first is $2\pi k^4$ times the photon spectral function $\rho(k^2)$ whereas the second is $2\pi W_1(q^2,k^2)$. Often these functions are expressed in terms of total cross sections: for example, writing $s = k^2$,

$$\rho(s) = \frac{\sigma_{e^+e^-}(s)}{16\pi^3\alpha^2} \xrightarrow{s\to\infty} \frac{R}{12\pi^2 s} \tag{3}$$

and

$$W_1(q^2,s) = \frac{(s-\mu^2)}{4\pi^2\alpha}\,\sigma_T(q^2,s) \tag{4}$$

where $\sigma_T(q^2,s)$ is the total transverse virtual photoabsorption cross section for pions. Note that only final states with $J^P = 1^-$ contribute to both $\sigma_{e^+e^-}$ and σ_T. With these definitions we can express (2) in the form

$$\mid \mathrm{Im}G(q^2,s) \mid^2 \leq \frac{4\pi^2 s^2 \rho(s) W_1(q^2,s)}{(s-\mu^2-q^2)^2 - 4q^2\mu^2} \ . \tag{5}$$

This inequality can best be exploited by writing dispersion relations for $G(q^2,k^2)$

in the variable k^2 keeping q^2 fixed. Below, we shall show that an unsubtracted form rigorously converges for $q^2 = 0$ and almost certainly converges for $q^2 \neq 0$, so, if there are no "arbitrary" real parts to G, we conclude that

$$| G(q^2, k^2 | \leq 2 \int_{4\mu^2}^{\infty} \frac{s\rho^{\frac{1}{2}}(s) W_1^{\frac{1}{2}}(q^2, s)\, ds}{(s-k^2)\left[(s-\mu^2-q^2)^2 - 4q^2\mu^2\right]^{\frac{1}{2}}} \tag{6}$$

(a) Corrections to the VD model (lower bound for R): First consider the case where $q^2 = k^2 = 0$. G(0, 0) is directly related to the $\pi^0 \rightarrow 2\gamma$ decay width (and thus, for massless pions, to the anomaly, S). The inequality (6) reduces to

$$| G(0,0) | \leq \frac{1}{4\pi^{5/2}\alpha^{3/2}} \int_{4\mu^2}^{\infty} \left[\frac{\sigma_{e^+e^-}(s)\,\sigma_T(0,s)}{s-\mu^2}\right]^{1/2} ds \,. \tag{7}$$

Because only $J = 1$ states contribute, neither of the cross sections can fall slower than $1/s$ as $s \rightarrow \infty$, so the convergence of the integral is ensured. Now, for $s \lesssim s_0 \simeq 1\ \text{GeV}^2$, the spectrum of intermediate states is dominated by the vector mesons. Furthermore, isospin conservation allows us to restrict these states to be isoscalar only (thus raising the threshold to $9\mu^2$), the "external" photon being purely isovector (or vice-versa). Identifying these contributions as a generalization of the VD model allows us to express (7) in the form of a bound for the corrections to this model:

$$| G(0,0) - G_{VD}(0,0) | \leq \frac{1}{\pi^2} \left(\frac{R}{12\alpha}\right)^{\frac{1}{2}} \int_{s_0}^{\infty} \sigma_T^{\frac{1}{2}}(0,s)\, \frac{ds}{s} \,. \tag{8}$$

For simplicity (and in accord with current theoretical prejudices[6]) we have taken R to be a constant for $s > s_0$. We can estimate $\sigma_T(0, s)$ by projecting out the $J = 1$ contribution from the total $\gamma\pi^0$ cross section, σ_0. This is best done in terms of

W_1; for example, the contribution to the Jth partial wave is

$$W_1^J(q^2, s) = \frac{2J+1}{4q^2_{CM}} \int_0^{-4q^2_{CM}} W_1(q^2, s, t)\, d^J_{++}(t)\, dt \tag{9}$$

where $W_1(q^2, s, t)$ is the imaginary part of the non-forward Compton scattering amplitude at momentum transfer $\sqrt{t}$. Experimentally it is known that for the proton $W_1(0, s, t)$ behaves asymptotically like $W_1(0, s)\, e^{bt}$ where b is a constant. We can use the factorization of Regge amplitudes to estimate that $W_1(0, s)$ ($\propto \sigma_0$) is roughly 3/5 that for the proton and that $b \sim 4\, \mathrm{GeV}/c^{-2}$. Putting all this together we find that $|G(0,0) - G_{VD}(0,0)| \lesssim 0.19\, R^{1/2}\mathrm{GeV}^{-1}$. To see what this means we note that straightforward calculations give $|G(0,0| \simeq 0.27\ \mathrm{GeV}^{-1}$ and $|G_{VD}(0,0)| \simeq 0.35\ \mathrm{GeV}^{-1}$ so the inequality is safely satisfied provided $R \gtrsim 1/8$, a condition met by both quark models as well as by experiment. Note that if $R \rightarrow m_0^2/s$ rather than like a constant the inequality is satisfied provided $m_0 \gtrsim 700$ MeV.

(b) <u>Possible bounds for S</u>: consider the inequality (7) in the limit where the "target" pion mass vanishes. We shall assume that $\sigma_T(0, s)$ does not change drastically in this limit, an assumption analogous to the one involved in the Adler-Weisberger sum rule. It is possible that such a continuation is smooth even though the continuation for G is not (as in ref. 4). In that case a useful bound on S can be obtained; we find $|S| \lesssim 0.6 + 0.3\, R^{1/2}$ which is well satisfied by both quark models. If we were to take the somewhat unconventional viewpoint that S is to be identified with the high energy continuum contributions only (as suggested by the work of ref. 4) then the inequality is more stringent, viz. $|S| < 0.3\, R^{1/2}$ which comes close to ruling out coloured quarks (e.g. for $R \simeq 2$, $|S| \lesssim 0.42$).

(c) <u>Bound on $G(0, k^2)$</u>: Setting only q^2 equal to zero in (6) leads to a bound for $G(0,k^2)$:

$$| G(0,k^2) | \leq \frac{1}{4\pi^{5/2}\alpha^{3/2}} \int_{4\mu^2}^{\infty} \left[\frac{\sigma_{e^+e^-}(s)\,\sigma_T(0,s)}{s-\mu^2}\right]^{1/2} \frac{ds}{s-k^2} \quad . \qquad (10)$$

If we were to integrate only up to a finite value of s then it is clear that the integral behaves asymptotically like $1/k^2$. The asymptotic form of the integrand, however, is $s^{-1/2}(s-k^2)^{-1}$ which can be trivially integrated to give a $1/\sqrt{k^2}$ behavior for large k^2. Thus $G(0,k^2)$ must fall at least as fast as $1/\sqrt{k^2}$. A straightforward calculation along the lines used in (b) above leads to

$$\lim_{k^2 \to -\infty} | G(0,k^2) - G_{VD}(0,k^2) | \leq \; 0.3 \; (R/-k^2)^{1/2} \, GeV^{-1} .$$

The inequality can also be exploited the other way around, i.e. by setting $k^2 = 0$ in (6) rather than q^2. The low-lying states now contain the form factors of the $V\pi\gamma$ vertex functions which are, of course, unknowns. However, by an argument similar to the previous one, they certainly fall at least as fast as $1/\sqrt{k^2}$. Using the scaling property of W_1 (i.e. that, for $s \gtrsim s_0$, $\lim_{-k^2 \to \infty} W_1(k^2, s) = F_1(\omega)$ where $\omega \equiv 1 - s/k^2$),[6] we find that $| G(0,k^2) | \lesssim \sum_V | f_V G_{V\pi\gamma}(0,k^2) | + 0.045 \; R^{1/2}/-k^2$. The bound on the continuum contribution here is to be compared with a value of $0.2/k^2$ for the VD model.

Finally we note that if $\sigma_{e^+e^-}(s) \xrightarrow{s\to\infty} 1/s^2$ rather than $1/s$ then these bounds are improved by a factor $1/\sqrt{-k^2}$ (up to logarithms).

(d) Bound on K: The constant K can be defined via the equal time commutator[2,7]

$$\left[j_i^a(x), j_j^b(0)\right] \; = \; iK d_{abc}\, \epsilon_{ijk} A_c^k \, \delta^{(3)}(\underset{\sim}{x}) + \ldots \qquad (11)$$

where a, b, c refer to SU(3) indices. The extra terms in (11) contain operators with quantum numbers differing from those of the pion. We can use Bjorken's asymptotic "theorem"[6] to relate this commutator to the behavior of $G(q^2, k^2)$ in the limit where $q^2 = k^2 \to \infty$: this "theorem" requires that

$$\lim_{q^2 = k^2 \to \infty} G(q^2, k^2) = \frac{2}{3} \frac{f_\pi K}{q^2} \tag{12}$$

A bound on K can thus be obtained by taking the corresponding limit of (6). We first note that states with finite s ($< s_0$, say) give a contribution to the bound which behaves like $1/q^4$ and so are not relevant. All that is required for the bound is the asymptotic behavior of the integrand; thus

$$|K| \leq \lim_{q^2 \to -\infty} \frac{3}{f_\pi} q^2 \int_{s_0}^{\infty} \frac{s\rho^{1/2}(s)\, W_1^{1/2}(q^2, s)}{(s - q^2)^2}\, ds\ . \tag{13}$$

Recall that we only require the $J=1$ contributions to W_1 and that these can be calculated from Eq. (9) provided we know $W_1(s, q^2, t)$. For simplicity we again characterize its t-dependence by e^{bt}, where now b can depend upon both q^2 and s, although for small values of q^2 (and $s \geq s_0$) it is known to be essentially s independent. We shall assume that when $q^2 \to -\infty$, $b(q^2)$ reaches some finite non-zero limit, b_0. Such a behavior seems to be a consequence of both parton and light cone dominance models[8,9] where, for example, we find that $W_1(q^2, s, t)$ scales to a function $F_1(\omega, t)$ which has a non-trivial t-dependence (i.e. $b_0 \neq 0$). This is, of course, strongly supported by the existence of Fubini-type sum rules. Further support for the non-vanishing of b_0 can be found in ref. 9. With these assumptions (13) can be expressed as

$$|K| \leq \frac{3}{2\pi} \frac{1}{f_\pi} \left(\frac{R}{b_0}\right)^{1/2} \int_0^1 (1-x)\, F_1^{1/2}(x)\, dx\ . \tag{14}$$

It is clear that the integral is mostly sensitive to the behavior of $F_1(x)$ near $x=0$ (i.e. the "Regge" region) so it is reasonable to use factorization to estimate that $F_1^\pi(x) \xrightarrow{x \text{ small}} 3/5\ F_2^p(s)$. We thus find that[5] $|K| \lesssim 1.3\ (R/b_0)^{1/2}$ where b_0 is to be expressed in GeV^{-2}. Taking $R=2$ and $b_0 = 4$ (as in real γp scattering) leads to

$|K| \lesssim 0.92$ which, if taken seriously, would rule out the usual kinds of quark models. In fact several authors[9] believe that $b_0 < b \ (q^2 = 0)$. Adapting their arguments we can estimate b_0 from πp and pp scattering; we find $b_0 \simeq 3$ so that the bound becomes $|K| \lesssim 0.75 \ R^{1/2}$ which is satisfied (just!) by the new quark model (but not by the old one).

(e) Theorem for K: If $\int_{-\infty}^{0} W_1(q^2, s, t)\, dt$ scales to a finite function $\mathscr{F}(\omega)$ whose growth at large ω is less than ω^2, and if $\rho(s)$ falls asymptotically like $1/s$,[10] then K is finite. However, if $\rho(s)$ falls faster than $1/s$ then K vanishes.

The proof of this theorem follows almost immediately from Eqs. (9) and (13), for, together these give

$$|K| \leq C \lim_{q^2 \to -\infty} q^2 \int_{s_0}^{\infty} \frac{\mathscr{F}^{1/2}(\omega)\, s^{1/2(3-\beta)}}{(s - q^2)^3}\, ds \tag{15}$$

where C is a constant and $\rho(s)$ is taken to behave like $1/s^{\beta}$ for $s \to \infty$. Transforming (15) to the scaling variable ω leads to

$$|K| \leq C \lim_{-q^2 \to \infty} (q^2)^{1/2(1-\beta)} \int_{1-s_0/q^2}^{\infty} \frac{\mathscr{F}^{1/2}(\omega)(\omega-1)^{1/2(3-\beta)}}{\omega^3}\, d\omega \tag{16}$$

When $\beta = 1$ the right hand side is a well-defined constant (whose size we attempted to estimate in (d) above) whereas if $\beta < 1$, it clearly vanishes[13] QED.

Note that if the conditions of this theorem are met then according to the Crewther relation[2], when $\beta < 1$, S vanishes. Such a case has been examined by Adler et al.[7] who investigated the Crewther analysis in the Gell-Mann-Low limit of QED (where Z_3 is finite, so $\beta < 1$) and found the situation inconsistent. It would appear from our work that if a theory has a finite Z_3 and an anomaly then the Crewther analysis will break down unless that theory implies a remarkable behavior in the large t non-forward Compton scattering of massive photons.

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10. Recall that the restriction to $J=1$ states only implies that $\rho(s)$ cannot fall slower than $1/s$, so, provided the first assumption is valid, K is finite.
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