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MULTIPLICITY DISTRIBUTION IN HIGH ENERGY COLLISION

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ERRATA

Page 6: The first two equations at the top of the page should read:

$$\frac{\beta}{\sqrt{\kappa_2}} = 1 + (0.005 \pm 0.03)$$

and

$$\frac{\gamma}{\sqrt{\kappa_2}} = 1 + (0.05 \pm 0.08).$$

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ABSTRACT

It is shown that the asymptotic multiplicity distribution approaches a Gaussian distribution. The agreement with the experimental data indicates that the higher correlations should not be so strong.

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Charged multiplicity distribution data is easily obtained in high energy collision experiments and has been discussed extensively in the literature. The existence of correlations in the pp collision data up to 300 GeV clearly excludes the Poisson distribution, the simplest possible distribution. Nevertheless, the data shows a simple form for the multiplicity distribution and suggests that there is some regularity behind it. In this note, we show that the form of the asymptotic distribution is Gaussian, and that it indicates a weak correlation among the produced particles.

The basis of the analysis is a theorem[2, 3] which states that if the higher correlations are not too strong (the precise meaning is specified later[4]), the asymptotic distribution approaches the normal (Gaussian) distribution. This is an analogue of the central limit theorem in statistics and probability theory[5]. More explicitly, the normalized charged multiplicity distribution can be expressed as[3]

$$P_n \equiv \frac{\sigma_n}{\sigma_{inel}} = \frac{1}{\sqrt{2\pi}\,\beta} \exp\left[-\frac{(n-m)^2}{2\gamma^2}\right] \left\{1 + \sum_{k=3}^{\infty} a_k \left(\frac{n-m}{\gamma}\right)^k\right\} \tag{1}$$

where all parameters are determined in terms of the cumulants κ_r or the correlation moments f_r defined by[5]

$$\sum_{n=0}^{\infty} e^{int} P_n = \exp\left[\sum_{r=1}^{\infty} \frac{\kappa_r (it)^r}{r!}\right] = \exp\left[\sum_{r=1}^{\infty} \frac{f_r (e^{it}-1)^r}{r!}\right], \tag{2}$$

i.e.

$$\beta = \sqrt{\kappa_2}\left(1 + \frac{1}{12}\frac{\kappa_3^2}{\kappa_2^3} - \frac{1}{8}\frac{\kappa_4}{\kappa_2^2} + O\left(\frac{1}{\kappa_2^2}\right)\right)$$

$$\gamma = \sqrt{\kappa_2}\left(1 + \frac{1}{4}\frac{\kappa_3^2}{\kappa_2^3} - \frac{1}{4}\frac{\kappa_4}{\kappa_2^2} + O\left(\frac{1}{\kappa_2^2}\right)\right) \qquad (3)$$

$$m = \kappa_1 - \frac{1}{2}\frac{\kappa_3}{\kappa_2} + O\left(\frac{1}{\kappa_2}\right)$$

and

$$a_3 = O\left(\frac{1}{\sqrt{\kappa_2}}\right) , \quad a_{4,6} = O\left(\frac{1}{\kappa_2}\right)$$

$$a_{3\ell-4,3\ell-2,3\ell} = O\left(\kappa_2^{-\ell/2}\right), \quad \ell \geq 3 .$$

The cumulants, the correlation moments, and the dispersion moments $\overline{(n-\bar{n})^k}$ are related to each other by

$$\kappa_1 = f_1 = \bar{n} , \quad \kappa_2 = f_1 + f_2 = \overline{(n-\bar{n})^2} ,$$

$$\kappa_3 = f_1 + 3f_2 + f_3 = \overline{(n-\bar{n})^3} , \qquad (4)$$

$$\kappa_4 = f_1 + 7f_2 + 6f_3 + f_4 = \overline{(n-\bar{n})^4} - 3\left(\overline{(n-\bar{n})^2}\right)^2 , \text{ etc.}$$

The assumptions which lead to Eqs. (1) and (3) are [2, 3] that: a) $\kappa_2 \to \infty$, b) the ratio κ_ℓ/κ_2 ($\ell > 2$) are bounded [4], and c) $|(n-\bar{n})/\kappa_2| < \pi$.

At extremely high energy, therefore, we expect to have a Gaussian distribution

$$P_n \xrightarrow[s \to \infty]{} \frac{1}{\sqrt{2\pi}\,\beta} \exp\left[-\frac{(n-m)^2}{2\gamma^2}\right] \qquad (5)$$

with

$$m \underset{s\to\infty}{\longrightarrow} \bar{n} - \frac{1}{2} \frac{\overline{(n-\bar{n})^3}}{\overline{(n-\bar{n})^2}} \tag{6}$$

and

$$\beta = \frac{1}{\sqrt{2\pi}} \frac{\sigma_{inel}}{\sigma_m} \underset{s\to\infty}{\longrightarrow} \sqrt{\overline{(n-\bar{n})^2}} \tag{7}$$

$$\underset{s\to\infty}{\longrightarrow} \gamma \tag{8}$$

where σ_m stands for the maximum value of the distribution function[6]. However, it can be recognized immediately that the convergence to the limiting form, Eq. (5)-(8), is rather slow. The reason for this is that the normal distribution with $\beta = \gamma$ is normalized in the internal $(-\infty, \infty)$ and the integral over the unphysical internal $(-\infty, 0)$ is not negligible at the present energy. Moreover, the speed of the convergence of the parameters β and γ to the asymptotic value $\sqrt{\overline{(n-\bar{n})^2}}$ is not the same as is seen from Eq. (3).

In order to obtain a more effective formula with predictive power, we <u>impose</u> the normalization condition[7, 8] on Eq. (5),

$$1 = \int_0^\infty P_n \, dn = \frac{\gamma}{\sqrt{2\pi}\,\beta} \int_{-\frac{m}{\gamma}}^{\infty} e^{-t^2/2} \, dt \tag{9}$$

i.e.

$$\frac{\beta}{m} = g(\gamma/m) = \frac{1}{2} \frac{\gamma}{m} \left[1 + \mathrm{Erf}\left(\frac{m}{\sqrt{2}\,\gamma} \right) \right] . \tag{10}$$

The function g(x) is depicted in Figure 1. It is easily observed that

$$1 < \gamma/\beta < 2 \tag{11}$$

where the lower (upper) bound corresponds to the limit $\gamma/m \to 0(\infty)$.

Assuming the asymptotic form (5) and the condition (10), we analyze the experimental data with the following procedures:

1. Consider the negative charged multiplicity, $n_- = \frac{n_{ch}}{2} - 1$.

2. Find the modal multiplicity m_- and the maximum cross section σ_{m_-} from the three largest cross sections using the Gaussian form. If the middle point of the three corresponds to the largest, as is the case for $E_L \geq 200$ GeV, the parabolla approximation is good enough. (We avoid using the $n_- = 0$ point because of an ambiguity due to the elastic cross section.) Disregard the γ obtained here.

3. Determine β by Eq. (7) and γ by Eq. (10) or Figure 1.

4. Plot $\log_{10}(\sigma_{n_-}/\sigma_{m_-})$ against $\log_{10}e\,(n_- - m_-)^2/2\gamma^2$.

5. The data points should approach the asymptotic limit, the straight line with the gradient -1.

The last step may be replaced by:

5'. Plot $\log_{10}(\sigma_{n_-}/\sigma_{m_-})$ against $(n_- - m_-)^2$. Determine γ^2 by the gradient of an expected straight line and compare the result with that obtained in step 4.

The result of the analysis based on the procedures 1-5 is shown in Figure 2 and Table 1. Some discussion is in order.

i) While the data points for 50 GeV given in Figure 2 are slightly off the expected asymptotic line; those for 69-300 GeV fall quite well on it. Even for 50 GeV, the data falls on a straight line. Thus, a slight change of the slope parameter γ within its experimental error seems to restore the agreement between the data and prediction.

ii) The asymptotic relation (7) is well satisfied already at the present energy, while the others, Eqs. (6) and (8) are not. In order to understand the difference in the speed of convergence, we estimate the parameters β and γ using Eq. (3) and the experimental data for the cumulants. At 303 GeV, we obtain

$$\frac{\beta}{\sqrt{\kappa_2}} = 1 + (0.07 \pm 0.03)$$

and

$$\frac{\gamma}{\sqrt{\kappa_2}} = 1 + (0.18 \pm 0.08) \; .$$

This is consistent with the values given in Table 1.

iii) The relation (7) is asymptotically equivalent to that obtained by Weisberger[9], $\sqrt{\kappa_2}\, P_{\bar{n}} = 1/\sqrt{2\pi}$ although the convergence to (7) seems faster.

iv) The experimental values for (γ/m, β/m) are moving down on the curve of Fig. 1 as energy increases. This seems to indicate that the data points are moving towards the limit $\gamma/m \to 0$ and $\gamma/\beta \to 1$. (This case may be called the weak two-body correlation model[4].)

v) However, the possibility of having the condition

$$\frac{\gamma}{m} \xrightarrow[s \to \infty]{} a \neq 0 \tag{13}$$

is not excluded. If that is the case, we obtain the KNO scaling law[10] with the Gaussian scaling function,

$$m\, P_n = \frac{1}{\sqrt{2\pi}\, g(a)} \, e^{-\frac{1}{2a^2}\left(\frac{n}{m} - 1\right)^2} \tag{14}$$

vi) The step 5 is preferable to 5' since the asymptotic form (1) or (5) is the best approximation around the modal point. Besides, the predictive power is more evident if step 5 is used. If the accuracy of experimental data at higher energies is improved, we may be able to detect the polynomial term which we have neglected. In particular, the a_3 term, which is of the order $0(1/\sqrt{\kappa_2})$, may be detected from the asymmetry of the curve. (The present accuracy does not permit us to detect such asymmetry.)

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References and Footnotes

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2. J.B.S. Haldane, Biometrika 32 (1942) 294.

3. Y. Tomozawa, Asymptotic Multiplicity Distribution and Analogue of the Central Limit Theorem (to be published).

4. The multiperipheral model or some field theoretical model satisfies the condition. See e.g. W. Frazer et al., Rev. Mod. Phys. 44 (1972) 284 and references quoted therein.

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6. m is called the modal multiplicity, and its usefulness is demonstrated in the analysis of B. R. Webber, Phys. Letters 42B (1972) 69.

7. After the completion of the work, the author came across the works by G. D. Kaiser, Nucl. Phys. B44 (1972) 171, and G. W. Parry and P. Rotelli, Application of the Truncated Gaussian to the Inelastic pp Charged Multiplicity Distribution, Triests preprint (IC/73/3), in which a similar analysis was made. The author believes, however, that the treatment of the present article is more transparent in presenting the reasoning of using the Gaussian form, the analysis of the experimental data, and suggesting a possible correction term.

8. $$\mathrm{Erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad .$$

9. W. I. Weisberger, Asymptotic Multiplicity Distributions in High Energy Collisions (preprint, 1973).

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Figure Captions

Figure 1 The curve for $\beta/m = g(\gamma/m)$, Eq. (10).

Figure 2 The negative charge multiplicity distribution. The solid line represents the expected asymptotic limit.

Table Caption

Table 1 The values for the parameters.

TABLE 1

$E_{(GeV)}$	50	69	102	205	303
m_-	0.96	1.19	1.21	2.0	2.24
β	1.32 ± 0.12	1.44 ± 0.06	1.59 ± 0.15	1.88 ± 0.15	2.21 ± 0.20
γ	1.90 ± 0.17	1.99 ± 0.08	2.26 ± 0.22	2.35 ± 0.19	2.90 ± 0.26
$\kappa_1 = f_1$	1.66 ± 0.07	1.95 ± 0.04	2.17 ± 0.07	2.82 ± 0.08	3.43 ± 0.10
$\kappa_2 = \overline{(n-\bar{n})^2}$	1,67 ± 0.11	2.10 ± 0.07	2.56 ± 0.12	3.77 ± 0.22	4.80 ± 0.29
$\kappa_3 = \overline{(n-\bar{n})^3}$	1.39 ± 0.32	2.07 ± 0.21	2.60 ± 1.31	5.24 ± 0.99	7.07 ± 1.60
$\beta/\sqrt{\kappa_2}$	1.02 ± 0.12	0.99 ± 0.05	1.00 ± 0.13	0.97 ± 0.11	1.01 ± 0.12
γ/β	1.44 ± 0.13	1.38 ± 0.06	1.42 ± 0.13	1.25 ± 0.12	1.31 ± 0.12
γ/m	1.98 ± 0.12	1.67 ± 0.07	1.87 ± 0.18	1.18 ± 0.10	1.29 ± 0.12
$(\kappa_3/2\kappa_2)\frac{1}{\bar{n}-m}$	0.60 ± 0.23	0.64 ± 0.13	0.53 ± 0.33	0.84 ± 0.29	0.61 ± 0.18

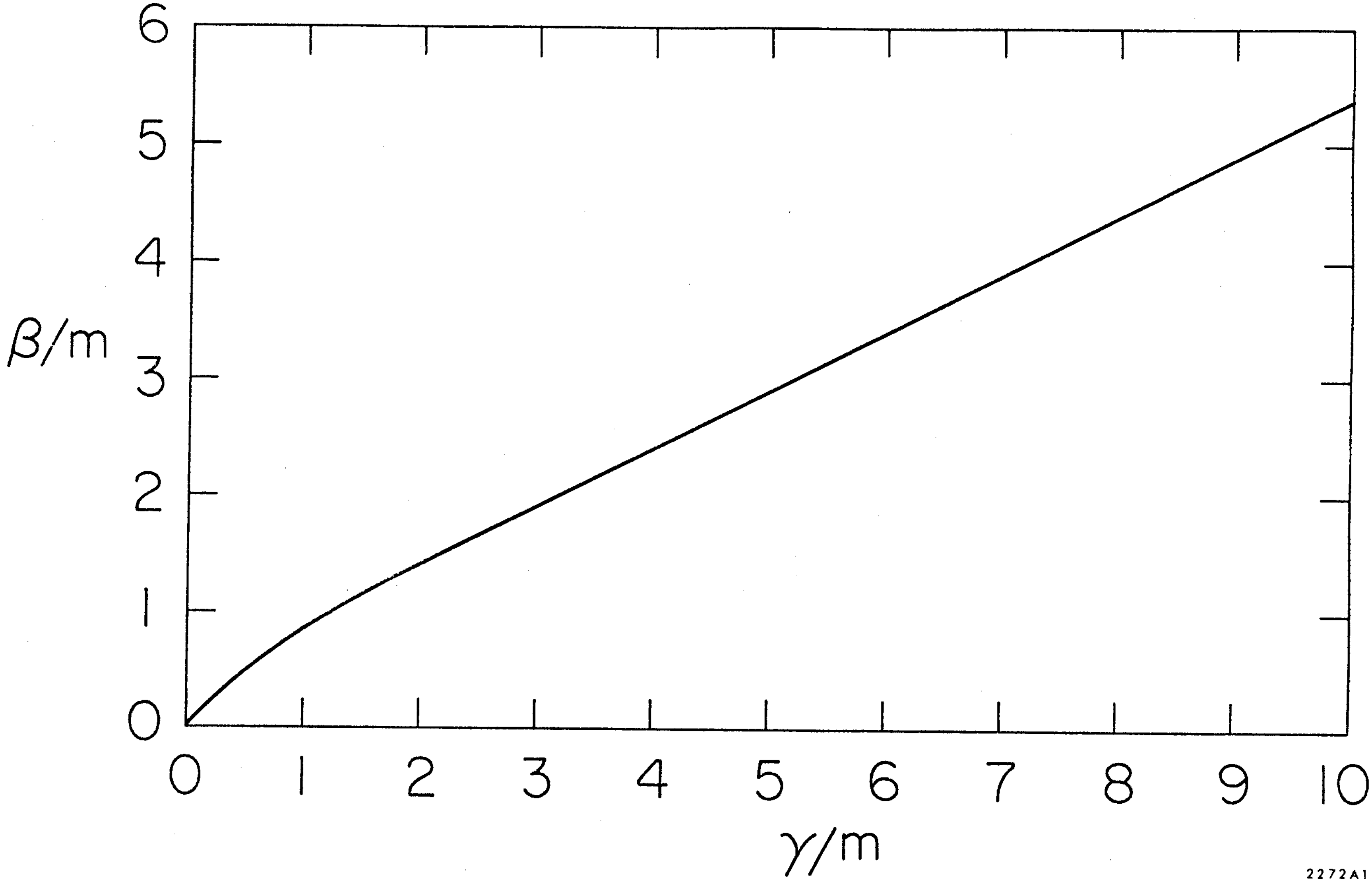

Fig. 1

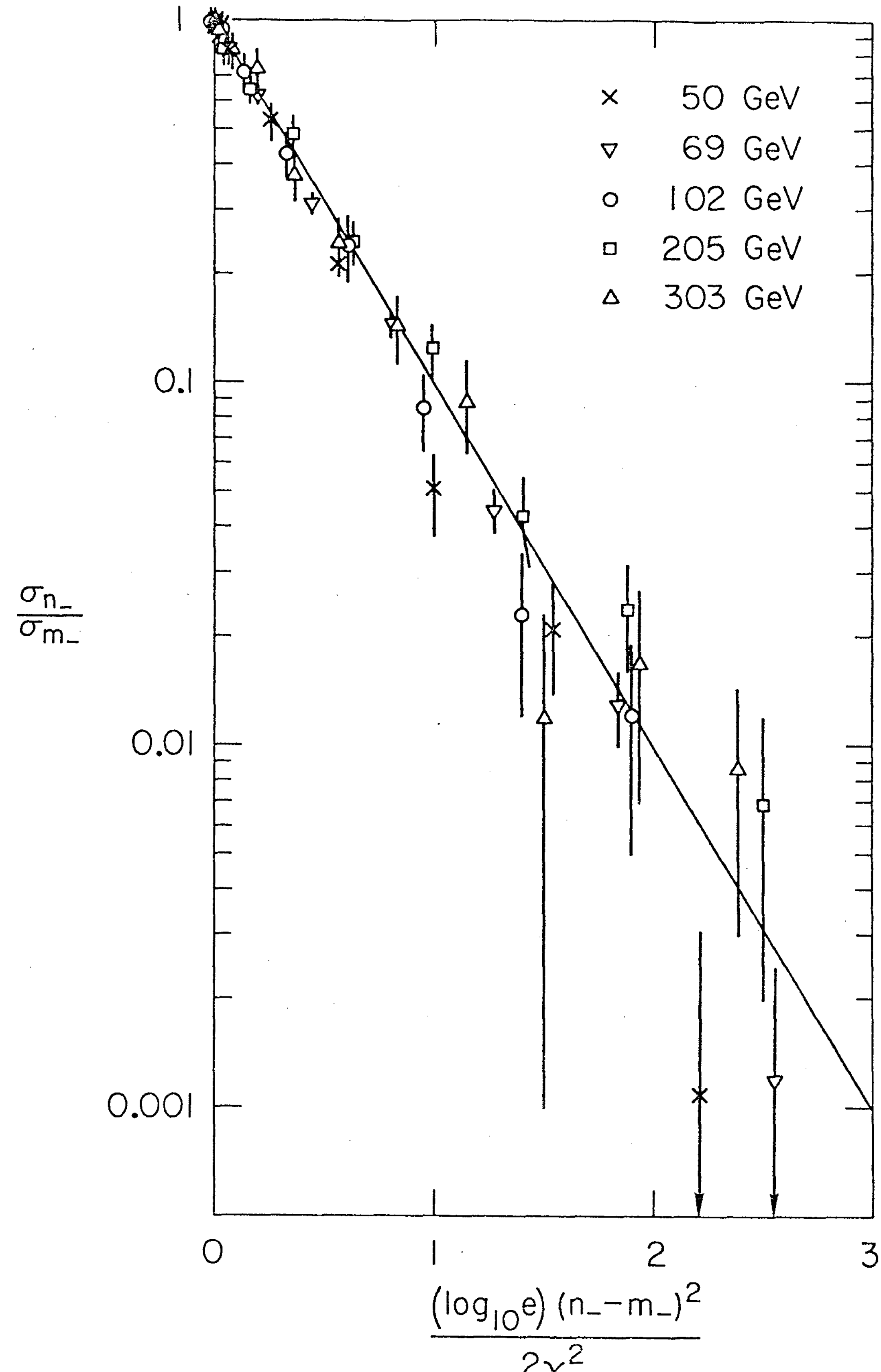
50 GeV
69 GeV
102 GeV
205 GeV
303 GeV
1
0.1
0.01
0.001
$\frac{\sigma_{n_-}}{\sigma_{m_-}}$
0
1
2
3
$\frac{(\log_{10}e)\,(n_- - m_-)^2}{2\gamma^2}$

Fig. 2

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