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COMMENTS ON THE DERIVATION OF THE COVARIANT FEYNMAN RULES IN THE GAUGE THEORIES*

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ABSTRACT

It is pointed out that Weinberg's derivation of the covariant Feynman rules in the canonical formalism contains two errors in handling the fermions, and, in fact, these two errors cancel each other.

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Recently, Weinberg[1] has given a general formalism for the gauge theories of weak and electromagnetic interactions in the unitary gauge, where all the unphysical particles have been eliminated. In particular, the canonical quantization procedure is carried out and a set of covariant Feynman rules are derived. These results provide a very convenient framework for the discussions of various interesting topics in the unified weak and electromagnetic interactions. In this note, we would like to point out that Weinberg's derivation of the covariant Feynman rules contains two errors, which invalidate the arguments leading to the results. However, due to the cancelling nature of these mistakes, the general conclusions remain valid. Also, because only the part of derivation involving fermions needs to be modified, Weinberg's derivations are still applicable to the class of theories without fermions. More specifically, the first error is that in the field equation for Yang-Mills fields $A_{\alpha\nu}$, a term of the form $i\overline{\psi}\gamma_\nu t^\alpha\psi$ has been left out. This extra term will generate a term of the form $-\frac{1}{2}\Omega^{-1}_{\alpha\beta}(\phi)\overline{\psi}\gamma_0 t^\alpha\psi\overline{\psi}\gamma^0 t^\beta\psi$ in the Hamiltonian density, similar to the Coulomb term in the ordinary QED.[2] The second error is that the summation of the diagrams due to the non-covariant part of the propagators is incomplete. Besides those diagrams given in Weinberg's paper, another set of diagrams (see Figure 4) should be included. These diagrams sum up to a term $+\frac{1}{2}\Omega^{-1}_{\alpha\beta}(\phi)\overline{\psi}\gamma_0 t^\alpha\psi\overline{\psi}\gamma^0 t^\beta\psi$, which cancels exactly the extra term in the Hamiltonian density mentioned above. Hence the final form of the Feynman rule is still covariant as claimed by Weinberg.

Let us illustrate these points in detail. The Lagrangian desnity is given by

$$\mathscr{L} = -\frac{1}{4}F_{\alpha\mu\nu}F_\alpha^{\mu\nu} - \frac{1}{2}(D_\mu\phi, D^\mu\phi) - \overline{\psi}\gamma^\mu D_\mu\psi - \overline{\psi}m_0\psi - \mathscr{P}(\phi) - \overline{\psi}(\Gamma, \phi)\psi \tag{1}$$

with

$$F^{\alpha}_{\mu\nu} \equiv \partial_\mu A^{\alpha}_{\nu} - \partial_\nu A^{\alpha}_{\mu} - C_{\alpha\beta\gamma} A^{\beta}_{\mu} A^{\gamma}_{\nu} \tag{2}$$

$$(D_\mu \phi)_p = \partial_\mu \phi_p + i(\theta^{\alpha})_{pq} \phi_q A^{\alpha}_{\mu} \tag{3}$$

$$(D_\mu \psi)_n = \partial_\mu \psi_n + i(t^{\alpha})_{nm} \psi_m A^{\alpha}_{\mu} \, . \tag{4}$$

The field equation for $A_{\beta\nu}$ is then given by

$$\begin{aligned} 0 &= \partial_\mu \left(\frac{\delta \mathscr{L}}{\delta(\partial_\mu A_{\beta\nu})} \right) - \frac{\delta \mathscr{L}}{\delta A_{\beta\nu}} \\ &= -\partial_\mu F^{\mu\nu}_{\beta} + F^{\mu\nu}_{\alpha} C_{\alpha\beta\gamma} A_{\gamma\mu} + i(\theta_\beta \phi, D^\nu \phi) + i\bar{\psi}\gamma^\nu t^\beta \psi \, . \end{aligned} \tag{5}$$

The last term in Eq. (5), coming from the term $\bar{\psi}\gamma^\mu D_\mu \psi = \bar{\psi}\gamma^\mu(\partial_\mu + i\, t^\alpha A^\alpha_\mu)\psi$ in the Lagrangian was left out in reference 1 (see Eq. (4.4) of reference 1). This term will modify the Hamiltonian density as follows;

$$\begin{aligned} \mathscr{H} &= \tfrac{1}{2}\,\Omega_{\alpha\beta}(\phi) A_{\alpha 0} A_{\beta 0} + \tfrac{1}{2}\vec{P}_\alpha \cdot \vec{P}_\alpha + \tfrac{1}{2}\pi_a \pi_a + \tfrac{1}{4} F_{\alpha ij} F^{ij}_{\alpha} + \tfrac{1}{2}\vec{\nabla}\phi_a \vec{\nabla}\phi_a + i\vec{A}_\alpha \cdot \vec{\nabla}\phi_a (\theta_\alpha)_{ab}\phi_b \\ &+ \tfrac{1}{2}\vec{A}_\alpha \cdot \vec{A}_\beta \left[\Omega_{\alpha\beta}(\phi) - (\theta_\alpha)_{ac}(\theta_\beta)_{bc}\phi_a\phi_b\right] + \bar{\psi}\,\vec{\gamma}\cdot\vec{\nabla}\psi + i\bar{\psi}\vec{\gamma}\, t_\alpha \psi \vec{A}_\alpha + \bar{\psi} m_0 \psi + \mathscr{P}(\phi) + \bar{\psi}(\Gamma, \phi)\psi \end{aligned} \tag{6}$$

where

$$A_{\gamma 0} = \Omega^{-1}_{\gamma\alpha}(\phi)\left[\vec{\nabla}\cdot\vec{P}_\alpha - C_{\alpha\beta\gamma}\vec{P}_\gamma \cdot \vec{A}_\beta + i(\theta_\alpha)_{ab}\pi_a \phi_b + i\bar{\psi}\gamma^0 t^\alpha \psi\right] \tag{7}$$

and $\vec{P}_\alpha, \pi_a$ are the canonical momenta of the vector field and scalar fields, respectively. In comparing with the corresponding equation in reference 1, we see that there is an extra term of the form

$$\mathscr{H}_e = -\tfrac{1}{2}\,\Omega^{-1}_{\alpha\beta}(\phi)(\bar{\psi}\gamma_0 t_\alpha \psi)(\bar{\psi}\gamma_0 t_\beta \psi). \tag{8}$$

We can now proceed, as in reference 1, to go over the interaction representation to work out the propagators and derive the Feynman rules. Since the quadratic part of the Hamiltonian remains unchanged, we will get the same results for the propagators as reference 1. Only the interaction Hamiltonian has an extra term given by Eq. (8). Using the same notations as reference 1, we write the interaction Hamiltonian as

$$\mathscr{H}' = \epsilon(\alpha) + \mathscr{H}_e + \mathscr{F}_N(\alpha)\beta_N + \tfrac{1}{2}\,G_{NM}(\alpha)\,\beta_N\beta_M \tag{9}$$

where $\epsilon(\alpha)$, $\mathscr{F}_N(\alpha)$, $G_{NM}(\alpha)$, and β_N, are the same quantities as defined in reference 1. We can now sum up the effects due to the non-covariant part of the propagators as done in reference 1. This amounts to adding a term of the following form to $\mathscr{H}'$,

$$-\tfrac{1}{2}\,\delta^4(0)\,\ln \mathrm{Det}\left[1-\mathscr{B}G(\alpha)\right] \tag{10}$$

and the following replacements

$$G(\alpha) \rightarrow G(\alpha)\left[1-\mathscr{B}G(\alpha)\right]^{-1} \tag{11}$$

$$\mathscr{F}(\alpha) \rightarrow \mathscr{F}(\alpha)\left[1-\mathscr{B}G(\alpha)\right]^{-1} = \left[1-\mathscr{B}G(\alpha)\right]^{-1}\mathscr{F}(\alpha) \tag{12}$$

where $\mathscr{B}$ is the non-covariant part of the propagator, i.e.

$$< T(\beta_N(x)\beta_M(y)) >_0 \; = \; < T^*(\beta_N(x)\beta_M(y)) >_0 + i\mathscr{B}_{NM}\,\delta^4(x-y)\,. \tag{13}$$

These three terms (10), (11), (12), are the results of summing up diagrams shown in Figures 1, 2, and 3, respectively. However, there is also a contribution coming from the diagrams in Figure 4. This contribution was left out in reference 1. It is easy to see that these diagrams sum up to

$$\begin{aligned} &\mathscr{F}\mathscr{B}\mathscr{F} + \mathscr{F}\mathscr{B}G\mathscr{B}\mathscr{F} + \mathscr{F}\mathscr{B}G\mathscr{B}G\mathscr{B}\mathscr{F} + \ldots \\ &= \mathscr{F}\mathscr{B}(1 - G\mathscr{B})^{-1}\mathscr{F} . \end{aligned} \tag{14}$$

After carrying out the matrix multiplications, we get

$$\mathscr{F}_N\left[\mathscr{B}(1 - G\mathscr{B})^{-1}\right]_{NM}\mathscr{F}_M = \tfrac{1}{2}\Omega^{-1}_{\alpha\beta}(\phi)(\overline{\psi}\gamma_0 t_\alpha\psi)(\overline{\psi}\gamma_0 t_\beta\psi) . \tag{15}$$

Thus, if we drop the non-covariant terms in the propagators, we can replace the interaction (9) by

$$\mathscr{H}'_{\text{eff}} = -\mathscr{L}' - \frac{i}{2}\delta^4(0)\,\ell n\,\text{Det}\left[1 - \mathscr{B}G(\alpha)\right] \tag{16}$$

with

$$\begin{aligned} -\mathscr{L}' \equiv\ & \epsilon(\alpha) + \mathscr{H}_e + \mathscr{F}_N\left[\mathscr{B}(1 - G\mathscr{B})^{-1}\right]_{NM}\mathscr{F}_M + \left(\mathscr{F}(\alpha)\left[1 - \mathscr{B}G(\alpha)\right]^{-1}\right)_N \beta_N \\ & + \tfrac{1}{2}\left(G(\alpha)\left[1 - \mathscr{B}G(\alpha)\right]^{-1}\right)_{NM}\beta_N\beta_M . \end{aligned} \tag{17}$$

From Eq. (14) and the expression for $\mathscr{H}_e$ in Eq. (8), we see that the second and third term in Eq. (17) cancel each other, while all the other terms sum to a covariant expression, just as in reference 1,

$$\mathscr{L}' = \tfrac{1}{2} A_{\alpha\mu} A^{\mu}_{\beta} \left[(\theta_\alpha)_{ab} (\theta_\beta)_{ac} \phi'_b \phi'_c - \Omega_{\alpha\beta}(\lambda + \phi') + \mu^2_{\alpha\beta} \right]$$

$$+ \tfrac{1}{4} C_{\alpha\beta\gamma} C_{\alpha\delta\epsilon} A_{\beta\mu} A_{\gamma\nu} A^{\mu}_{\delta} A^{\nu}_{\epsilon} - \tfrac{1}{2} C_{\alpha\beta\gamma} F_{\alpha\mu\nu} A^{\mu}_{\beta} A^{\nu}_{\gamma} + (\theta_\alpha)_{ab} \phi'_b \partial_\mu \phi'_a A^{\mu}_{\alpha} \tag{18}$$

$$+ i \bar{\psi} \gamma^{\mu} t_{\alpha} \psi A_{\alpha\mu} + \mathscr{P}(\lambda + \phi') - \tfrac{1}{2} M^2_{ab} \phi'_a \phi'_b + \bar{\psi} (\Gamma, \phi') \psi \ .$$

Therefore, the resulting Feynman rules are covariant, in the unitary gauge.

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Figure Captions

References

1. S. Weinberg, "General Theory of Broken Local Symmetries", MIT preprint (1972)(to be published in Phys. Rev.).
2. See, for example, J. Bjorken and S. Drell, "Relativistic Quantum Fields", McGraw-Hill Book Company (1965).

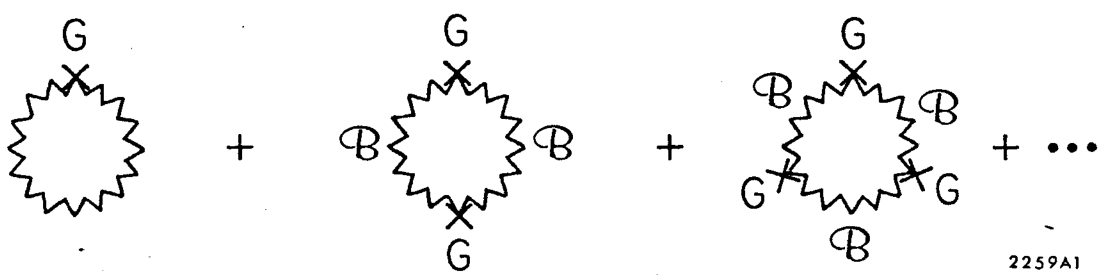

Fig. 1

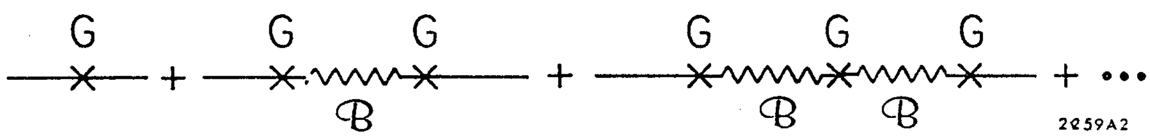

Fig. 2

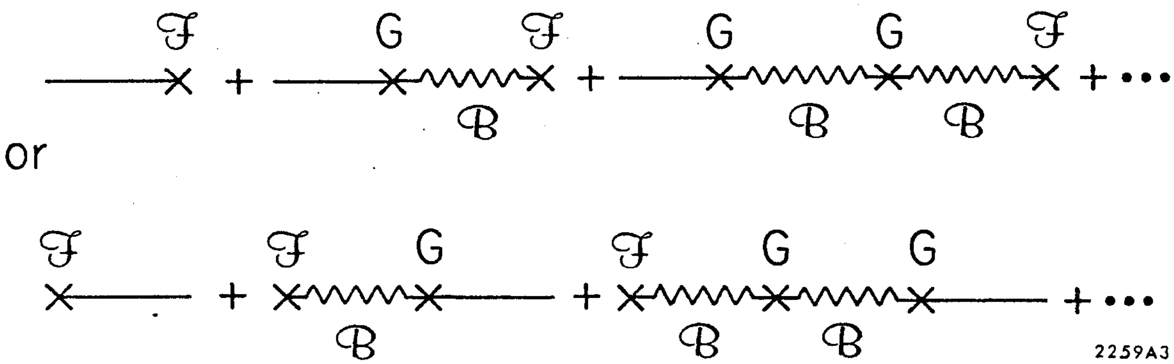

Fig. 3

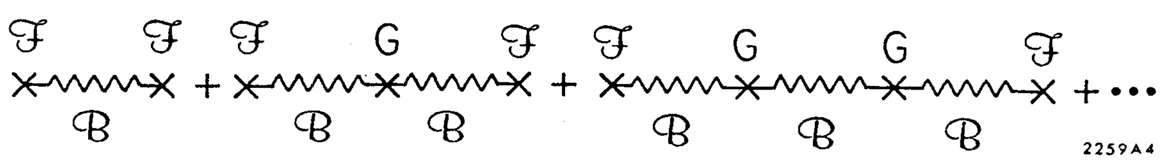

Fig. 4