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NEW RIGOROUS UPPER BOUND FOR THE RENORMALIZATION CONSTANT OF THE NUCLEON*

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ABSTRACT

From the analyticity properties of the nucleon propagator and the improper πNN vertex function a new upper bound is established for Z_2, the wave renormalization constant of the nucleon, in terms of the pion-nucleon coupling constant g and the P_{11} and S_{11} elastic πN phase shifts. The result is:

$$Z_2^{-1} - 1 \geq 0.096(g^2/4\pi); \;\; Z_2 \leq 0.42,$$

representing improvements by factors of 8 and 2 respectively over a previous bound obtained by Drell, Finn and Hearn. In addition the phase of the one-nucleon-irreducible P_{11} πN partial-wave amplitude, in the elastic region, is calculated in an N/D approximation. The result of this calculation strongly suggests the existence of a zero of the nucleon propagator function, the possibility of which has been widely discussed in connection with the validity of an upper bound on $g^2/4\pi$.

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I. INTRODUCTION

In this paper we consider what may be learned about the renormalization constant of the nucleon and the nucleon propagator from analyticity and unitarity. The starting points for our investigations are the lower bound for the nucleon spectral function given by πN intermediate states, and the phase of the improper vertex function, given in certain regions by elastic unitarity. Our principal results are a new rigorous upper bound of 0.42 for Z_2, the nucleon wave function renormalization constant, and a very strong suggestion of a zero in the nucleon propagator function, which if present invalidates an upper bound for the strong coupling constant given by Geshkenbein and Ioffe.(1)

The renormalization constant, Z_2, is necessarily a field-theoretic quantity. In a world in which both nucleon and pion are elementary, with their fields appearing in the underlying Lagrangian, Z_2 measures the extent to which the nucleon field is renormalized by the interaction. It may be interpreted as the probability for finding a "bare" nucleon in the dressed, physical particle. There are, however, many indications that the nucleon may not be elementary, but rather, composite. We cite what are perhaps the three most obvious intimations of compositeness: The taxonomic success of the quark model in describing the spectrum of hadrons, the continued observation of falling nucleon electromagnetic form factors, and the intuitive understanding of deep-inelastic electron scattering (DIES) afforded by parton models. Since constituents (quarks and/or partons, for example) have, if they exist, successfully eluded all attempts at direct observation, it becomes a question of importance to decide to what degree and with what rigour we can determine the compositeness or elementarity of the nucleon.

In the case of the deuteron, the low energy features of the NN interaction argue strongly in favour of $Z = 0$, relegating the deuteron to the level of a bound state, as discussed by Weinberg.[2] Unfortunately such an analysis has not proved feasible for the nucleon itself - essentially because of the absence of observable candidates for its constituents. A general classification of particles has been given by Ida[3] in terms of the asymptotic behaviour of Green's functions, but this does not readily permit a decision on the status of the nucleon.

Gell-Mann and Zachariasen[4] argued that in a renormalizable field theory, taken to all orders in perturbation theory, the asymptotic value of the nucleon's charge form factor is given by Z_2. Thus the observed rapidly falling nucleon electromagnetic form factors suggest a very small and possible zero value for Z_2. It is difficult, however, to translate this result into a rigorous upper bound for Z_2, given the form factors at finite momentum transfer and no knowledge of the underlying field theory.

There remains the rôle of Z_2 in DIES. The condition $Z_2 = 0$ is part of the input to Drell, Levy and Yan's[5] parton model and is interpreted by them as representing an entirely composite nucleon. Cooper and Pagels,[6] West[7] and more recently[8] one of the present authors (D.J.B.) have attempted to set bounds on Z_2, given information about the structure functions in DIES and the asymptotic behaviour of the nucleon form factors. It was shown in reference (8) that upper bounds for Z_2 between 0 and 0.3 result from a variety of assumptions about possible subtraction constants in the sideways dispersion relations for the nucleon form factors and about the behaviour of $R(\omega)$ in the Bjorken limit. ($R(\omega)$ is the ratio of longitudinal to transverse photoabsorption cross sections for a virtual photon of energy ν and four momentum q, incident on a nucleon, mass m, at rest, in the limit $\nu \to \infty$, $\omega = -2m\nu/q^2$, fixed.)

Subsequently it has been claimed by Hirayama and Ishida[9] that $Z_2 = 0$, if $R(\infty) = 0$, given certain restrictions on the subtraction constants and on the structure functions near $\omega = 1$. Unfortunately all these attempts to bound Z_2 from DIES information involve high energy regions which we are at present only on the verge of exploring and require assumptions which owe more to the necessity for mathematical precision than they do to physical intuition. They are also very weak applications of unitarity since they involve inequalities in terms of only the $J^{\pi} = \frac{1}{2}^{+}$ contributions to DIES.

There does exist, however, one rigorous, non-trivial bound for Z_2, derived by Drell, Finn and Hearn[10]

$$Z_2^{-1} \geq 1 + (g^2/4\pi) \,/\, 85.6\,; \; Z_2 \leq .86 \,. \qquad (1.1)$$

Our principal new result is

$$Z_2^{-1} \geq 1 + (g^2/4\pi) \,/\, 10.4\,; \; Z_2 \leq .42, \qquad (1.2)$$

with the only assumption being that the improper πN vertex function of Bincer[11] is free of a particularly vicious class of essential singularities at infinity. Whilst our new result may not appear spectacular in the light of the suspicion that $Z_2 = 0$ it represents an improvement by a factor of 8 in the lower bound for the continuum contributions to the sum rule for Z_2^{-1}.

An interesting by-product of our search for an improved rigorous bound on Z_2 is a very strong indication that the nucleon propagator function has a zero, corresponding to a pole of the proper πN vertex function. The existence of such a zero, and, a fortiori, the validity of upper bounds on $g^2/4\pi$ which assume the absence of zeroes, have been a subject of considerable discussion in the past. A very clear review is given by Okubo.[12]

Geshkenbein and Ioffe,[1, 13] and Meiman[14] proved that in the absence of zeroes $g_2/4\pi \leq 85.6$. They also assumed that the propagator at least satisfies a once-subtracted dispersion relation, but Okubo[12] has recently shown this to be an inessential restriction. We improve the bound to $g^2/4\pi \leq 57.6$, in the absence of zeroes. Much more significantly we have obtained the approximate bound $g^2/4\pi \leq 15.3$ using an N/D model of Ida[15] to calculate the phase of the one-nucleon-irreducible P_{11} πN partial-wave amplitude, in the elastic region. We believe this calculation to be relatively reliable, in the absence of zeroes, and argue that the near saturation of our bound is a very strong indication of at least one zero, thereby invalidating the Geshkenbein-Ioffe bound on the strong coupling constant. We find further support for the existence of a zero by reexamining a model-dependent bound on Z_2 given by Ida.[15]

The paper is organized as follows: In Section II we develop the requisite formal preliminaries, defining the nucleon propagator and vertex functions, deducing the basic unitarity bound, and using elastic unitarity to determine phases of the vertex functions. In Section III we deduce a set of bounds on Z_2, our only assumption being the absence of certain essential singularities at infinity in the improper vertex function. The previous result of Drell et al.[10] is a particular case of our work and is appreciably improved by a knowledge of the P_{11} and S_{11} elastic phase shifts. We also consider whether an approximate current algebra result of Suura and Simmons[16] helps in bounding Z_2, but find it to be of negligible value when used in conjunction with the phase shifts. Finally in Section IV we develop two sets of bounds, one for $g^2/4\pi$ and one for Z_2, assuming the absence of zeroes of the propagator. After an N/D calculation we show that these bounds are so restrictive as virtually to require a zero. Certain purely mathematical details are contained in the Appendix.

II. FORMAL PRELIMINARIES

In this Section we define the nucleon propagator function $\Delta(W)$ and the proper and improper πN vertex functions $\Gamma(W)$ and $K(W)$ and deduce a unitarity inequality by considering the contribution of πN intermediate states to the nucleon spectral function $\rho(W)$. In addition we relate the phases of $K(W)$ and $\Gamma(W)$, on their elastic cuts, to the phases of the complete and one-nucleon-irreducible P_{11} and S_{11} πN partial-wave amplitudes.

We define the propagator function $\Delta(W)$ in terms of the renormalized nucleon Feynman propagator $S_F'(p)$, with $p^2 = W^2$, by

$$i\, S_F'(p) = \tfrac{1}{2}[(1 + \not{p}/W)\Delta(W) + (1 - \not{p}/W)\Delta(-W)], \qquad (2.1)$$

so that for real, positive W

$$\lim_{\epsilon \to 0^+} \operatorname{Im} \Delta\,(\pm(W + i\epsilon)) = \pi\,(\pm W\rho_1(W^2) + \rho_2(W^2)), \qquad (2.2)$$

where $\rho_{1,2}(W^2)$ are the conventional renormalized nucleon spectral functions,[10] which satisfy

$$W^2 \rho_1(W^2) \geq |W\rho_2(W^2)|. \qquad (2.3)$$

Combining Eq. (2.2) and Inequality (2.3), we obtain

$$\rho(W) = \frac{1}{\pi} \operatorname{Im}\Delta(W + i\epsilon) \geq 0, \qquad (2.4)$$

for <u>all</u> real W.

If $\Delta(W)$ is bounded by some negative power of W as $|W| \to \infty$ it satisfies the unsubtracted dispersion relation,

$$\Delta(W) = \frac{1}{m - W} + \left\{ \int_{-\infty}^{-(m+\mu)} + \int_{(m+\mu)}^{\infty} \right\} \frac{dW'\rho(W')}{W' - W}, \qquad (2.5)$$

where m and μ are the nucleon and pion masses.

Given the dispersion relation (2.5), Z_2 may be evaluated in terms of $\rho(W)$. We have

$$< 0 | \hat{\Psi}(x) | ps > \; = \; (Z_2)^{\frac{1}{2}} < 0 | \Psi(x) | ps > \; , \qquad (2.6)$$

where $\hat{\Psi}(x)$ and $\Psi(x)$ are the unrenormalized and renormalized nucleon interpolating fields. Z_2 is then given by the sum rule[17]

$$Z_2^{-1} - 1 = \left\{ \int_{-\infty}^{-(m+\mu)} + \int_{(m+\mu)}^{\infty} \right\} dW \rho(W) \, . \qquad (2.7)$$

If the integral of Eq. (2.7) fails to converge, we have the formal result $Z_2 = 0$, which has been interpreted as indicating compositeness or the absence of the field from the underlying Lagrangian.[18] In Section III we adopt Eq. (2.7) as a <u>definition</u> of Z_2 and are able to deduce a nonzero upper bound without assuming the validity of an unsubtracted dispersion relation for $\Delta(W)$. This is achieved by finding a lower bound for $\rho(W)$ in terms of the improper off-shell πN vertex function $K(W)$, as defined by Bincer.[11]

Consider the coupling of a nucleon (momentum p', spin s) and pion (momentum q, isopin α) to an off-shell nucleon:

$$< 0 | \eta(0) | p's, \, q\alpha > =$$

$$\tfrac{1}{2} [\, (1 + \not{p}/W) K(W) + (1 - \not{p}/W) K(-W) \,] \; i g \gamma^5 \tau_\alpha u(p' s), \qquad (2.8)$$

where

$\eta(x) = (i\not{\partial} + m)\Psi(x)$, $p = p' + q$, $p^2 = W^2$ and we normalize $K(m) = 1$ so that g is the strong coupling constant.[19]

The nucleon spectral function is given by

$$(W - m)^2 \ \rho(W) \ = \ (2\pi)^3 \sum_n \delta^4(p - n) \tag{2.9}$$
$$\times \ \mathrm{Trace}[<0|\eta(0)|n><n|\bar{\eta}(0)|0>(\not{p} + W)/4|W|],$$

for $|W| \geq m + \mu$. Keeping only πN intermediate states we find

$$|W - m|\pi\rho(W) \geq (g^2/4\pi)|K(W)|^2 h(W), \tag{2.10}$$

where

$$h(W) \ = \ \frac{3[(W - m)^2 - \mu^2]^{\frac{3}{2}} [(W + m)^2 - \mu^2]^{\frac{1}{2}}}{8 \ W^3 \ (W - m)}, \tag{2.11}$$

and we have equality for $m + 2\mu \geq |W| \geq m + \mu$.

Inequality (2.10) is the basis of previous bounds on Z_2 and g. To it we add information about the phase of K(W) on its elastic cuts. As shown by Bincer,[11] K(W) is a real analytic function of W with cuts $(-\infty, -(m+\mu))$ and $(m+\mu, \infty)$ and for $m+2\mu \geq |W| \geq m+\mu$ its phase on the cuts is given by elastic unitarity:

$$\mathrm{Im}\, K(\pm(W + i\epsilon)) \ = \ [T(\pm W)]^* K(\pm(W + i\epsilon)), \tag{2.12}$$

where

$$T(W) \ = \ \sin\delta_P(W) \ \exp(i\delta_P(W)), \tag{2.13}$$

and

$$\delta_P(W) \ = \ \delta_S(-W), \tag{2.14}$$

with $\delta_{P,S}(W)$ the P_{11} and S_{11} πN phase shifts. Thus if we define $\delta(W)$ to be the phase of K(W) on the upper lips of the cuts:

$$K(W + i\epsilon) \ = \ \exp(i\delta(W))|K(W)|, \tag{2.15}$$

then

$$\delta_P(W) = \delta(W), \; \delta_S(W) = -\delta(-W), \tag{2.16}$$

modulo π, for $m + 2\mu \geq W \geq m + \mu$.

Equations (2.7), (2.16) and Inequality (2.10) are sufficient for our rigorous bound on Z_2, proved in Section III.

In Section IV we address ourselves to the question of the existence of zeroes in the propagator function. The absence of zeroes permits one to deduce new bounds on Z_2 and bounds on g.[12,15] It will be useful to consider the function

$$Z(W) = [(m - W)\,\Delta(W)]^{-1}, \tag{2.17}$$

which satisfies $Z(m) = 1$ and has the same cut structure as $\Delta(W)$. If we assume the validity of the dispersion relation (2.5),

$$Z_2 = \lim_{|W| \to \infty} Z(W). \tag{2.18}$$

Moreover on the cuts

$$\mathrm{Im}\, Z(W + i\epsilon) = \pi\rho(W)\,(W - m)\,|Z(W)|^2, \tag{2.19}$$

and zeroes of $\Delta(W)$ correspond to poles of $Z(W)$. There can be at most two poles, located on the real line $m + \mu > W > -(m + \mu)$ and if both poles are present they lie on either side of the point $W = m$. This follows from the positivity of $d\Delta(W)/dW$ on the real line, implied by Eq. (2.5). Hence

$$Z(W) = 1 + (W - m)\left[\sum_i \frac{C_i}{W_i - W} + \left\{\int_{-\infty}^{-(m+\mu)} + \int_{(m+\mu)}^{\infty}\right\} \frac{dW'\rho(W')}{W' - W}\,|Z(W')|^2\right] \tag{2.20}$$

and the residues C_i are positive and, at most, two in number. From Eq. (2.18)

$$1 - Z_2 = \left\{ \int_{-\infty}^{-(m+\mu)} + \int_{(m+\mu)}^{\infty} \right\} dW \rho(W) |Z(W)|^2 + \sum_i C_i. \qquad (2.21)$$

Equation (2.21) and Inequaltiy (2.10) allow us to give an upper bound on Z_2 given information about the proper vertex function

$$\Gamma(W) = K(W)\, Z\,(W), \qquad (2.22)$$

which satisfies $\Gamma(m) = 1$.

As shown by Ida,[20] $\eta(W)$, the phase of $\Gamma(W)$ on the upper lips of its cuts, is given, in the elastic regions, by the phases of the P_{11} and S_{11} one-nucleon-irreducible πN partial-wave amplitudes. For simplicity consider the P-wave cut, $m + 2\mu \geq W \geq m + \mu$. We have

$$\begin{aligned} T(W) &= T_R(W) + T_{IR}(W) \\ &= \sin \delta_P(W) \exp(i\delta_P(W)), \end{aligned} \qquad (2.23)$$

and the reducible part is given by

$$\begin{aligned} T_R(W) &= -(g^2/4\pi) h(W) K(W) Z(W) K(W) \\ &= -\sin\beta(W) \exp(i[2\delta_P(W) + \beta(W)]), \end{aligned} \qquad (2.24)$$

where $\beta(W)$ is the phase of $Z(W)$ on the upper lips of its cuts and we have used (2.10) as an equality. Solving for $T_{IR}(W)$ we find

$$T_{IR}(W) = \sin\eta_P(W) \exp(i\eta_P(W)), \qquad (2.25)$$

and $\eta(W) = \eta_P(W)$, modulo π. The proof for negative W is similar.

Thus in analogy with Eq. (2.16) we have

$$\eta_P(W) = \eta(W), \; \eta_S(W) = -\eta(-W), \tag{2.26}$$

modulo π, for $m + 2\mu \geq W \geq m + \mu$.

This result is the basis for our improved bounds in Section IV.

III. BOUNDS FOR Z_2

In this Section we use the experimental value of the strong coupling constant and approximate information about the elastic P_{11} and S_{11} phase shifts to obtain the bound $Z_2 \leq 0.42$ to be compared with the previous result of Drell, Finn and Hearn,[10] $Z_2 \leq 0.86$. In addition we consider what improvement results from assuming the validity of the current algebra sum rule of Suura and Simmons,[16]

$$K(-m) \approx 1/g_A(0), \tag{3.1}$$

where $g_A(t)$ is the nucleon axial-vector form factor. This sum rule results from assuming simple equal-time commutation relations for the weak currents and the nucleon field. Even if these assumptions are valid the sum rule is only exact in the limit of zero pion mass. One might, however, hope that the corrections are small, as in the case of the Goldberger-Treiman relation where they are only about 8%. It turns out that the current algebra result (3.1) by itself gives some improvement over the result of Drell et al., but is of negligible value when combined with phase-shift information. Our final bound does not assume the current algebra result.

The problem we solve here, stated in its generality, is to find an upper bound on Z_2 given the values $K(W_i)$ for some real W_i, such that

$$-(m + \mu) < W_i < (m + \mu) \qquad i = 1, n$$

and information about the phase of K(W) on its cuts in the regions

$$a \geq W \geq m + \mu \quad \text{and} \quad b \geq -W \geq (m + \mu).$$

(In our case $a, b \leq m + 2\mu$ and $n = 1$ or 2 with $W_1 = m$ and $W_2 = -m$.)

To solve this problem we perform the mapping[12]

$$\left(\frac{1 - z}{1 + z}\right) = \left[\frac{(a - W)\,(b + m)}{(a - m)\,(b + W)}\right]^{\frac{1}{2}}, \tag{3.2}$$

which takes the points $W = -b$, m, a to $z = -1, 0, 1$, respectively, and maps the upper lips of the cuts $(-\infty, -b)$ and (a, ∞) to the upper half of the unit circle, $z = e^{i\theta}$, $\pi \geq \theta \geq 0$, and the lower lips to the lower half.

Now consider the function

$$F(z) = \frac{K(W)}{P(W, a)\, S(W, b)}, \tag{3.3}$$

where

$$P(W, a) = \exp\left\{\frac{1}{\pi}\int_{(m+\mu)}^{a} \frac{dW'}{W' - W}\,\delta_P(W')\right\}, \tag{3.4}$$

$$S(W, b) = \exp\left\{\frac{1}{\pi}\int_{(m+\mu)}^{b} \frac{dW'}{W' + W}\,\delta_S(W')\right\}. \tag{3.5}$$

By construction the phases of K(W) and P(W, a) are equal (modulo π) on the cut $(m+\mu, a)$ and the phases of K(W) and S(W, b) are equal (modulo π) on the cut $(-b, -(m+\mu))$, by Eq. (2.16). In addition, neither P(W, a) nor S(W, b) have zeroes inside the unit circle. (Note that $\delta_{P,S}(W)$ vanish like $[W-(m+\mu)]^{\frac{3}{2}, \frac{1}{2}}$ as $W \to m+\mu$.) Hence F(z) is analytic inside the unit circle.

From Eq. (2.7) and Inequality (2.10) we have

$$(g^2/4\pi)\,\frac{Z_2}{1-Z_2} \;\leq\; 1/I \quad , \tag{3.6}$$

where

$$I = \frac{1}{\pi}\left\{\int_{-\infty}^{-b} + \int_{a}^{\infty}\right\}\frac{dW}{|W-m|}\;|K(W)|^2 h(W) \quad . \tag{3.7}$$

We rewrite the integral as

$$I = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \;\; |F(e^{i\theta})|^2 g(\theta) \quad , \tag{3.8}$$

where, from Eqs. (3.3) and (3.7),

$$g(\theta) = h(W)P^2(W, a)S^2(W, b)\;\left[\frac{(W-a)(W+b)}{(a-m)(b+m)}\right]^{\frac{1}{2}} , \tag{3.9}$$

for $W \geq a$ and $W \leq -b$, and we have used

$$\left(\frac{dW}{d\theta}\right)^2 = (W-m)^2\;\frac{(W-a)(W+b)}{(a-m)(b+m)} \quad , \tag{3.10}$$

which follows from the mapping (3.2) with $z = e^{i\theta}$.

For the integral (3.7) to be finite it is sufficient that $|K(W)|$ be bounded by some negative power of $|W|$ as $|W| \to \infty$, Im W = 0. In what follows we merely require that $|K(W)|$ grows more slowly than $\exp(\epsilon |W|)$, for any positive ϵ, as $|W| \to \infty$ in an arbitrary direction. With this restriction we may deduce a finite lower bound for the integral, if it exists, giving a nonzero upper bound for Z_2. If the integral diverges, $Z_2 = 0$ in any case.

The problem has thus been reduced to finding a lower bound for I given the values $F(z_i)$ at the real points z_i, $i = 1, n$. The phases no longer appear explicitly, since they have been absorbed into the weight function $g(\theta)$. This problem has been solved by one of us (V. B.) in reference (21). An outline of the method is sketched in the Appendix. Here we merely state the result for $n = 1$ and $n = 2$.

For $n = 1$

$$I \geq J^2(z_1) \tag{3.11}$$

and for $n = 2$

$$I \geq J^2(z_1) + \left(\frac{J(z_2) - \left[1 - \alpha_{12}^2\right]^{\frac{1}{2}} J(z_1)}{\alpha_{12}} \right)^2 , \tag{3.12}$$

where

$$\alpha_{12} = (z_1 - z_2)/(1 - z_1 z_2) \quad , \tag{3.13}$$

$$J(z) = \left[1 - z^2\right]^{\frac{1}{2}} F(z)\, D(z) \quad , \tag{3.14}$$

with

$$D(z) = \exp \left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} d\theta \left(\frac{e^{i\theta} + z}{e^{i\theta} - z} \right) \ln g(\theta) \right\} \quad . \tag{3.15}$$

Finally we express our general result in terms of convenient functions of W. We find

$$\alpha_{12} = \frac{\left[(a - W_2)(b + W_1)\right]^{\frac{1}{2}} - \left[(a - W_1)(b + W_2)\right]^{\frac{1}{2}}}{\left[(a - W_2)(b + W_1)\right]^{\frac{1}{2}} + \left[(a - W_1)(b + W_2)\right]^{\frac{1}{2}}} \quad . \tag{3.16}$$

The evaluation of D(z) by Eqs. (3.15), (3.9) and (2.11) is straightforward and is outlined in the Appendix. We find

$$J(z) = 2K(W)\left[G(W,a,b)\Delta_S(W,a,b)\Delta_P(W,a,b)\left(\left[\frac{a-m}{a-W}\right]^{\frac{1}{2}} + \left[\frac{b+m}{b+W}\right]^{\frac{1}{2}}\right)\right]^{-1} , \tag{3.17}$$

where

$$\Delta_P(W,a,b) = \exp\left\{\frac{1}{\pi}\int_{(m+\mu)}^{a} \frac{dW'}{W'-W}\,\delta_P(W')\left[\frac{(a-W)(b+W)}{(a-W')(b+W')}\right]^{\frac{1}{2}}\right\}, \tag{3.18}$$

$$\Delta_S(W,a,b) = \exp\left\{\frac{1}{\pi}\int_{(m+\mu)}^{b} \frac{dW'}{W'+W}\,\delta_S(W')\left[\frac{(a-W)(b+W)}{(a+W')(b-W')}\right]^{\frac{1}{2}}\right\} , \tag{3.19}$$

$$G(W,a,b) = \frac{[8/3]^{\frac{1}{2}}\,[C(W,0)]^3\,[C(W,m)]}{\left[C(W,m+\mu)\,C(W,m-\mu)\right]^{\frac{3}{2}}\left[C(W,-(m+\mu))C(W,-(m-\mu))\right]^{\frac{1}{2}}} , \tag{3.20}$$

with

$$C(W,c) = \left[(a-W)(b+c)\right]^{\frac{1}{2}} + \left[(a-c)(b+W)\right]^{\frac{1}{2}} , \tag{3.21}$$

for $a \geq c \geq -b$.

These regrettably extended expressions constitute the general solution of our problem. We remark some simple properties of our auxiliary functions which will simplify the deriviation of bounds in Section IV and may help to render our present result more transparent. Firstly, $C(W,c)$ is a real analytic function of W, with cuts $(-\infty, -b)$ and (a, ∞), has no zeroes, and satisfies

$$|C(W,c)|^2 = |W-c|(a+b) \tag{3.22}$$

on the cuts. From this it follows that $G(W,a,b)$ enjoys the same analyticity properties, and on the cuts

$$|G(W,a,b)|^{-2} = h(W) \quad , \tag{3.23}$$

by comparison of Eqs. (3.20) and (2.11). Secondly, the function $[\Delta_S(W,a,b)\ \Delta_P(W,a,b)]$ has unit modulus on the cuts $(-\infty,-b)$ and (a,∞) and its phase on the cuts $(-b,-(m+\mu))$ and $(m+\mu, a)$ is equal to the phase of $K(W)$, modulo π.

We now consider special cases of the result, Eqs. (3.11) to (3.21).

First we restrict ourselves to the use of the normalization condition $K(m) = 1$. Combining Inequalities (3.6) and (3.11) with Equation (3.17), we have

$$(g^2/4\pi)\frac{Z_2}{1-Z_2} \leq G^2(m,a,b)\ \Delta_S^2(m,a,b)\ \Delta_P^2(m,a,b) \quad . \tag{3.24}$$

Let us suppose we are only given upper bounds on the phases:

$$\begin{aligned} \delta_P(W) &\leq \alpha\ , \quad \text{for} \quad a \geq W \geq m+\mu \quad , \\ \delta_S(W) &\leq \beta\ , \quad \text{for} \quad b \geq W \geq m+\mu \quad , \end{aligned} \tag{3.25}$$

then from Eqs. (3.18) and (3.19) we obtain

$$\ln \Delta_P (m, a, b) \leq (\alpha/\pi) \ln (1/z^+(a, b)), \tag{3.26}$$

$$\ln \Delta_S (m, a, b) \leq (\beta/\pi) \ln (1/z^-(a, b)), \tag{3.27}$$

where the points $z = \pm z^{\pm}(a, b)$ correspond to $W = \pm(m + \mu)$, and we have equality if and only if $\delta_{P,S}(W)$ are everywhere equal to their upper bounds.

In Fig. 1 we have plotted the elastic P_{11} and S_{11} phases as given, in parametric form, by Roper et al.[22] This is a best fit to the data with $W \leq m + 3\mu$ and agrees well, in the elastic region, $W \leq m + 2\mu$, with the single-energy determinations tabulated by Almehed and Lovelace in the CERN Sept. 1971 analysis.[23] We set $\alpha = 0$ and $\beta = 10^o$ and evaluate the bound (3.24) in the four cases of interest:

for $a = b = m + \mu$,

$$(g^2/4\pi) \frac{Z_2}{1 - Z_2} \leq G^2(m, m + \mu, m + \mu) = 85.6 , \tag{3.28a}$$

for $a = m + \mu$, $b = m + 2\mu$,

$$(g^2/4\pi) \frac{Z_2}{1 - Z_2} \leq 82.2 [1.14]^{(2\beta/\pi)} = 83.4 , \tag{3.28b}$$

for $a = m + 2\mu$, $b = m + \mu$,

$$(g^2/4\pi) \frac{Z_2}{1 - Z_2} \leq 11.1 [5.33]^{(2\alpha/\pi)} = 11.1 , \tag{3.28c}$$

for $a = b = m + 2\mu$

$$(g^2/4\pi) \frac{Z_2}{1 - Z_2} \leq 10.5 [1.19]^{(2\beta/\pi)} [5.36]^{(2\alpha/\pi)} = 10.7. \tag{3.28d}$$

Inequality (3.28a) is the old result of Drell et al.,[10] which has been rederived by Okubo.[12] It can be seen that little improvement comes from the inclusion of the S_{11} phase shift; it is the smallness of the P_{11} phase shift which gives the large improvement of bound (3.28c) upon bound (3.28a). The sensitivity to the P_{11} phase shift may be readily understood by the closeness of our datum point, W = m, to the elastic P-wave cut.

Next we use the parametrized phase shifts in Inequality (3.24). Setting $a = b = m + 2\mu$, we find

$$(g^2/4\pi)\frac{Z_2}{1 - Z_2} \leq 10.4 \ . \tag{3.28e}$$

Finally, we experiment with using $K(-m) \approx 1/g_A(0)$, where[24] $g_A(0) = (1.226\pm0.011)$. If we set $K(-m) = 1/1.226$, and $a = b = m + \mu$, so that the phase shifts are not required,

$$(g^2/4\pi)\frac{Z_2}{1 - Z_2} \leq 47.3 \ , \tag{3.28f}$$

whereas with $a = b = m + 2\mu$ and the parametrized phase shifts,

$$(g^2/4\pi)\frac{Z_2}{1 - Z_2} \leq 10.0 \ . \tag{3.28g}$$

It can be seen that the current algebra result, by itself, gives a significant improvement over the old result (compare bounds (3.28a) and (3.28f)), but is of little value when combined with the phase shifts (compare bounds (3.28e) and (3.28g)). Accordingly we state our final result without any assumption about K(-m).

From (3.28e) we obtain

$$Z_2 \leq .413^{+.010}_{-.007} \ , \tag{3.29}$$

as against the previous best rigorous bound,[10,12] (3.28a),

$$Z_2 \leq .853 \begin{smallmatrix} +.006 \\ -.004 \end{smallmatrix} . \tag{3.30}$$

The errors given arise from three sources. First there are the experimental errors in g. We take[25] $g^2/4\pi = (14.73 \pm 0.29)$. Second there are the phase shifts. We allow for variations comparable to the small discrepancies between references (22) and (23). Lastly, there is the appropriate value for the pion to nucleon mass ratio. Here we have to consider isospin breaking, since our result is only valid in the limit of exact isospin symmetry, which presumably corresponds to ignoring the electromagnetic interaction. Isospin assumptions are necessary to determine the coupling constants and phases involving π^o. For the numerical results given in Inequalities (3.28a) to (3.28g) we made the somewhat arbitrary choice

$$\begin{aligned} \mu^2 &= \frac{1}{3}\left(2\mu^2_{\pi^+} + \mu^2_{\pi^o}\right) , \\ m &= \frac{1}{2}\left(m_p + m_n\right) , \end{aligned} \tag{3.31}$$

but the errors given above allow for the whole range

$$\mu_{\pi^+}/m_p \geq \mu/m \geq \mu_{\pi^o}/m_n , \tag{3.32}$$

which should adequately represent the sensitivity of our result to isospin breaking.

In conclusion, we have improved the rigorous upper bound on Z_2 from .86 to .42 using πN elastic phase shifts. We consider this a radical improvement, since it corresponds to an improvement by a factor of 8 in the lower bound for $Z_2^{-1}-1$. Our only assumption is that $|K(W)| < \exp(\epsilon |W|)$, for any $\epsilon > 0$, as $|W| \to \infty$.

IV. ZEROES OF THE PROPAGATOR

In this Section we improve the upper bound for $g^2/4\pi$ originally given by Geshkenbein and Ioffe,[1] assuming the absences of zeroes of $\Delta(W)$. This is done by including the phases of the proper vertex function, $\Gamma(W)$, on its elastic cuts, which are given by the phases of the one-nucleon-irreducible P_{11} and S_{11} partial-wave amplitudes. These phases are, of course, not experimentally accessible. We calculate $\eta_P(W)$, the phase of the P_{11} irreducible amplitude in the elastic region, using an N/D approximation of Ida.[15] In addition we critically reexamine the Lehmann-Symanzik-Zimmermann (LSZ) sum rule, Eq. (2.21), which was considered by Ida. Our conclusion is that $\Delta(W)$ has a zero.

First consider the function

$$R(z) = \frac{(g^2/4\pi)K(W)\Gamma(W)}{G^2(W,a,b)\Delta_S(W,a,b)\Delta_P(W,a,b)H_S(W,a,b)H_P(W,a,b)} , \quad (4.1)$$

where, in analogy with Eqs. (3.18) and (3.19),

$$H_P(W,a,b) = \exp\left\{ \frac{1}{\pi} \int_{(m+\mu)}^{a} \frac{dW'}{W'-W}\, \eta_P(W') \left[\frac{(a-W)(b+W)}{(a-W')(b+W')}\right]^{\frac{1}{2}} \right\}, \quad (4.2)$$

$$H_S(W,a,b) = \exp\left\{ \frac{1}{\pi} \int_{(m+\mu)}^{b} \frac{dW'}{W'+W}\, \eta_S(W') \left[\frac{(a-W)(b+W)}{(a+W')(b-W')}\right]^{\frac{1}{2}} \right\}, \quad (4.3)$$

and $m+2\mu \geq a, b \geq m+\mu$. We are using the same mapping as before, given by Eq. (3.2). Then $R(z)$ is meromorphic inside the unit circle, having poles corresponding to the zeroes of $\Delta(W)$ (if any). By virtue of Eq. (3.23),

$$|R(e^{i\theta})| = (g^2/4\pi)|K(W)\Gamma(W)|h(W) , \quad (4.4)$$

and the unitarity inequality (2.10) takes a particularly simple form:

$$|R(e^{i\theta})| = |r(W)\sin\beta(W)| \leq 1 , \tag{4.5}$$

where r(W) is the fraction of the nucleon spectral function contributed by πN intermediate states and $\beta(W)$ is the phase of Z(W) on the upper lips of its cuts. If we now assume $\Delta(W)$ has no zeroes, R(z) is analytic inside the unit circle. We now invoke the generalized maximum modulus theorem (see, for example, ref. (12)) which requires

$$|R(z)| \leq 1 , \tag{4.6}$$

inside the unit circle, provided that K(W) does not grow like

$$|K(W)| \propto \exp\left\{c|W|^{\gamma}\right\} \tag{4.7}$$

with $c > 0$ and $\gamma \geq 1$, as $W \to \infty$ along any straight radial direction. Rejecting such a pathological essential singularity as (4.7) we conclude that, in particular,

$$|R(0)| \leq 1, \tag{4.8}$$

or, by Eq. (4.1),

$$g^2/4\pi \leq G^2(m,a,b)\Delta_S(m,a,b)\Delta_P(m,a,b)H_S(m,a,b)H_P(m,a,b) . \tag{4.9}$$

Inequality (4.9) should be compared with Inequality (3.24). The former involves the phase $\eta(W)$ but not Z_2, and for the latter vice versa.

Setting $a = b = m + \mu$ we obtain the old result of Geshkenbein and Ioffe,[1]

$$g^2/4\pi \leq G^2(m,\ m+\mu,\ m+\mu) = 85.6 . \tag{4.10}$$

There is a third class of inequalities, involving both $\eta(W)$ and Z_2, which follows from the LSZ sum rule (2.21) and the unitarity inequality. We have

$$1 \geq 1 - Z_2 \geq \frac{1}{\pi} \left\{ \int_{-\infty}^{-b} + \int_{a}^{\infty} \right\} \frac{dW}{|W-m|} \, |\Gamma(W)|^2 h(W) \, , \tag{4.11}$$

which, by the same arguments as in Section III, yields

$$(g^2/4\pi)\frac{1}{1-Z_2} \leq G^2(m,a,b)H_S^2(m,a,b)H_P^2(m,a,b) \, , \tag{4.12}$$

provided $\Delta(W)$ has no zeroes. Setting $a = b = m + \mu$ we obtain

$$(g^2/4\pi)\frac{1}{1-Z_2} \leq 85.6 \, , \tag{4.13}$$

which is more restrictive than the bound of Drell et al., Inequality (3.28a), but has been obtained with much more restrictive assumptions.

We now concentrate on Inequality (4.9), which, in general, involves the phases $\eta_{P,S}(W)$.

Our first new result comes from the fact that the phases $\beta_{P,S}(W) = \eta_{P,S}(W) - \delta_{P,S}(W)$ satisfy

$$\pi \geq \beta_{P,S}(W) \geq 0 \quad \text{for} \quad m + 2\mu \geq W \geq m + \mu \, ,$$

using the fact that the $\rho(W)$ is positive, and assuming that the propagator has neither poles nor zeroes on the elastic cuts.

Hence

$$H_{P,S}(m,a,b) \leq \Delta_{P,S}(m,a,b) \left[\frac{1}{z^{\pm}(a,b)} \right] . \tag{4.14}$$

We combine Inequalities (4.12) and (4.14) and use the parametrized phase shifts $\delta_{P,S}(W)$. The optimal values of a and b turn out to be $m + 1.8\,\mu$ and $m + \mu$, respectively, giving

$$g^2/4\pi \leq 57.6 , \tag{4.15}$$

which is some small improvement on the Geshkenbein-Ioffe result. However, we consider bounds on g of little interest (since g is well determined experimentally) unless they are violated or else perilously close to saturation, thereby forcing us to examine one of our assumptions (such as the absence of zeroes). Accordingly we resort to a model for the first time, enabling a calculation of $\eta_P(W)$ in the elastic region. Yet again in this Section we assume the absence of zeroes so that the irreducible partial-wave amplitude

$$\begin{aligned} T_{IR}(W) &= \sin \eta_P(W) \exp (i\eta_P(W)) \\ &= N(W)/D(W) \end{aligned} \tag{4.16}$$

has the same analyticity properties as the full amplitude T(W), apart, of course, from having lost the direct channel nucleon pole. (If $\Delta(W)$ had zeroes, $T_{IR}(W)$ would have poles.) To calculate $\eta_P(W)$ we use Ida's N/D model,[15] in which the dynamical left-hand singularities of N are approximated by two poles, representing nucleon and P_{33} exchange. Details are given by Ida and by Frautschi and Walecka.[26]

The result may be given compactly as follows:

$$\cot \eta_P(W) = \mathrm{Re}D(W)/N(W) , \tag{4.17}$$

$$D(W) = 1 - \frac{(W-m)}{\pi} \int_{(m+\mu)}^{\infty} \frac{dW' N(W')}{(W'-m)(W'-W)} , \tag{4.18}$$

$$N(W) = (g^2/4\pi)\, h(W) \left[1+8D(W_1) \left(\frac{W-m}{W-W_1} \right) \right] /9 \quad . \tag{4.19}$$

The first term in Eq. (4.19) approximates the short cut from $W = (m^2-\mu^2)/m$ to $W = (m^2+2\mu^2)^{\frac{1}{2}}$, coming from nucleon exchange. (Note that h(W) has a pole at W = m.) The second pole term approximates the short cut from W = .59 m to W = .76 m, coming from P_{33} exchange. We take[26] W_1 = .68 m and the effective resonance coupling has been taken from dispersion theory.[27] The evaluation of the principal value integral presents no difficulty, since we deform the contour to one parallel to the imaginary W axis. The output, $\eta_P(W)$, is plotted in Fig. 2.

It is important to realize that we are using Ansatz (4.19) quite circumspectly. We require $\eta_P(W)$ in the region $m + 2\mu \geq W \geq m + \mu$, which is very close to the approximated short cuts. We attempt no calculation of $\eta_S(W)$, where ρ exchange is important.[15] Setting $a = m + 2\mu$, $b = m + \mu$ in Inequality (4.9), we obtain

$$g^2/4\pi \leq G^2(m, m+2\mu,\ m+\mu)\Delta_P(m,\ m+2\mu,\ m+\mu)\ H_P(m,\ m+2\mu,\ m+\mu) \tag{4.20}$$

$$\approx 11.1 \times 0.99 \times 1.39 \ = \ 15.3 \ ,$$

to be compared with the experimental value, $g^2/4\pi = (14.73 \pm .29)$.

Whilst Inequality (4.20) is not actually violated it almost certainly requires at least one zero in the propagator, since in obtaining it we have neglected all contributions to the nucleon spectral function except πN, and have ignored the real part of the propagator. By way of illustration we consider the inequality

$$\ln|R(0)| \ \leq \frac{1}{\pi} \int_0^{\pi} d\theta \ \ln|R(e^{i\theta})| \quad , \tag{4.21}$$

of which the maximum modulus theorem is a weakened version. It can be seen that requiring the absence of essential singularities such as (4.7) ensures that the upper semicircle at infinity in the W plane does not contribute to the integral. Also (4.21) becomes an equality if K(W) has no zeroes away from the cuts.

Combining Inequality (4.21) with Eq. (4.5), and accepting the phase calculation, we obtain

$$g^2/4\pi \leq 15.3 \exp\left\{ \frac{1}{\pi} \int_0^\pi d\theta \ \ln|r(W) \sin\beta(W)| \right\} , \qquad (4.22)$$

where Fig. 3 illustrates the relation between W and θ. Inequality (4.22) requires that the mean value, $< r \sin\beta >$, as weighted by the integral, must exceed .95, even allowing the smallest experimentally acceptable value of $g^2/4\pi$. This seems overwhelmingly unlikely. For example, in the range W = (1470 ± 100) MeV ($\theta \approx (80 \pm 10)^\circ$) where the Roper resonance is important for πN scattering we would also expect it to be the dominant contribution to the nucleon spectral function, suggesting $r(W) \approx 0.6$, the elasticity of the Roper. This, by itself, is sufficient to violate Inequality (4.22). Even more, there is no good reason to suppose r(W) will be any larger for larger values of W ($\theta = 90^\circ - 140^\circ$) or that Δ(W) will be so predominantly imaginary.

If the N/D calculation is reliable in the absence of zeroes, it seems very difficult to escape the conclusion that Δ(W) has at least one zero. The only plausible way out would be to maintain that we have considerably underestimated $\eta_p(W)$. However, $\eta_p(W)$ is constrained to vanish like $[W-(m+\mu)]^{\frac{3}{2}}$ at threshold and in our calculation already has a large value (76°) at inelastic threshold.

Geshkenbein and Ioffe[1] originally argued against the existence of a zero of Δ(W) on the grounds that it corresponds to a pole in the reducible πN scattering amplitude and the full amplitude may not have such a pole. However, Goebel and Sakita[28] were the first to point out that such a pole is ghost-like and must be cancelled by higher order Feynman diagrams. In fact Drell et al.[10] demonstrated the existence of a zero in a generalized Lee model, with two fermions, V and W, coupling to N and θ and a suitable choice of coupling constant for the unstable W particle. In conclusion, there appears to be no compelling argument against zeroes. They merely correspond to ghost poles in the reducible partial-wave amplitude, which are cancelled by the irreducible partial-wave amplitude.

Finally we consider our LSZ inequality (4.12). Setting $a = m + 2\mu$, $b = m + \mu$, we obtain

$$(g^2/4\pi)\frac{1}{1 - Z_2} \leq 11.1 \times [1.39]^2 \tag{4.23}$$

which gives $Z_2 \leq .31$, <u>in the absence of zeroes</u>, and using the N/D approximation. Of course we do not now believe this bound, but it is instructive to compare it with the result of Ida, $Z_2 < .20$, which was obtained from the more restrictive LSZ inequality (4.11) (setting $a = b = m + \mu$). Now Inequality (4.11) is valid in the presence of zeroes. It merely assumes that $\Delta(W)$ satisfies an unsubtracted dispersion relation (in which case it is a Herglotz function).[29] Ida's method was to assume that both $\Delta(W)$ and $K(W)$ are free of zeroes, so that $\Gamma(W)$ has an Omnes representation in terms of its phase, $\eta(W)$. He then set $\eta(W)$ equal to $\eta_P(W)$, as calculated in the N/D approximation, for $m+2.5\,\mu \geq W \geq m+\mu$. This, we have suggested, is probably reliable, at least for $m + 2\mu \geq W$. However, he then chose a form for $\eta(W)$ in the regions $-(m + \mu) \geq W \geq -\infty$ and $\infty \geq W \geq m + 2.5\,\mu$ such as to give the <u>least</u> restrictive upper bound for Z_2, giving the justification that otherwise there was a danger of deducing a negative upper bound for Z_2, which is impermissible. We consider his arbitrary determination of $\eta(W)$ outside the elastic P-wave region as unjustified and are relatively unconcerned about an apparent violation of the LSZ inequality in his model, since it proceeds from an assumption (the absence of zeroes of $\Delta(W)$) which we have shown to be scarcely tenable, merely by the much more trustworthy determination of $\eta(W)$ in the elastic P-wave region.

That we are able to conclude $Z_2 \leq .31$ from Inequality (4.11), assuming no zeroes, is further suggestive of the existence of zeroes, since we actually neglect even the πN contributions in the elastic region, where, according to Ida, a major contribution to the LSZ sum rule is to be found. The smallness of our upper bound is therefore highly suspicious and would have cast doubts on our assumptions even if we had not been able to give the much cleaner argument based on our more restrictive version of the Geshkenbein and Ioffe bound on $g^2/4\pi$.

It is interesting that the improvement of the upper bound for $g^2/4\pi$ (in the absence of zeroes) that results from including the phase $\eta(W)$ is akin to the improvements of bounds on π-π scattering amplitudes obtained by Łukaszuk and Martin[30] who also use phase information.

In conclusion, at least one of the following situations obtains:

(i) K(W) has an essential singularity at infinity of the form of Eq. (4.7);

(ii) $\Delta(W)$ is predominantly imaginary over a large range of W, its imaginary part is given predominantly by πN intermediate states over a large range of W and the N/D calculation considerably underestimates the phase of the irreducible partial-wave amplitude on the elastic P_{11} cut;

(iii) $\Delta(W)$ has a zero.

We decide in favour of (iii), knowing of no compelling physical argument to the contrary. This conclusion invalidates all bounds on g or Z_2 derived in this Section. It in no way affects our rigorous bound on Z_2 derived in Section III.

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APPENDIX

Here we find a lower bound for

$$I = \frac{1}{\pi}\int_0^\pi d\theta \, |F(e^{i\theta})|^2 g(\theta) \quad < \quad \infty$$

given the values $x_i = F(z_i)$, for the real points z_i, $i = 1, n$. Further details are given in reference (21).

We expand F(z) in terms of the orthogonal polynomials $\phi_\nu(z)$, defined by the weight function $g(\theta)$:

$$F(z) = \sum_\nu F_\nu \phi_\nu(z),$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \; \phi_\nu^*(e^{i\theta}) \; \phi_\mu(e^{i\theta}) \, g(\theta) = \delta_{\nu\mu} ,$$

so that

$$x_i = \sum_\nu F_\nu \phi_\nu (z_i) ,$$

$$I = \sum_\nu F_\nu^2 .$$

It is now required to find the extremum of I for fixed values of x_i. This is easily done using Lagrange multipliers. Solving the equation

$$\frac{\delta}{\delta F_\nu} (I - \sum_i \lambda_i x_i) = 0$$

we obtain

$$F_\nu = \sum_i \lambda_i \phi_\nu(z_i)$$

giving

$$I \geq \sum_i \lambda_i x_i \qquad (A.1)$$

where

$$x_i = \sum_j a_{ij} \lambda_j \qquad (A.2)$$

and

$$a_{ij} = \sum_{\nu} \phi_\nu(z_i)\phi_\nu(z_j)$$

$$= \left[(1 - z_i z_j)\, D(z_i)\, D(z_j)\right]^{-1} \qquad (A.3)$$

with D(z) the characteristic function for the set of polynomials, given by

$$D(z) = \exp\left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} d\theta \left(\frac{e^{i\theta} + z}{e^{i\theta} - z} \right) \ln g(\theta) \right\} . \qquad (A.4)$$

Combining Eqs. (A.1) to (A.3) we have

$$I \geq \sum_{i,j} J(z_i)\, M_{ij}\, J(z_j) ,$$

where

$$J(z) = \left[1 - z^2\right]^{\frac{1}{2}} F(z)\, D(z) , \qquad (A.5)$$

and M is a matrix whose inverse is given by

$$M_{ij}^{-1} = \frac{\left[(1 - z_i^2)\,(1 - z_j^2)\right]^{\frac{1}{2}}}{(1 - z_i z_j)} = \left[1 - \alpha_{ij}^2\right]^{\frac{1}{2}}$$

where

$$\alpha_{ij} = (z_i - z_j)/(1 - z_i z_j) . \qquad (A.6)$$

For $n = 1$, M is just unity, so that

$$I \geq J^2(z_1) \qquad (A.7)$$

and for $n = 2$,

$$M = \begin{pmatrix} 1 & -\left[1 - \alpha_{12}^2\right]^{\frac{1}{2}} \\ -\left[1 - \alpha_{12}^2\right]^{\frac{1}{2}} & 1 \end{pmatrix} \Big/ \alpha_{12}^2 ,$$

so that

$$I \geq \left[J^2(z_1) + J^2(z_2) - 2\left[1 - \alpha_{12}^2\right]^{\frac{1}{2}} J(z_1)\, J(z_2)\right] \Big/ \alpha_{12}^2$$

$$= J^2(z_1) + \left(\frac{J(z_2) - \left[1 - \alpha_{12}^2\right]^{\frac{1}{2}} J(z_1)}{\alpha_{12}}\right)^2 \quad . \qquad (A.8)$$

Equations (A.4) to (A.8) are those used in Section III.

In deriving these bounds it is necessary to assume that $\ln|F(z)|$ is integrable on $|z| = 1$, which corresponds to our restriction that $|K(W)| < \exp(\epsilon |W|)$, for any positive ϵ, as $|W| \to \infty$.

Finally we sketch the evaluation of D(z) in terms of $g(\theta)$. If we write

$$g(\theta) = \prod_i g_i(\theta)$$

$$D(z) = \prod_i D_i(z)$$

then the various $g_i(\theta)$ which occur in our case are of the form

$$g_i(\theta) = K > 0,$$

$$g_i(\theta) = |W - c|, \quad a \geq c \geq -b,$$

$$g_i(\theta) = \exp \frac{1}{\pi} \int_{-\beta}^{\alpha} \frac{dW'}{W' - W} \delta(W'), \text{ where } \alpha = a,\ \delta(-\beta) = 0, \text{ or } \beta = b,\ \delta(\alpha) = 0,$$

for which the corresponding $D_i(z)$ are

$$D_i(z) = K^{\frac{1}{2}},$$

$$D_i(z) = \left[\frac{(a - W)(b + c)}{(a + b)}\right]^{\frac{1}{2}} + \left[\frac{(a - c)(b + W)}{(a + b)}\right]^{\frac{1}{2}},$$

$$D_i(z) = \exp \left\{ \frac{1}{2\pi} \int_{-\beta}^{\alpha} \frac{dW'}{W' - W} \, \delta(W') \left(1 - \left[\frac{(a - W)(b + W)}{(a - W')(b + W')} \right]^{\frac{1}{2}} \right) \right\} .$$

In each case $D_i(z)$ is a real analytic function of z inside the unit circle, has no zeroes inside the unit circle and satisfies

$$|D_i(e^{i\theta})|^2 = g_i(\theta)$$

on the unit circle.

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FIGURE CAPTIONS

1. The P_{11} and S_{11} πN phase shifts in the elastic region, as given in solution 24, table III of reference (22).

2. The phase η_P of the irreducible P_{11} πN partial-wave amplitude, as calculated from Eqs. (4.17) to (4.19).

3. The unit circle with values of $(W-m)/\mu$ given by the mapping (3.2) with $a = m+2\mu$, $b = m+\mu$. The shaded area corresponds to the range of θ for $W = (1470\pm100)$ MeV.

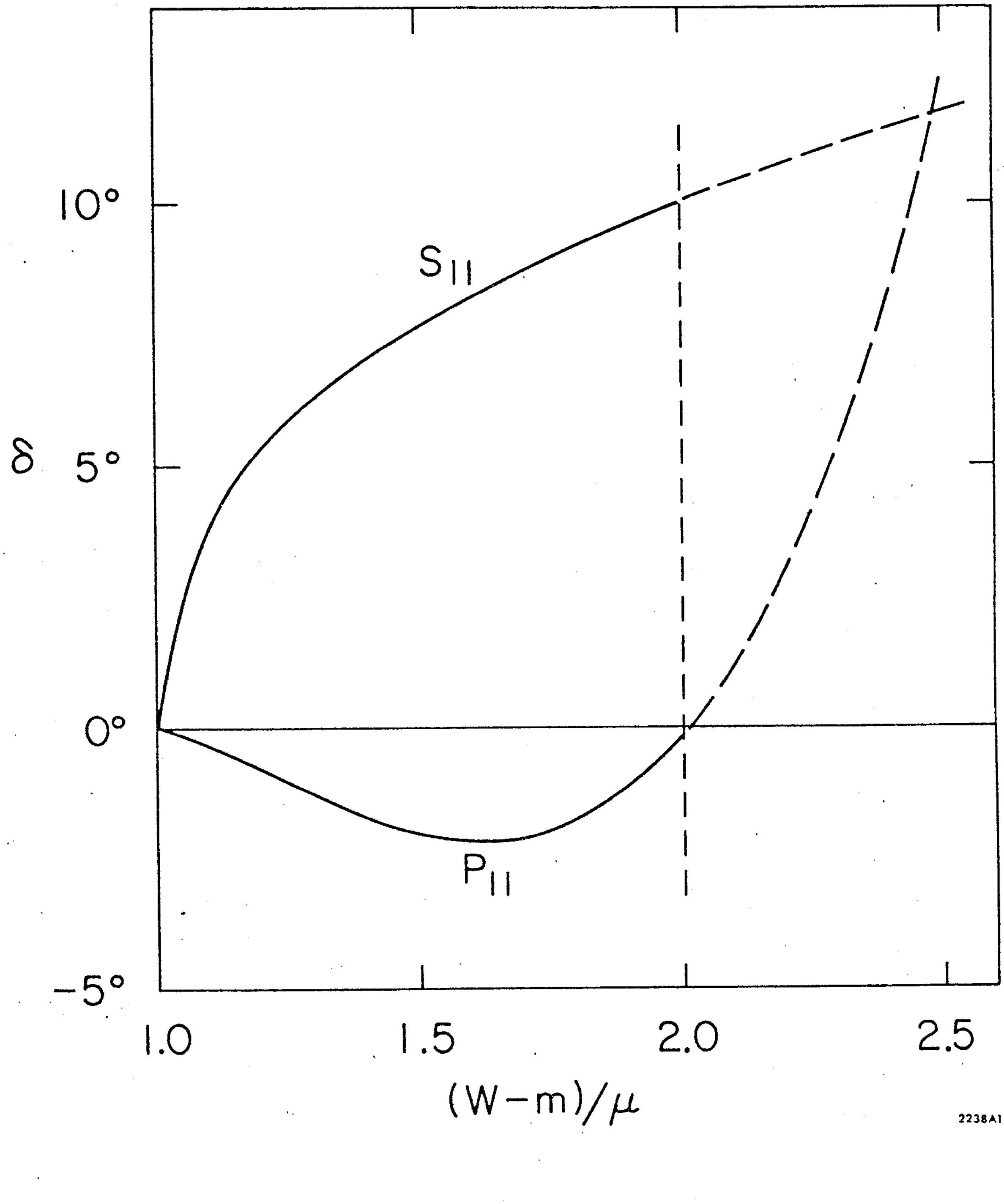

Fig. 1

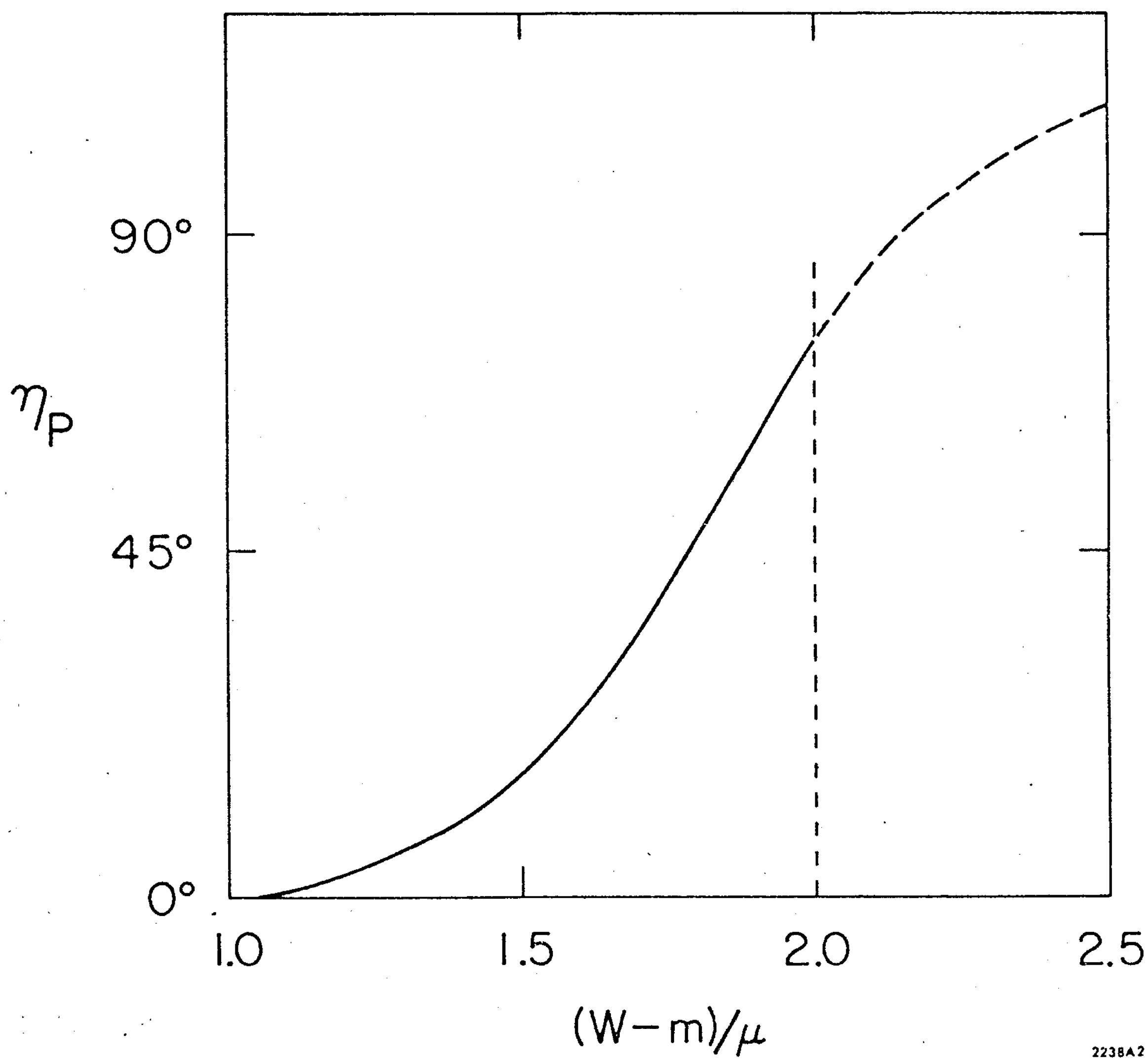

Fig. 2

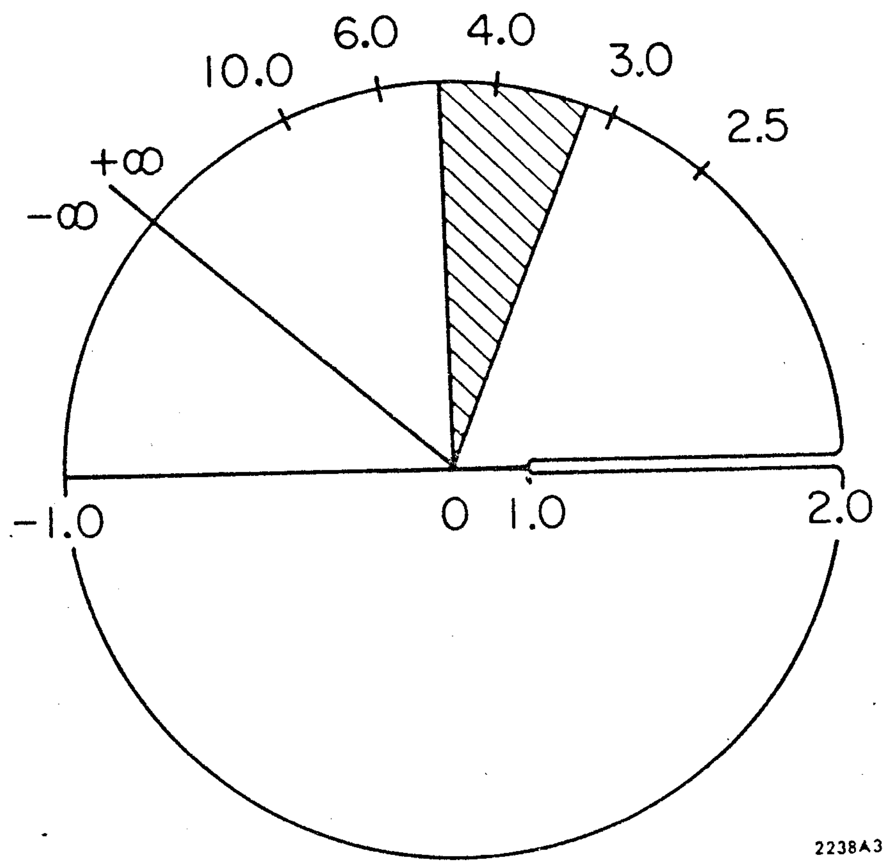

Fig. 3