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NUMERICAL SOLUTION OF THE ELECTRON DIFFUSION EQUATION*

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ABSTRACT

A numerical solution to the integro-differential equation describing the energy distribution of a beam of electrons which has passed through matter, losing energy by radiation only, has been obtained utilizing a finite difference mesh method. Solutions were obtained for thicknesses of up to 0.1 radiation lengths for a complete screening approximation to the energy loss equation. The accuracy of the method was checked by comparison of results with known solutions to the diffusion equation. The formulas of Mo-Tsai and Tsai for electron straggling distributions were compared to the numerical results. Good agreement was found near the high energy end of the distribution, the numerical results being within two percent of the theoretical predictions. At the low energy end of the distribution, the numerical results differ from those predicted by as much as eight percent at thicknesses of 0.1 radiation length. The disagreement was found to be proportional to thickness traversed by the beam.

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1. Introduction

A problem of interest in electron scattering experiments is that of determining the energy loss of electrons due to bremsstrahlung in the target material before and after the scattering process of interest. The formulas of Bethe and Heitler[1, 2] and of Eyges[3] can be shown to satisfy the electron diffusion equation but are derived from expressions which only approximate the bremsstrahlung spectrum shape. Recently, Mo and Tsai[4] proposed an expression which, while utilizing a more accurate approximation to the energy loss spectrum, appears to be impossible to check analytically except by examining its normalization properties. This has stimulated interest in obtaining a solution to the electron diffusion equation numerically in order to test the accuracy of the formulas in use.

The problem which has been considered is the distribution in energy of electrons of uniform initial energy which have penetrated t radiation lengths and which lose energy by radiation only. Our interest was motivated because this distribution is required in the radiative corrections in inelastic electron scattering; in this application t is typically less than 0.1 radiation lengths, and this is the greatest thickness which has been investigated numerically. The energy distribution is always normalized to unity. Electrons created by pair-production, for example, are not included in the energy distribution under consideration.

A method utilizing a finite difference mesh has met with some success; the accuracy was checked by comparison with known solutions to the diffusion equation. The method of solution, as well as the results of comparing the various analytic expressions with numerical results are discussed below.

II. The Diffusion Equation

The derivation of the diffusion equation given below follows that of Rossi and Greison[5], but ignores pair production processes since we are only interested in solutions for values of t up to 0.1 radiation lengths.

Let $\pi(E,t)\,dE$ represent the probability of an electron having energy between E and E + dE at thickness t. $\phi(v)\,dvdt$ represents the probability for an electron of energy E to radiate in thickness dt a photon of energy between k and k+dk, where v = k/E.

Electrons with energy E', larger than E, can enter the interval between E and E+dE by radiating part of their energy (k=E'-E). The increase in $\pi(E,t)$ at t+dt is:

$$dEdt \int_E^\infty \pi(E',t)\ \phi((E'-E)/E')\ dE'/E'$$

Electrons initially in the interval between E and E+dE can leave this interval by radiation loss. The decrease in $\pi(E,t)$ at t+dt is:

$$dEdt\ \pi(E,t) \int_0^E \phi((E-E')/E)\ dE'/E$$

so that the change in $\pi(E,t)$ with thickness dt is represented by the difference between the two integrals:

$$\frac{\partial \pi}{\partial t} = -\ \pi(E,t) \int_0^E \phi((E-E')/E)\ dE'/E$$

$$+ \int_E^\infty \pi(E',t)\ \phi((E'-E)/E')\ dE'/E'$$

The upper limit of infinity in the second integral can be replaced by the highest energy contained in the distribution in a practical case. The diffusion equation is often written in a slightly different form:

$$\frac{\partial \pi}{\partial t} = -\int_0^1 \left(\pi(E,t) - \frac{1}{(1-v)} \, \pi\left(\frac{E}{1-v}, t\right) \right) \phi(v) \, dv \qquad (2)$$

using

$$E' = \frac{E}{1-v} \, .$$

Equation (2) is the form of the diffusion equation most convenient for the application of Mellin transforms, while (1) is suitable for finite difference mesh solutions which depend on values of $\pi(E,t)$ computed at regular intervals in E and t.

Solutions to the above equations are of interest in electron scattering experiments at values of t up to 0.1 radiation length. The expression used for $\phi(v)$ in the numerical solution was that corresponding to complete screening,

$$\phi(v) = b\,(1/v - 1) + v \qquad b \approx 4/3 \qquad (3)$$

which is valid for small energy loss and very close to the true shape of the bremsstrahlung spectrum for large energy loss at energies in the GeV range[4] (Fig. 1). Expressions involving arbitrary screening are impractical in the type of numerical solution discussed here because of the comparitively long time required to compute values for $\phi(v)$.

Bethe and Heitler[1, 2] used an expression which approximates the case of partial screening in obtaining their analytic expression for straggling:

$$\phi(v) = -\frac{b}{\log\,(1-v)} = b/\log\,(E/E') \qquad (4)$$

While Eyges[3] used a more general form:

$$\phi(v) = -b(1-v)^a/\log(1-v) = b\,(E'/E)^a/\log(E/E') \qquad (5)$$

In (4) and (5), b is the value of $v\,\phi(v)$ for $v=0$. The logarithms are to the base e. Expression (5) is the same as (4) for the case $a=0$. The various ϕ functions can be compared in Fig. 2, where values of $v\phi$ have been plotted vs. v. Figure 1 shows some results from Rossi's calculations of $v\phi$ for variable screening [5].

Eyges[3] obtained the following solution to (2) for the loss function (5) by a Mellin transform:

$$\pi(E_0, E, t) = \frac{1}{E_0}\,(1+a)^{bt}\,\frac{E}{E}^{a}\,\frac{\left[\log(E_0/E)\right]^{bt-1}}{\Gamma(bt)} \qquad (6)$$

where $\pi(E_0, E, t)dE$ is the probability for finding an electron with energy between E and E+dE at thickness t radiation lengths when the initial energy at $t=0$ is E_0. This allows comparison of numerical and analytic results for various values of the parameter a in (5). It can be seen that values of a between .25 and .5 describe the loss function best in the low v region, while negative values of a, like -.25 are a better approximation in the region of v near to one. However, in the diffusion equation, the value of v in the gain integrand ranges from zero to $(1 - E/E_0)$, while the loss integrand always ranges over the entire range of v from zero to one, so that it is impossible to find an optimum value for a except for a specific energy (see Fig. 5).

The expression given by Mo-Tsai is, in the present notation:

$$\pi(E_0, E, t) = \frac{t\,\phi\left(\frac{E_0-E}{E_0}\right)\,\log\left(\frac{E_0}{E}\right)^{bt}}{E_0}$$

If (4) is substituted for ϕ in the above expression we get:

$$\pi(E_0, E, t) = \frac{bt\left(\log\left(\frac{E_0}{E}\right)\right)^{bt^{-1}}}{E_0}$$

While Bethe-Heitler and Eyges give for this same loss function (4) the analytic result:

$$\pi(E_0, E, t) = \frac{bt\left(\log\left(\frac{E_0}{E}\right)\right)^{bt^{-1}}}{E_0\ \Gamma(1+bt)}$$

which is normalized to unity when integrated from E=0 to E=E_0. Since the Mo-Tsai expression was derived for small bt, the gamma function was approximated by one. In the present study, we have restored $\Gamma(1+bt)$ to the denominator since at values of $t \approx .1$, $\Gamma(1+bt) \approx .94$.

Although the Mellin transformation can be applied to (2) when the loss function corresponds to (3), the inverse transformation requires solution of the integral[3,5]:

$$\int_{\delta-i\infty}^{\delta+i\infty} e^{\left[y(s+1) - b\,\psi(s+1)\,t + \frac{t}{(s+1)(s+2)}\right]} ds$$

where $y = \log(E_0/E)$, and ψ is the logarithmic derivative of the gamma function. Eyges solved the inversion integral corresponding to the loss function (4) by completing the contour in the negative half plane with a circle and applying the residue theorem. Since the poles at s=-1 and s=-2 in the above integral are not regular singular points, the residue theorem can not be used for loss function (3). Also, ψ has an infinite number of poles and zeroes in the negative half plane. Saddle point solutions[3] are inaccurate for the case of small t, of interest here, and solving the integral by numerical methods appears difficult. Hence, there is need for another approach.

III. Numerical Method of Solution

In obtaining a numerical solution to the diffusion equation, we make use of the fact that $\pi(E,t)$ is known at t=0. The integrals in (1) may be evaluated numerically by a method to be discussed below. An approximate solution can be obtained at t=dt by applying Euler's method[6], which is equivalent to applying a first order Taylor expansion over a small interval, dt:

$$\pi(E, dt) = \pi(E, 0) + dt \frac{\partial \pi(E, 0)}{\partial t} \tag{7}$$

A mesh, or grid is set up in E and t as shown in Fig. 3. Mesh points are taken at equal intervals in energy so that they may be used as sample points in the numerical evaluation of the integrals in (1). Equation (7) is applied to every mesh point in column 2 of the mesh. Points in column 3 may be computed by using points in column 2 as sample points in the integrals. The process is repeated until the desired value of t is reached and all points in energy have been computed. Any initial distribution can be handled this way, but we are interested in the case of a beam of electrons with an initial distribution at t = 0 of

$$\pi(E, 0) = \delta(E_0 - E)$$

where E_0 is the initial energy.

This distribution is approximated in column 1 of the mesh by assigning a value of 1/dE to the top, left-hand corner mesh point P_{11}. All others are initially set equal to zero. dE is the separation of mesh points in energy. Thus, the delta function is approximated by a triangular distribution in column 1. 1/dE is chosen as the value for P_{11} to normalize the distribution. In each column j we should have:

$$\sum_{i=1}^{NE} P_{ij}\, dE = 1$$

where NE is the total number of energy points.

Equation (1) is a more useful form than (2) for a mesh solution. The apparent pole at v = 1 in (2) corresponds to an infinite upper limit of energy and is not of importance here. One can rewrite (2) as follows:

$$\frac{\partial \pi}{\partial t} = -\pi \int_0^1 \phi(v)\, dv + \int_0^{1-E/E_0} \pi\left(\frac{E}{1-v}, t\right) \frac{dv}{(1-v)}$$

where we have ignored the region above E_0 since $\pi(E,t)$ is zero for $E>E_0$.

For the case of the lower limit, we note that $\phi(v)$, in Eqs. (3), (4), and (5), behaves like 1/v and diverges as v goes to 0. However, the difference between the integrals in (1) or in the terms of the integrand in (2) remains finite[5].

Equation (1) can be rewritten as follows:

$$\frac{\partial \pi}{\partial t}(E,t) = -\pi(E,t) \int_\delta^E \phi\left(\frac{E'}{E}\right) \frac{dE'}{E} + \int_{E+\delta}^{E_0} \pi(E',t)\, \phi\left(\frac{E'-E}{E'}\right) \frac{dE'}{E'}$$

$$-\pi(E,t) \int_0^\delta \phi\left(\frac{E'}{E}\right) \frac{dE'}{E} + \int_E^{E+\delta} \pi(E',t)\, \phi\left(\frac{E'-E}{E'}\right) \frac{dE'}{E'} \qquad (8)$$

We can consider the last two integrals separately, making a change of variable in the first of $E'=\epsilon$, and in the second of $E'=E+\epsilon$ and combining the terms into one integral:

$$\int_0^\delta \left\{ \pi(E+\epsilon, t)\, \phi\left(\frac{\epsilon}{E+\epsilon}\right) \frac{1}{(E+\epsilon)} - \pi(E,t)\, \phi\left(\frac{\epsilon}{E}\right) \frac{1}{E} \right\} dE$$

For small ϵ

$$\pi(E+\epsilon, t) = \pi(E,t) + \epsilon \frac{\partial \pi}{\partial E} + \frac{1}{2} \epsilon^2 \frac{\partial^2 \pi}{\partial E^2} + \ldots$$

and

$$\phi\left(\frac{\epsilon}{E+\epsilon}\right) \approx b\left(\frac{E+\epsilon}{\epsilon} - 1\right) = b\,\frac{E}{\epsilon}$$

$$\phi\left(\frac{\epsilon}{E}\right) \approx b\left(\frac{E}{\epsilon} - 1\right) = b\,\frac{E-\epsilon}{\epsilon}$$

Dropping second order terms in ϵ, the integral becomes:

$$b\int_0^{\delta}\left\{\pi(E,t) + \epsilon\,\frac{\partial\pi(E,t)}{\partial E}\left(\frac{E}{\epsilon} - \pi(E,t)\left(\frac{E-\epsilon}{\epsilon}\right)\right\}\right. d\epsilon =$$

$$b\int_0^{\delta}\left\{\pi(E,t) + E\,\frac{\partial\pi(E,t)}{\partial E}\right\} d\epsilon \approx b\,\delta\left\{\pi(E,t) + E\,\frac{\partial\pi(E,t)}{\partial E}\right\} \qquad (9)$$

which tends to 0 in the limit as $\delta \to 0$ for $E < E_0$.

As a first approximation, the first two integrals in (8) can be evaluated by rectangular rule[6]:

$$y(x)\,dx \sim h\left(y(A) + y(A+h) + y(A+2h) + \ldots + y\,A + (n-1)\,h\right)$$

$$h = \frac{B-A}{n-1}$$

The value of h is determined by the value of dE in the mesh. Expression (9) is added to compensate for the error introduced by the finite size of δ(=dE). If rectangular rule is used, together with Euler's method (7), the method of solution is easy to visualize, but not very accurate, the error in rectangular rule integration being approximately:

$$\frac{h}{2}\,(B-A)\,\frac{dy(\xi)}{dx} \qquad A \leq \xi \leq B$$

With this method, one can consider the probability distribution as bined in E, with bins at a given value of t losing to bins of lower E values at t+dt, while gaining from higher bins at t - dt (Fig. 3). The first attempt at solving (1)

utilized this method and was moderately successful. The chief drawback of the method is that h must be made very small, i.e., a large number of points must be taken in E and t to minimize error.

In the numerical solution under discussion here, the first two integrals of (7) were evaluated by the trapezoidal rule:

$$\int_A^B y(x)\,dx \approx \frac{h}{2}\left(y(A) + 2y(A+h) + 2y(A+2h) + \ldots + y(A+(n-1)h)\right)$$

$$h = \frac{B-A}{n-1}$$

where n is the number of equally spaced sample points. The error in trapezoidal integration is:

$$-\frac{h^2}{12}(B-A)\frac{d^2y(\xi)}{dx^2} \qquad A \leq \xi \leq B$$

where ξ is some value of x between the limits of integration. The integrand in the gain integral is strongly peaked due to the nature of $\pi(E, t)$ near the upper limit, and the integrands of the gain and loss integrals are peaked near the lower limits due to the nature of $\phi(v)$. In the case of the lower limits, there is some tendency for the error to cancel since we are taking the difference of the two integrals. When E is near 0 fewer mesh points are available for computation of the loss integral, while when E is near E_0, fewer mesh points are available for computation of the gain integral. The net result of all of these effects can be seen in Fig. 4, where numerical results for the case:

$$\phi(v) = \frac{-b(1-v)^a}{\log(1-v)}$$

have been compared to the analytic solution obtained by Eyges. The effect of changing the mesh size in the E direction by a factor of two is shown in Fig. 4.

The expression (9) is added to the sum of the first two integrals in (8) to compensate for the finite lower limit of integration. The partial of π with respect to E is computed by a difference approximation:

$$\frac{\partial \pi(E,t)}{\partial E} = \frac{\pi(E+dE,t) - \pi(E,t)}{dE}$$

which shows an obvious bias towards higher values of E. This form is chosen because the partial originated in the Taylor expansion of $\pi(E,t)$ in the direction of increasing energy.

The Euler method for integration along lines of constant E has an error per step of[6]

$$\frac{(dt)^2}{2} \frac{\partial^2 \pi(E,\eta)}{\partial t^2} \qquad t < \eta < t + dt$$

and values of dt as low as .0001 are required in the present case, which of course means 1000 steps in t.

A considerable improvement in accuracy can be obtained by using one of the so-called predictor-corrector methods.[6] We first use:

$$\pi(E,t+dt) = \pi(E,t-dt) + 2(dt) \frac{\partial \pi(E,t)}{\partial t}$$

to predict a value of $\pi(E,t+dt)$, then use the result obtained for $\pi(E,t+dt)$ to correct by the expression:

$$\pi(E,t+dt) = \pi(E,t) + \frac{1}{2}(dt) \left(\frac{\partial \pi(E,t+dt)}{\partial t} + \frac{\partial \pi(E,t)}{\partial t} \right)$$

This method allows the use of a larger step size, dt, so that the number of energy points can be increased without increasing computation times drastically. The error for the first (predictor) is:

$$\frac{(dt)^3}{3} \frac{\partial^3 \pi(E,\eta)}{\partial t^3}$$

while the error for the second (corrector) is:

$$-\frac{(dt)^3}{12}\,\frac{\partial^3\pi(E,\eta)}{\partial t^3}$$

The difference between the corrected and predicted values is:

$$-\frac{5\,(dt)^3}{12}\,\frac{\partial^3\pi(E,\eta)}{\partial t^3}$$

If $\partial^3\pi/\partial t^3$ does not change sign in the interval, then the two values obtained are on opposite sides of the true value (for that step). Thus by accumulating differences between the predictor and the corrector, some feeling for the error may be obtained. Unfortunately, errors in the integrals over E give wrong values for $\partial\pi/\partial t$, so that this procedure is only useful in determining an optimum value of dt and cannot be used in estimating the error in the final result except perhaps in the region $E \sim E_0/2$ where there is a tendency for error in the integrals to cancel. The value of dt was .004 for the results reported here.

Since the evaluation of the integrals in Eq. (1) is the most difficult part of the solution, accuracy might be improved by using some higher order numerical technique. Trapezoidal rule was chosen because it is simple, not requiring either an even or odd number of sample points, and handles the triangular approximation to the delta function starting condition in a way which is clearly understood. As the solution progresses in t, the normalization over E remains close to unity. Higher order methods are often equivalent to the analytic integration of some interpolation polynomial, which may not behave properly in the peaked region $E \sim E_0$.

Our procedure then, is to compute values of $\pi(E, t)$ in column 2 of the mesh by using Euler's method after obtaining $\partial\pi/\partial t$ by applying the trapezoidal rule to the integrals in (1), using mesh points from column 1 as sample points. We

can start with any initial distribution, but for the purposes of this discussion, we start with the triangular distribution shown in Fig. 3. After mesh points in column 2 are computed, succeeding columns can be computed by the predictor-corrector method, since it requires two previous points.

IV. Numerical Results

A computer program was written from the foregoing considerations and run on the SLAC IBM 360/91 machine. Double precision was used to minimize roundoff error, since something like 100,000 integrations are performed. The loss integrals need only be computed once for each i value in the mesh, but the gain integrals must be recomputed for each ij point, and the number of sample points in the gain integrals is proportional to the number of energy points. Thus, the running time of the program is proportional to the product of the square of the number of points in energy and the number of points in thickness. When the complete screening approximation is used (3) the running time for the program is a little over 3 minutes on the IBM 360/91. When expressions like (4) and (5) are used as energy loss functions, the running time increases to 24 minutes due to the time required to compute the logarithms and exponentials.

The P_{ij} array which represents $\pi(E,t)$ in the program is a double precision array and would occupy 200,000 words of storage if dimensioned according to the total number of points in E and t. The maximum storage available on the IBM 360/91 is 75,000 words. For this reason, a solution is obtained for ten columns in t, and then re-started for another ten, the last column of a set becoming the first column of the next set.

The accuracy of the program was checked by using (5) as a loss function and comparing the results with (6). Figure 4 shows that the error is greatest

at values of E near 0 and near E_0 as might be expected from the considerations of the preceeding section. Other checks were made by varying dE and dt. Figure 4 also shows the effect of changing dE by a factor of two. The effects of changing dt by a factor of two are not large enough to be observed in the figure, being only .005% at $E/E_0 = .5$ and .04% at $E/E_0 = .9$. The results of comparison with (6) indicate that the solution obtained numerically is accurate to .02% over the range $E/E_0 = .15$ to $E/E_0 = .97$ and to .3% for E/E_0 up to .99 when all of the mesh points are calculated. However, the accuracy of the solution when (3) is used as a loss function may not be this good. The results obtained in this type of calculation are therefore, unreliable at values of E/E_0 greater than .95. If we believe that the solution obtained by applying the Mellin transformation to (2) is correct, then we know the solution approximately at $E \sim E_0$ since the ϕ functions all behave as $1/\nu$ in the low ν region which corresponds to $E \sim E_0$. The Eyges expression (6) with a=0 becomes equivalent to that of Mo-Tsai divided by $\Gamma(1+bt)$.

Thus, the numerical solution is most accurate in the middle energy region, while the various theories generally become equivalent in the region $E \sim E_0$, although (6) contains the factor $(1+a)^{bt}$ which has a 3 percent effect for a = .25 at t = .1. The effect of the ϕ function on the accuracy of the numerical results is known only for the case of (5) where values of a of -.25, 0, and .25 were used in comparison runs between analytic and numerical results. The behavior of the solution is the same as that shown in Fig. 4 for high and low E values, but the flat mid regions in E are somewhat different. At $E/E_0 = .5$, the error curve is about .002% below the axis for the case of a = 0. When a = .25, the error at $E/E_0 = .5$ becomes -.006%, while for a = -.25, it becomes +.02%.

One obvious way of improving the accuracy of the numerical solution is to use a finer mesh, i.e., use more points in the integrations. However, computation times increase nearly as the square of the number of energy points used. In order to compute the value of π at given values of E and t, the values of π at mesh points of higher E, but lower t, must be known. Thus the solution at a particular mesh point depends only on the points contained in the rectangular region of higher E and lower t. Points of lower energy do not contribute to the solution. This feature can be used to find π numerically at E values close to E_0. The same 400 point mesh is used, only with energy E on the lowest line instead of zero. The loss integrals are evaluated for each E value once at the beginning of the calculation and stored. In this way, numerical results were obtained at values of E/E_0 up to .999. Again, the accuracy of the numerical solution was checked by comparison with (6) when (5) was used as a loss function. At $E/E_0 = .999$, .05 percent agreement was obtained for $a = 0$.

Figure 5 shows a comparison between the numerical results for complete screening and the expression for straggling obtained by Eyges (6) for various values of the parameter a, and for $t = 0.1$ radiation length. The Bethe-Heitler formula (a=0) can be seen to deviate from the numerical results for complete screening up to 10% down to values of $E/E_0 = 0.3$, below which the error becomes greater. The formula (6) for $\pi(E,t)$ is too large at high values of E/E_0 due to overestimation of the gain integral in the low v region and is too small at low values because of underestimation of the gain integral in the high v region. The relative error in the loss integral is the same in both cases, as the absolute value of the loss integral is too large due to overestimation of ϕ in the low v region.

Note that vertical scale in Fig. 5 is a factor of twenty greater than that of Fig. 4. The deviations from unity in Fig. 5 show how the formulas of Bethe-Heitler and Eyges differ from the numerical results when the complete screening approximation (3) is used as a loss function in the numerical computation, and are due to the ϕ functions used in the theories, not errors in the numerical method.

The expression of Mo-Tsai[4] for complete screening modified by dividing by $\Gamma(1+bt)$:

$$\pi(E_0, E, t) = \frac{t\,\phi\left(1 - \frac{E}{E_0}\right)\left(\log\frac{E_0}{E}\right)^{bt}}{E_0\,\Gamma(1+bt)} \qquad (10)$$

has been compared to the numerical results for complete screening (Fig. 6). The error is proportional to t, vanishing as $t \to 0$, and is most severe at low values of E. For values of $E/E_0 > .65$, this modified expression of Mo-Tsai agrees with the numerical results within 1%, but the error grows to 9% at $E/E_0 = .1$.

G. Miller[7] has used the numerical results to find a correction to (10) for values of Z of 1, 13, and 29 and values of t up to 0.1 and E/E_0 from .1 to 1. His results are:

$$\pi(E_0, E, t) = \frac{t\,\phi\left(1 - \frac{E}{E_0}\right)\log\left(\frac{E_0}{E}\right)^{bt}}{E_0\,\Gamma(1+bt)}\left(1 + bt\;x(.53875 + x(-2.1938 + .9634x))\right) \qquad (11)$$

$$x = \frac{E_0 - E}{E_0}$$

As shown in Fig. 7, this expression agrees with the numerical results to within .6% over the ranges of E/E_0 from .1 to 1.0 for E_0 ranging from 1 to 20 GeV and agrees with the analytic solution of Eyges for $a=0$ in the limit as $E/E_0 \rightarrow 1$. Note that the correction factor becomes unity when $t=0$ and when $E=E_0$.

Tsai[8] has proposed a new expression based on numerical results using (10).

$$\pi(E_0, E, t) = \frac{t\,\phi\left(1 - \frac{E}{E_0}\right)\left(\frac{E_0 - E}{E_0}\right)^{bt}}{\Gamma(1\,(bt)} \qquad (12)$$

Again, this expression agrees with (6) in the limit as $E \rightarrow E_0$ for $a=0$. Comparison with numerical results is shown in Fig. (6). The difference is less than 2% for values of E/E_0 between .2 and .999. The large deviation below $E/E_0 = .2$ is due to the fact that the exponential term in (12) is no longer a good approximation to the log term in (10). One nice feature of (12) is that it can be integrated analytically over E for the complete screening case. The results of integrating (12) from $E=0$ to $E=E_0$ are that it is not normalized, the value obtained being .99 at $t=0.1$ rather than 1. This is consistent with Fig. 6 which shows that (12) is too low.

$$W(E_0, E, t) = \int_E^{E_0} \pi(E_0, E, t)_{Tsai}\, dE = \frac{t}{\Gamma(1+bt)}\left[b\,\frac{v^{bt}}{bt} - \frac{v^{bt+1}}{(bt+1)}\right) + \frac{v^{bt+2}}{(bt+2)}\Bigg]$$

where

$$v = 1 - \frac{E}{E_0}\Big) \quad .$$

The fact that most of the distribution is contained in the region $E \approx E_0$ may account for the fact that the total distribution is only one percent low, while the segment shown in Fig. 6 is generally two percent low.

The disagreement between the numerical results and expressions (10), (11), and (12) at $E/E_0 \approx 1.0$ and t=.1 radiation length is .6%. One would expect that these expressions would agree with numerical results within the known error of the program (.05%) at this energy, because (10) becomes identical to the Bethe-Heitler expression in the limit as $E \sim E_0$, and this is a known solution to the diffusion equation. However, Eyges' expression (6) is also an analytic solution and clearly depends on the shape of the loss function when $E \sim E_0$ (Fig. 5) because of the factor $(1+a)^{bt}$. Since all of the loss functions behave as 1/v in the low v region, it is tempting to assume that the probability distribution $\pi(E_0, E, t)$ at $E \sim E_0$ depends on small v terms and that the nature of $\phi(v)$ at high values of v is relatively unimportant. Equation (6) and the numerical results show that this is true only for very small values of t like t=.01.

V. Conclusion

The results shown in Figs. 5 and 6 indicate that the expressions given by Mo-Tsai[4] and Tsai[8] are useful approximations to the probability distribution describing electron straggling at radiator thicknesses up to 0.1 radiation lengths. The numerical results indicate that these expressions are approximations only, however, and do not represent true solutions to the electron diffusion equation. G. Miller[7] has given a correction for the Mo-Tsai expression which brings it into agreement with the numerical results to within .6% at thicknesses of 0.1 radiation length. The disagreement between all of the above expressions for electron straggling and the numerical results is proportional to the thickness of material traversed, so that at thicknesses of .01 radiation lengths, the maximum disagreement is less than one percent.

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LIST OF FIGURES

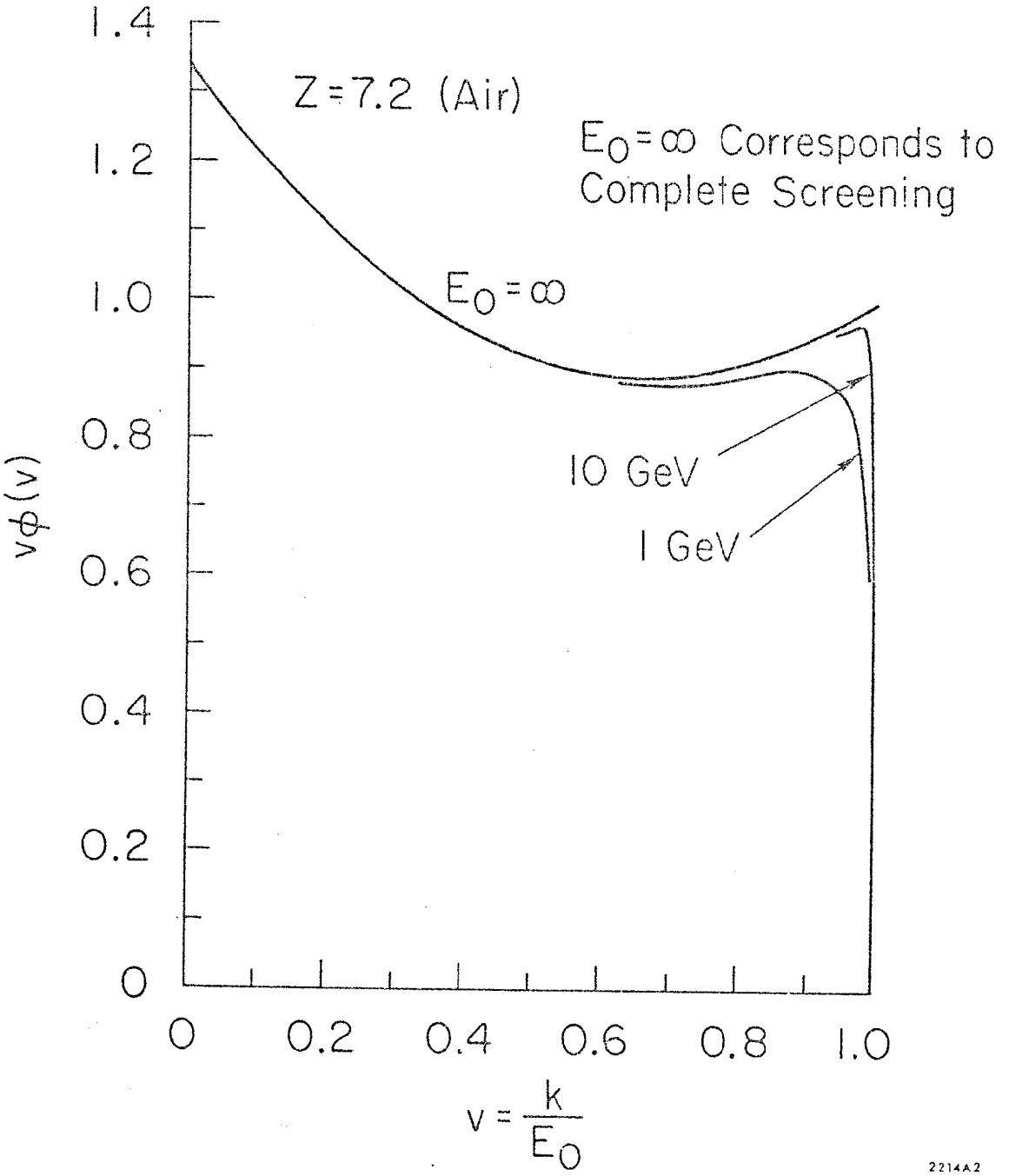
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
vφ(v)
Z=7.2 (Air)
E0=∞ Corresponds to
Complete Screening
E0=∞
10 GeV
1 GeV
0
0.2
0.4
0.6
0.8
1.0
v = k/E0
2214A2

Fig. 1

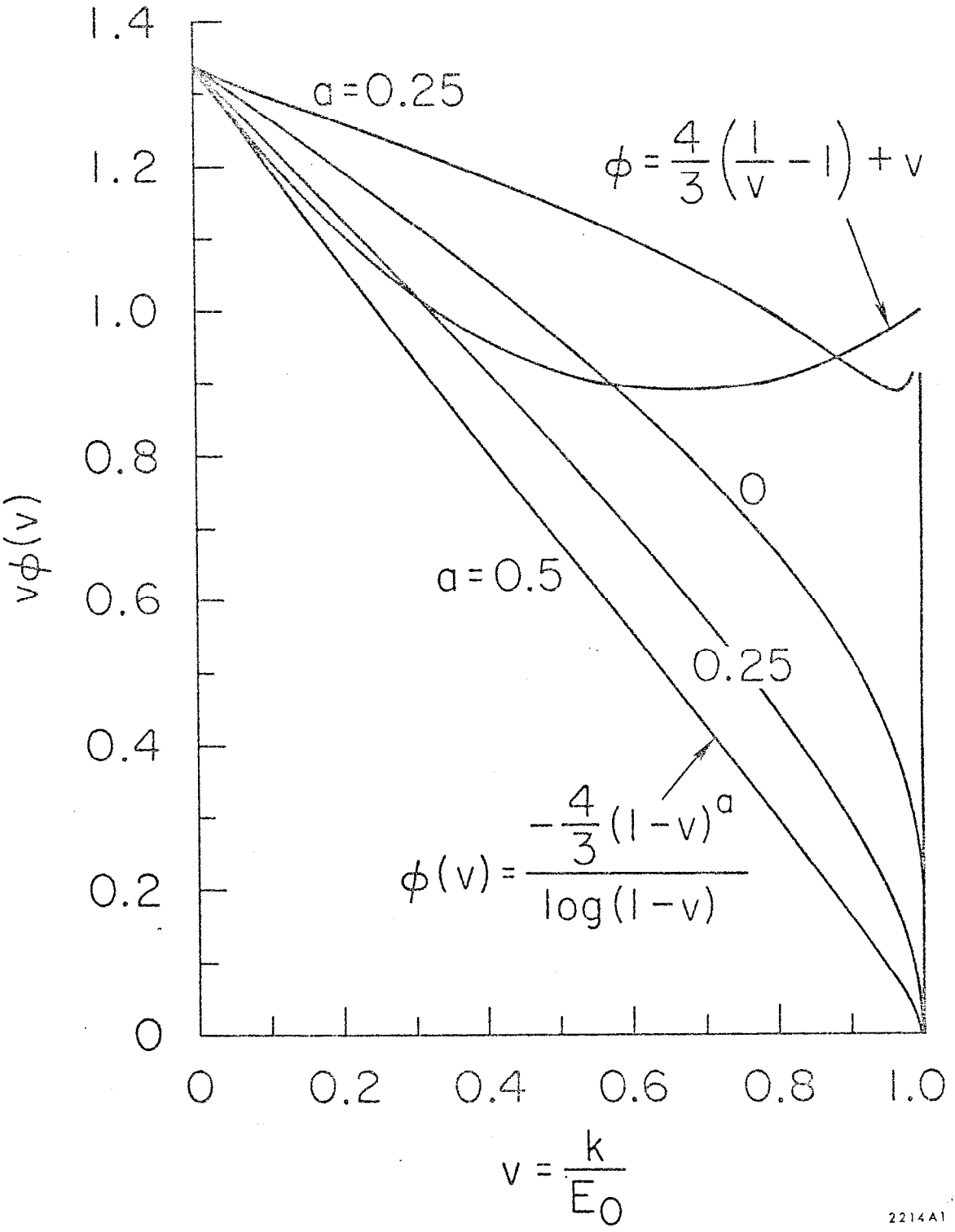
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1.0
vφ(v)
v = k/E0
a=0.25
φ = 4/3 (1/v − 1) + v
0
a=0.5
0.25
φ(v) = −4/3 (1−v)^a / log(1−v)
2214A1

Fig. 2

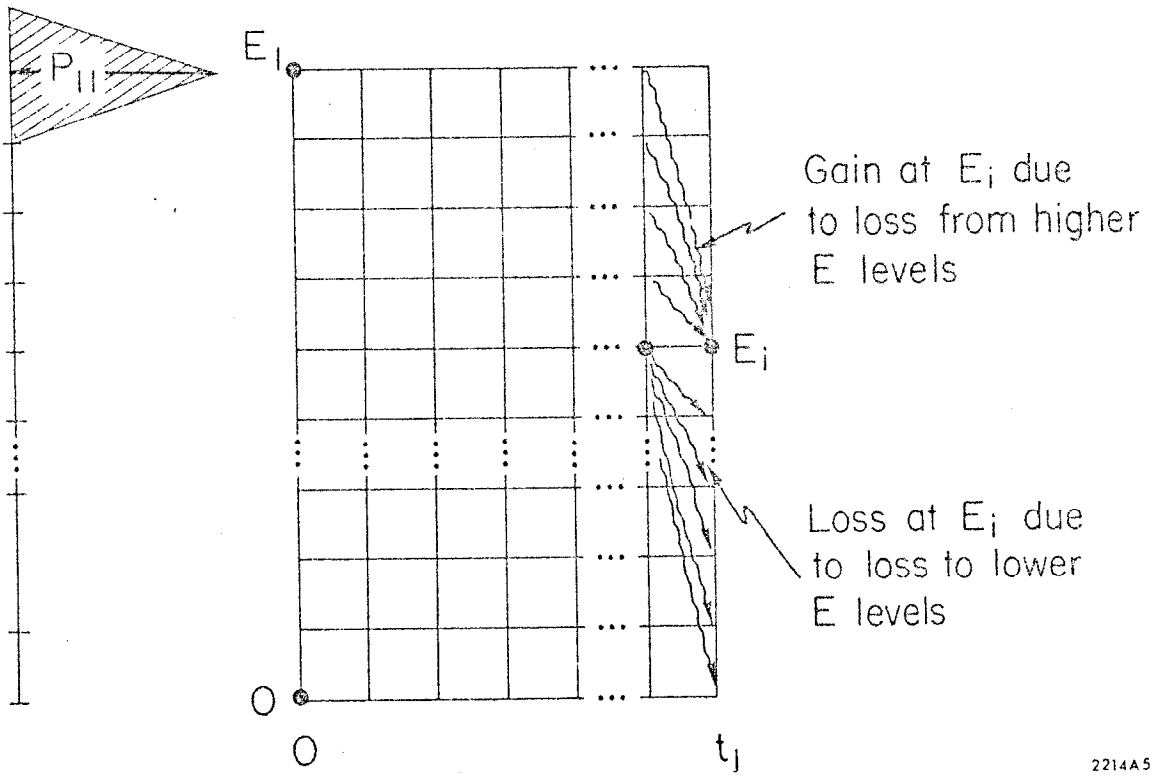
P_{II}
E_1
E_i
0
0
t_j
Gain at E_i due to loss from higher E levels
Loss at E_i due to loss to lower E levels
2214A5

Fig. 3

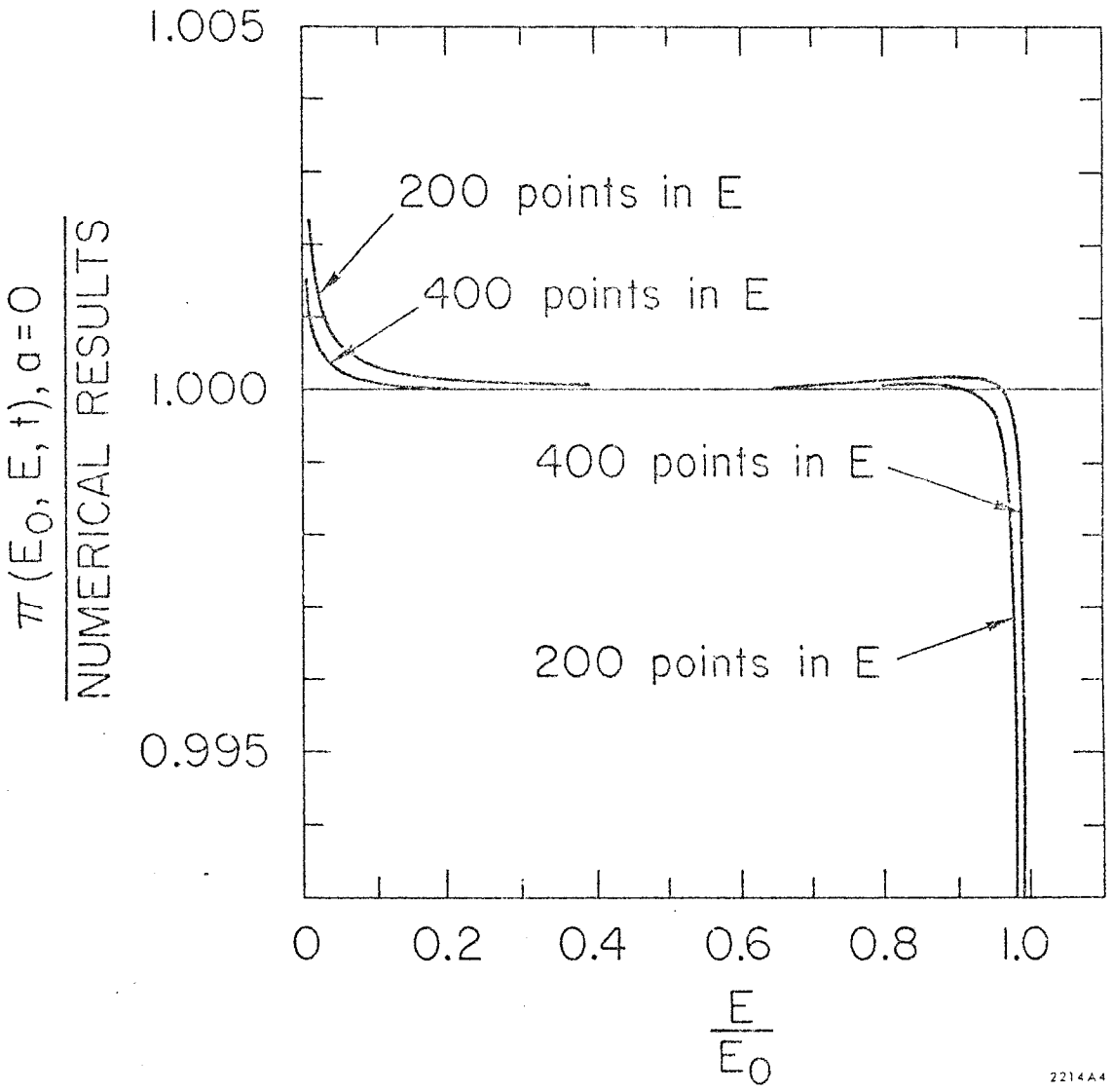
1.005
1.000
0.995
200 points in E
400 points in E
400 points in E
200 points in E
0
0.2
0.4
0.6
0.8
1.0
E/E0
π(E0, E, t), a=0 / NUMERICAL RESULTS
2214A4

Fig. 4

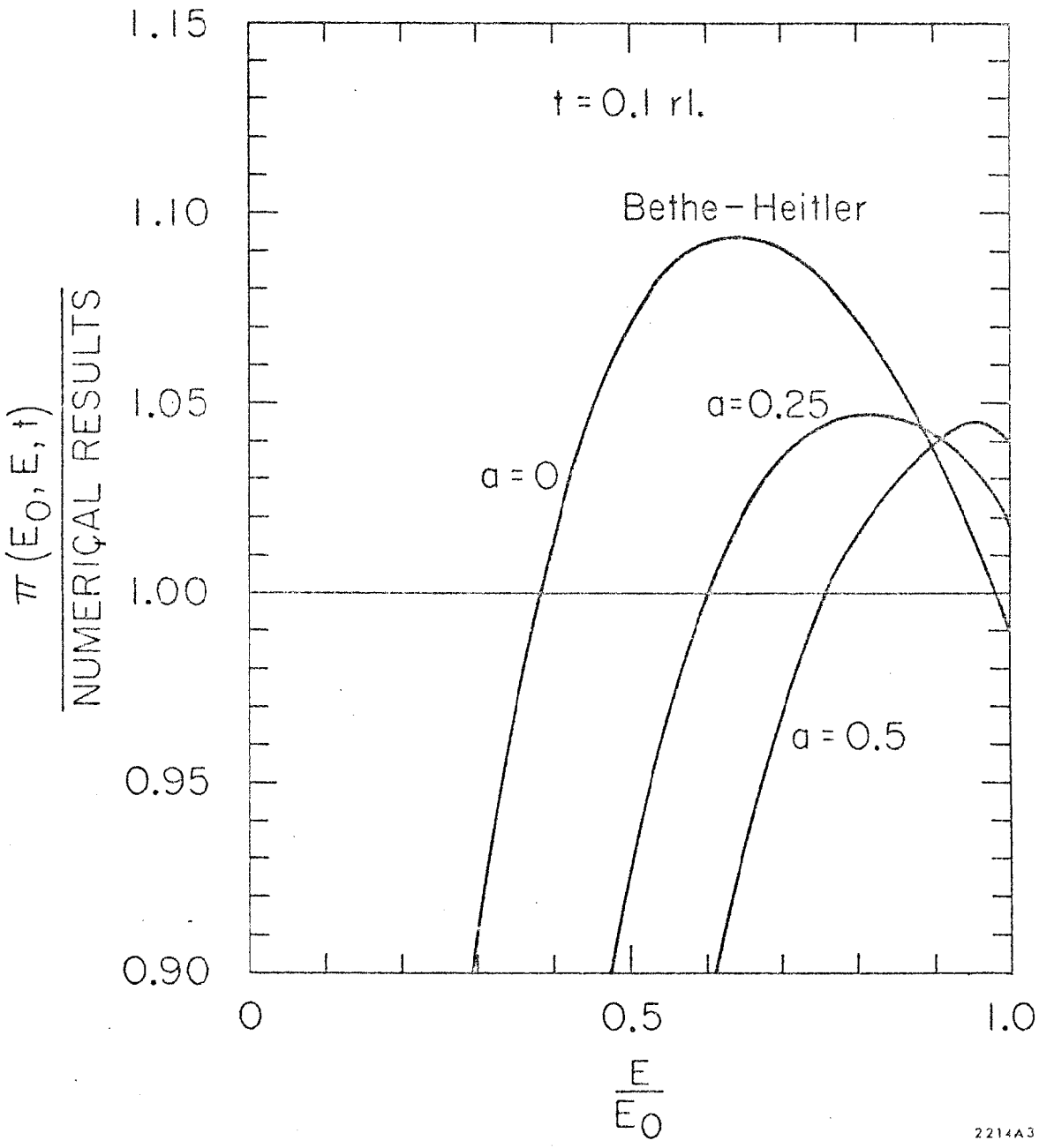
1.15
1.10
1.05
1.00
0.95
0.90
0
0.5
1.0
t = 0.1 rl.
Bethe–Heitler
a = 0
a = 0.25
a = 0.5
π (E0, E, t) / NUMERICAL RESULTS
E / E0
2214A3

Fig. 5

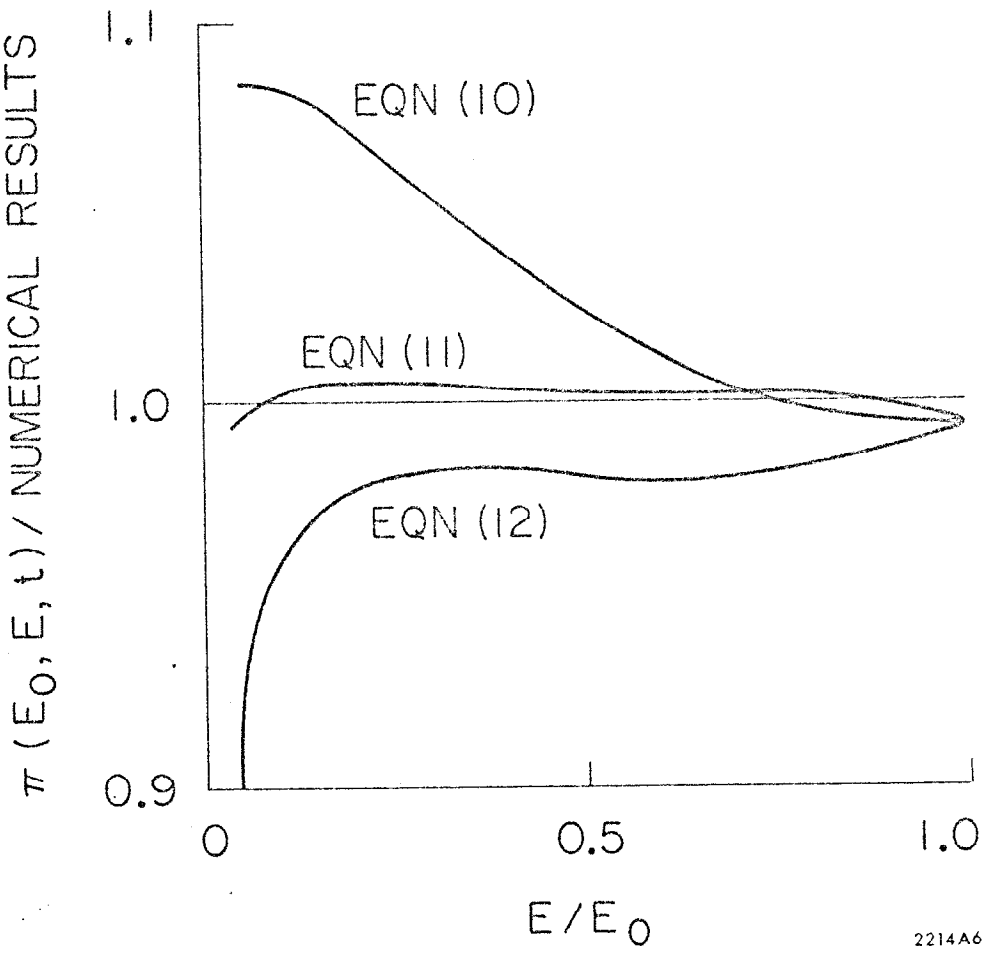
1.1
1.0
0.9
0
0.5
1.0
EQN (10)
EQN (11)
EQN (12)
π (E0, E, t) / NUMERICAL RESULTS
E/E0
2214A6

Fig. 6