SLAC-PUB-1127
(TH) and (EXP)
October 1972

BOUNDS ON POMERON AMPLITUDES AND THE POSITIONS OF THEIR ZEROES*

Yoram Avni† and Robert Savit

Stanford Linear Accelerator Center
Stanford University, Stanford California 94305

ABSTRACT

Rigorous lower bounds on the imaginary part of nonflip elastic amplitudes are derived under the assumption that their impact parameter representations are monotonically decreasing functions of b (central profiles). One result of these bounds is that at intermediate to high energies, typical pomeron dominated elastic nonflip amplitudes must not have zeroes for $|t| \lesssim 0.6$ BeV^2. A connection between the phenomenological absence of zeroes in these pomeron amplitudes at small $|t|$ and the conjectured central behavior of the profile is thus established.

(Submitted for publication.)

*Work supported by the U.S. Atomic Energy Commission

†Fulbright Scholar; on leave from the Weizmann Institute of Science, Rehovot, Israel

I. Introduction

Ever since the geometrical-absorptive picture was first applied to high-energy two-body scattering processes,[1] the impact parameter profile of elastic nonflip amplitudes has been viewed as a monotonically decreasing function in impact parameter (b) space. This profile is often referred to as a "central" profile. Such forms have been used extensively to calculate the effects of absorption on various "basic" exchanges, as well as to phenomenologically describe elastic amplitudes at high energies.

One of the more successful phenomenological models which uses geometrical language, and which has been applied with considerable success to two body processes at intermediate and high energies, is the dual absorptive model.[2] This model utilizes two-component duality. The diffractive-or pomeron exchange-part of any given amplitude is assumed to be central in b-space and to have no zeroes in t-space for $|t| \leq 1$ BeV^2. This last assumption is of a crucial importance in the phenomenological analyses to which the model has been applied, where it has become a common practice to parameterize pomeron exchange amplitudes by simple exponentials, e^{St}.

It has already been shown[3] that the two above assumptions—"centrality" in b and "no zeroes" in t—are in some sense independent. Slight modifications of the profile can effortlessly generate or eliminate zeroes in the amplitude. In fact, it is quite possible that the structure observed in $pp \to pp$ at $|t| \simeq 1.2$ BeV^2 (4) is caused by a zero in a pomeron exchange amplitude. It is, however, a remarkable phenomenological observation that pomeron exchange amplitudes do not exhibit zeroes or dips at values of $|t|$ smaller than ~ 0.8 BeV^2. We face therefore the interesting problem: what exactly is the relation between the property of "centrality" in b, and the location of the zeroes in t? A precise answer to this question will, of course, provide tests for monotonicity of the pomeron exchange profile. Some interest was recently added to this problem by conjectures that pomeron profiles may possess a peripheral component[5].

In this work we derive a lower bound on the imaginary part, $A(t)$, of helicity conserving elastic amplitudes which have monotonically decreasing profiles. For a given $A(t=0)$, which defines the strength of the interaction, and which is related to the total cross section, σ_T, by the optical theorem; and for a given slope $S = A'(t=0)/A(t=0)$, which defines the range of the interaction, we prove that for amplitudes which have monotonically decreasing imaginary parts in b-space,

$$A(t) \geq A(t=0)\frac{2J_1(R\sqrt{-t})}{R\sqrt{-t}} \qquad (1)$$

for values of t such that $R\sqrt{-t} \leq z \simeq 3.83$, the first zero of J_1, where J_n is the Bessel function of order n. R is defined by $R^2 = 8S$, and is the radius of a uniformly grey sphere which generates an amplitude with a slope S.

In other words, for realistic values of σ_T and S, the smallest A(t) (for the t range given above) which can be generated by a central profile, is generated by a grey sphere of constant opacity.

An immediate result is that $\bar{t}$, the position of the first zero of A(t), (if zeroes exist at all) is constrained by $R\sqrt{-\bar{t}} \geq z$. At intermediate energies, where the slope is 3-4 BeV^{-2}, corresponding to a radius of 1-1.2 fermi, we find $|\bar{t}| \geq 0.5-0.7\ BeV^2$. The absence of zeroes for $|t| < 0.5-0.7\ BeV^2$ is therefore a simple consequence of monotonicity.

We have proven our theorem by using the technique of Lagrange multipliers[6]. This method can be used to find bounds only if the lower (upper) bound is obtained as a minimum (maximum) in the space of profiles considered, which is not necessarily true in general, and must be checked in every specific calculation. This point has been overlooked in many previous calculations of bounds by the above technique, where it has been implicitly assumed that the bound is obtained in the space. Though in some calculations this assumption is a posteriori obviously justified, no formal proofs were given. We have overcome this difficulty by a technique which we describe in the following section, and which may be useful to formally complete the calculations of other bounds as well.

In section II of this paper we state and prove the mathematical theorem which gives the lower bound. In section III we discuss the constraints and phenomenological implications, and suggest future directions of inquiry.

II. Derivation of the Bounds

In this section we wish to present the two related theorems, theorems 1 and 2 below, which are the central results of this paper. Since the proofs of these theorems involve virtually identical arguments, we shall state and prove theorem 1 in detail, and merely state theorem 2. We would like, however, to eliminate the unpleasant possibility discussed in the introduction, namely, that the lower bound is not obtained as a minimum in the space we are considering. To prove that this does not happen in our problem, we proceed as follows: we first prove theorem 0 below, which is identical to theorem 1, except that the profiles are defined on a finite interval of b, $[0, x]$. In this case the bound is necessarily a minimum in the space considered and so the lowest local minimum is the lower bound. Using theorem 0, we then proceed to prove theorem 1.

In the spirit of geometrical models, we shall work with profiles which are functions of the continuous impact parameter, b. However, the proof can easily be translated into the more exact language of partial wave amplitudes labelled by the discontinuous parameter, ℓ. The discreet reader who fears that something may be lost in translation, may consult reference (7), where both treatments of a similar problem are discussed in detail.

In the impact parameter notation, our problem is to minimize the imaginary part of a helicity nonflip scattering amplitude

$$A(t) = \int_0^\infty 2b\, a(b)\, J_0\,(b\sqrt{-t})\, db, \quad t \leq 0$$

given the total cross section,

$$L \equiv \frac{\sigma_T}{2\pi} = A(0) = \int_0^\infty 2b\, a(b)\, db$$

and the value of the imaginary part of the amplitude at a point outside the physical region,

$$G \equiv A(t_1) = \int_0^\infty 2b\, a(b)\, I_0\,(b\sqrt{t_1})\, db\,, \quad 0 < t_1 \leq t_0$$

where t_0 is the value of the first continuum singularity in the t channel, I_n is the modified Bessel function of order n, and a(b) is the imaginary part of the profile

$$a(b) \equiv \text{Im}\, f(b) = \text{Im}\left[e^{i\delta(b)} \sin\delta(b)\right].$$

In addition to these constraints, we require that unitarity be obeyed,[8]

$$a(b) \geq a^2(b)$$

and we implement the central behavior of the pomeron by requiring

$$\dot{a}(b) \equiv \frac{d\,a(b)}{db} \leq 0.$$

Theorem 1 may then be stated as:

<u>Theorem 1:</u> If at some energy, values for σ_T and G are given, and if unitarity is obeyed and a(b) is a non-increasing function of b, then A(t), the absorptive part of the helicity nonflip scattering amplitude is bounded from below by

$$A(t) \geq 2cR^2\, \frac{J_1(R\sqrt{-t})}{R\sqrt{-t}}$$

in the t-range $-z^2 \leq tR^2 \leq 0$, where c and R are determined by

$$\frac{\sigma_T}{2\pi} = cR^2 \quad \text{and} \quad G = 2cR^2\, \frac{I_1(R\sqrt{t_1})}{R\sqrt{t_1}}$$

and z is the first zero of J_1, $z \simeq 3.83$.

To prove this, we first need to prove

Theorem 0: Let $\widetilde{H}(x, \widetilde{L}, \widetilde{G})$ be the space of functions, $\widetilde{a}(b)$, defined on $0 \le b \le x$, where $x > \frac{z}{\sqrt{-t}}$, such that

$$\text{(i)}\ 0 \le \widetilde{a}(b) \le 1$$

$$\text{(ii)}\ \dot{\widetilde{a}}(b) \equiv \frac{d\widetilde{a}}{db} \le 0$$

$$\text{(iii)}\ \widetilde{L} = \int_0^x 2b\, \widetilde{a}(b)db, \text{ and}$$

$$\text{(iv)}\ \widetilde{G} = \int_0^x 2b\ I_0(b\sqrt{t_1})\, \widetilde{a}(b)db.$$

Then, for $\widetilde{a}(b) \in \widetilde{H}$,

$$\widetilde{A}(t) \equiv \int_0^x 2b\ J_0(b\sqrt{-t})\, \widetilde{a}(b)db \ge 2\widetilde{c}\widetilde{R}^2 \frac{J_1(\widetilde{R}\sqrt{-t})}{\widetilde{R}\sqrt{-t}}$$

where R and $\widetilde{c}$ are determined by $\widetilde{L} = cR^2$ and $\widetilde{G} = 2c\widetilde{R}^2 \frac{I_1(\widetilde{R}\sqrt{t_1})}{\widetilde{R}\sqrt{t_1}}$ for the t-range $0 \le \widetilde{R}\sqrt{-t} \le z$.

Proof of theorem 0:

The proof involves four steps. First, we show that a lower bound exists, and we derive the necessary equations which describe the profile $\widetilde{a}(b)$ that minimizes $\widetilde{A}$, assuming it is piecewise differentiable. Next we present a function, $\widetilde{a}(b)$ which is a candidate for the correct solution. Third, after proving a lemma, we show that there is no other profile that satisfies the necessary conditions. Finally, we demonstrate that our candidate is consistent with the variational equations, thus proving that a piecewise differentiable minimizing function does exist.

To begin, we note that a lower bound to $\widetilde{A}$ certainly exists, because $|\widetilde{A}| \le \widetilde{L}$. We have explicitly proven that the lower bound is obtained as a minimum in $\widetilde{H}$, i.e., that there exists a profile $\widetilde{a}(b)$ in $\widetilde{H}$ such that the amplitude $\widetilde{A}(t)$ corresponding to it is equal to the lower bound. We will not present this proof in this paper, since it is based on involved and purely mathematical considerations. We only wish to remark that the finiteness of x plays an essential role in the proof; when $x = \infty$, there is no a priori reason why the bound should be a minimum in the space.

Now we wish to derive the variational equations for the lower bound. To do this, we use the Lagrange multiplier technique with inequality constraints.[6, 7] The auxilliary functional for our problem is

$$\mathscr{L} = -\int_0^x 2b\,\tilde{a}(b)\,J_0(b\sqrt{-t})\,db + \alpha\left[\tilde{L} - \int_0^x 2b\,\tilde{a}(b)\,db\right] \tag{2}$$

$$+\beta\left[\tilde{G} - \int_0^x 2b\,\tilde{a}(b)\,I_0(b\sqrt{t_1})\,db\right] + \int_0^x 2b\,\lambda(b)\left[\tilde{a}(b) - \tilde{a}^2(b)\right]db$$

$$-\int_0^x 2\,\phi(b)\,\dot{\tilde{a}}(b)\,db.$$

α and β are equality multipliers, and $\lambda(b)$ and $\phi(b)$ are non-negative inequality multipliers. The generalized coordinates of the variational problem are $\tilde{a}(b)$. In writing the above functional we have made the extremely mild assumption that the minimizing profile is piecewise differentiable. This assumption is sufficient in order to implement the monotonicity constraint into the Lagrange functional by using $\dot{\tilde{a}}(b)$. Applying the Euler-Lagrange equations to the functional $\mathscr{L}$, gives

$$-J_0(b\sqrt{-t}) - \alpha - \beta I_0(b\sqrt{t_1}) + \lambda(b)\,(1 - 2\tilde{a}(b)) + \frac{\dot{\phi}(b)}{b} = 0\,. \tag{3}$$

where $\dot{\phi}(b) \equiv \frac{d\,\phi(b)}{db}$.

As usual, it is convenient to define three sets for b, depending on the value of $\tilde{a}(b)$. If $\tilde{a}(b) = 1$, $b \in B_1$ and equation (3) becomes:

$$b \in B_1:\ \lambda(b) = \frac{\dot{\phi}(b)}{b} - J_0(b\sqrt{-t}) - \alpha - \beta I_0(b\sqrt{t_1}) \geq 0 \tag{4}$$

For $0 < \tilde{a}(b) < 1$, $b \in I$, $\lambda(b) = 0$ and we have

$$b \in I:\ \frac{\dot{\phi}(b)}{b} = J_0(b\sqrt{-t}) + \alpha + \beta I_0(b\sqrt{t_1}) \tag{5}$$

Finally, when $\tilde{a}(b) = 0$, $b \in B_0$, and then

$$b \in B_0 : \ \lambda(b) = \frac{-\dot{\phi}(b)}{b} + J_0(b\sqrt{-t}) + \alpha + \beta I_0(b\sqrt{t_1}) \geq 0 \tag{6}$$

The most general behavior $\tilde{a}(b)$ can have is regions where $\dot{\tilde{a}}(b) = 0$ alternating with regions where $\dot{\tilde{a}}(b) < 0$. Suppose $b \in I$. If $\dot{\tilde{a}}(b) < 0$, then $\phi(b) = 0$. Equation (5) indicates that for $b \in I, \dot{\phi}(b)$ cannot be zero except at isolated points since α and β do not depend on b. Therefore $\phi(b)$ cannot be zero except at isolated points and the most general behavior $\tilde{a}(b)$ can have for $b \in I$ is a series of flat plateaus with sharp edges. If $b \in B_1$, then $\tilde{a}(b) = 1$, and as soon as $\tilde{a}(b)$ starts to decrease, $b \in I$. Hence the transition from B_1 to I must also be a sharp drop off. Similarly, it is easy to see that a region decreasing from I to B_0 must be a sharp edge.

Let us now discuss the positions of these fall off points. First, consider a point connecting a plateau in I to another plateau in I. As shown by equation (5), $\phi(b)$ is continuous and infinitely differentiable for $b \in I$. Since $\phi(b) \geq 0$ everywhere, and since $\phi(b) = 0$ at the drop off points in I, it follows that $\dot{\phi}(b) = 0$ and $\ddot{\phi}(b) \geq 0$ at these points. Using expression (5), we can write these two conditions as

$$J_0(D\sqrt{-t}) + \alpha + \beta I_0(D\sqrt{t_1}) = 0 \tag{7}$$

$$-\sqrt{-t}\, J_1(D\sqrt{-t}) + \beta\sqrt{t_1}\, I_1(D\sqrt{t_1}) \geq 0 \tag{8}$$

where $b = D$ is a drop off point.

Now consider a transition point from B_1 to I. Approaching this point from the right, we have the explicit expression (5) for $\dot{\phi}(b)$. Since $\phi(b) = 0$ at this point, and since $\phi(b) \geq 0$ everywhere, the right-hand side of (5) must be non-negative here. Since $\tilde{a}(b)$ is identically equal to 1 in B_1, $\phi(b)$ may clearly be chosen to be differentiable in B_1. Expression (4) tells us that approaching the drop off point from the left, $\dot{\phi}(b)$ is greater than or equal to the right-hand side of (5). But, since $\phi(b) = 0$ at this point and is everywhere non-negative, we must have $\dot{\phi}(b) \leq 0$, approaching this fall-off point from the left. The only way to make all these statements consistent is to have $\dot{\phi}(b) = 0$ whether we approach from the left or the right. Since $\phi(b) = \dot{\phi}(b) = 0$ here, and since $\phi(b) \geq 0$ everywhere, it follows that $\ddot{\phi}(b) \geq 0$ when we approach from the left or right. (Note, however, that $\ddot{\phi}(b)$ may be discontinuous at this point.) Therefore, equations (7) and (8) apply to this transitionpoint also, with the understanding that (8) is obtainedby evaluating $\ddot{\phi}(b)$ as a limit from the right. One might have thought that $\phi(b)$ could have been discontinuous at the transition point from B_1 to I, thus allowing this point to exist without satisfying expresssions (7) and (8). That this is not the case can be intuitively understood by remembering that the drop-off point is a transition point when approached either from B_1 or I. Furthermore, expressions (7) and (8) can be obtained by taking the limit of the difference equations which govern the transition from B_1 to I when the problem is treated in the formalism of discrete partial wave amplitudes. These arguments are, of course, closely connected with the assumption that $\tilde{a}(b)$ is piecewise differentiable.

It is easy to see that similar arguments can be used in discussing the transition from I to B_0. Expressions (7) and (8) apply here too, where now $\ddot{\phi}(b)$ has been evaluated from the left.

Suppose now that there exists a plateau region in I, which begins at $b = D_1 \neq 0$ and ends at $b = D_2 \neq x$. Equations (7) and (8) must, of course, be valid at D_1 and D_2, but we also need $\phi(D_1) = \phi(D_2) = 0$. Integrating equation (5), this condition can be written as

$$\frac{b}{\sqrt{-t}} \; J_1(b\sqrt{-t}) + 1/2 \; \alpha \, b^2 + \beta \frac{b}{\sqrt{t_1}} \; I_1 \, (b\sqrt{t_1}) \Big|_{D_1}^{D_2} = 0 \tag{9}$$

We now notice that we can solve for α and β by using (7) evaluated at the points D_1 and D_2. These values can be inserted into (9), and we have a condition that must be fulfilled by the end points of a plateau. A remarkable feature of this condition is that it does not depend on the values of the constraints $\widetilde{L}$ and $\widetilde{G}$. Notice also that since there is no a priori reason to require $\phi(0)=0$ or $\phi(x)=0$, equation (9) must be modified by adding $-\phi(0)$ to the right hand side if $D_1=0$ and $\phi(x)$ if $D_2=x$.

Expressions (7), (8), and (9) govern the allowed positions of fall-off points for our minimization problem. Whether in general for a given value of -t and t_1 they uniquely determine one set of transition points is not clear. However, the fact that they are conditions which it is necessary to satisfy is enough for our purposes.

We seek, therefore, a solution which is a series of plateaus obeying the conditions (7), (8), and (9). The simplest profile for $\widetilde{a}(b)$ to assume is a single plateau of height $\widetilde{c}$ $(0\leq\widetilde{c}\leq1)$ falling to zero at a radius $\widetilde{R}$. It can easily be shown that if the given values of $\widetilde{L}$ and $\widetilde{G}$ can be fit by a profile $\widetilde{a}(b)$ which satisfies $0\leq\widetilde{a}(b)\leq1$ and $\dot{\widetilde{a}}(b)\leq0$, then there exist values for $\widetilde{R}$ and $0\leq\widetilde{c}\leq1$ such that a single plateau of height $\widetilde{c}$ and radius $\acute{R}$ will also fit $\widetilde{L}$ and $\widetilde{G}$. Physically this is clearly the case. This profile results in the expressions for $\widetilde{L}$, $\widetilde{G}$, and the lower bound given in the statement of theorem 0, above. A solution (if it exists) with n plateaus of heights $\{\widetilde{c}_i\}$ and radii $\{\widetilde{R}_i\}$ gives for the lower bound to A

$$\widetilde{A}\geq\sum_{i=1}^{n} 2\widetilde{d}_i\,\widetilde{R}_i^2\;\frac{J_1(\widetilde{R}_i\sqrt{-t})}{\widetilde{R}_i\sqrt{-t}} \qquad (10)$$

where $\widetilde{d}_i=\widetilde{c}_i-\widetilde{c}_{i+1}$, and $\widetilde{c}_{n+1}=0$.

Since the value of $\widetilde{A}$ at $t=0$ is fixed by the constraint $\widetilde{L}$, it can be seen from (10), that any solution with more than one plateau which falls below the one plateau solution for $0\leq\widetilde{R}\sqrt{-t}\leq z$ must necessarily have some members of the set $\{\widetilde{R}_i\}$ be greater than $\widetilde{R}$, the radius of the one plateau solution.

We shall now prove that for $0\leq\widetilde{R}\sqrt{-t}\leq z$ there are no n plateau solutions $(n>1)$ which satisfy the variational conditions. There are therefore no solutions of the type described in the last paragraph, and so the one plateau solution provides the only local mimimum in this region of t. To show this, we need to use the following lemma.

<u>lemma</u>: Suppose we can fit the given values of the constraints $\widetilde{L}$ and $\widetilde{G}$ with a one plateau profile function of height $\widetilde{c}(0 \le \widetilde{c} \le 1)$ and radius $\widetilde{R}$. Any other profile which consists of n flat plateaus (n>1) with sharp edges must have at least one fall-off point at some $b<\widetilde{R}$ in order to fit the same value of the ratio of the constraints, $\widetilde{G}/\widetilde{L}$.

<u>Proof</u>: Define $\widetilde{S} = \widetilde{G}/\widetilde{L}$. For a solution with n non-zero plateaus (n > 1), we have:

$$\widetilde{S} = \widetilde{S}_n \equiv \frac{\sum_{i=1}^{n} 2\widetilde{d}_i \widetilde{R}_i^2 \frac{I_1(\widetilde{R}_i\sqrt{t_1})}{\widetilde{R}_i\sqrt{t_1}}}{\sum_{i=1}^{n} \widetilde{d}_i \widetilde{R}_i^2} > \frac{2 I_1(\widetilde{R}_1\sqrt{t_1})}{\widetilde{R}_1\sqrt{t_1}} \frac{\Sigma \widetilde{d}_i \widetilde{R}_i^2}{\Sigma \widetilde{d}_i \widetilde{R}_i^2} = \frac{2 I_1(\widetilde{R}_1\sqrt{t_1})}{\widetilde{R}_1\sqrt{t_1}}$$

where $\widetilde{R}_1$ is the smallest member of $\{\widetilde{R}_i\}$. This follows because $\frac{I_1(y)}{y}$ is monotonically increasing function of y. Evaluating $\widetilde{S}$ for the one plateau solution with radius $\widetilde{R}$, we find

$$\widetilde{S} = \widetilde{S}_1 = \frac{2 I_1(\widetilde{R}\sqrt{t_1})}{\widetilde{R}\sqrt{t_1}} .$$

Hence, if $\widetilde{R}_1 \ge \widetilde{R}$, $\widetilde{S}_n > \widetilde{S}_1$, and the n plateau solution will not be able to fit the same values of the constraints as the one plateau solution. Therefore, $\widetilde{R}_1 < \widetilde{R}$.

To complete the proof of theorem 0, we need only show that for $-z^2/\widetilde{R}^2 \le t \le 0$ it is not possible to have the left end-point of a plateau at $0<b<\widetilde{R}$. To do this, we first refer again to expressions (7) and (8). Now, $\beta>0$ which can most easily be seen in our problem by referring to expression (8), and remembering that we need at least one fall-off point to occur when $J_1(b\sqrt{-t}) \ge 0$. Consider first equation (7) for the two end points of a plateau, $D_1 \ne 0$ and $D_2 \ne x$. Since I_0 is an increasing function of its argument, we must have

$$J_0(D_2\sqrt{-t}) < J_0(D_1\sqrt{-t}); \qquad D_2 > D_1 \tag{11}$$

in order for (7) to be satisfied at both end points. Now look at inequality (8). The left-hand side is the derivative of (7) with respect to b. This derivative is required to be non-negative at the end points, and the value of the function is required to be zero. Suppose these two equations are satisfied at the left end point of a plateau, D_1. In order for them to be satisfied again at D_2, there must be a region between D_1 and D_2 where the left side of (8) is negative. Call a point in this region P. Then we require,

$$\frac{\sqrt{-t}\, J_1(P\sqrt{-t})}{\sqrt{t_1}\, I_1(P\sqrt{t_1})} > \beta \geq \frac{\sqrt{-t}\, J_1(D_1\sqrt{-t})}{\sqrt{t_1}\, I_1(D_1\sqrt{t_1})} ,$$

or

$$\frac{\dfrac{J_1(P\sqrt{-t})}{P\sqrt{-t}}}{\dfrac{I_1(P\sqrt{t_1})}{P\sqrt{t_1}}} > \frac{\dfrac{J_1(D_1\sqrt{-t})}{D_1\sqrt{-t}}}{\dfrac{I_1(D_1\sqrt{t_1})}{D_1\sqrt{t_1}}} \quad ; \quad D_1 < P < D_2 .$$

Since $I_1(y)/y$ is a monotonically increasing function of y, it is necessary that

$$\frac{J_1(P\sqrt{-t})}{P\sqrt{-t}} > \frac{J_1(D_1\sqrt{-t})}{D_1\sqrt{-t}} , \quad D_1 < P < D_2 . \tag{12}$$

In figure 1, we have plotted $J_0(y)$ and $J_1(y)/y$. This curve can be read as $J_0(b\sqrt{-t})$ and $J_1(b\sqrt{-t})/b\sqrt{-t}$ plotted as a function of b for fixed t. For a value of b corresponding to $0<y<y_2$, we can not have the left end point of a plateau, since we will never be able to find a P which will satisfy (12). Now let us consider the interval $y_1<y<y_3$. If we begin a plateau in this region, in order to

satisfy (11) we will also have to end the plateau in this region. But, then we will not be able to satisfy (12). Since $y_1 < y_2$, this shows that we cannot have a transition between two non-zero plateaus when $0 < y \leq y_3$. Therefore, by the above lemma, we cannot have an intermediate plateau with end points, $D_1 \neq 0$ and $D_2 \neq x$.

The only other possibility to consider, is the existence of a plateau with end points $D_1 \neq 0$ and $D_2 = x$, leading to a two plateau solution with B_0 empty. We shall now show that such a profile cannot satisfy the variational equations. We can write the expressions for $\tilde{A}$, $\tilde{L}$, and $\tilde{G}$ generated by this profile as

$$\tilde{A} = 2\tilde{d}_1 \tilde{R}_1^2 \frac{J_1(\tilde{R}_1\sqrt{-t})}{\tilde{R}_1\sqrt{-t}} + 2\tilde{d}_2 x^2 \frac{J_1(x\sqrt{-t})}{x\sqrt{-t}}$$

$$\tilde{L} = \tilde{d}_1 \tilde{R}_1^2 + \tilde{d}_2 x^2$$

$$\tilde{G} = 2\tilde{d}_1 \tilde{R}_1^2 \frac{I_1(\tilde{R}_1\sqrt{t_1})}{\tilde{R}_1\sqrt{t_1}} + 2\tilde{d}_2 x^2 \frac{I_1(x\sqrt{t_1})}{x\sqrt{t_1}}$$

Now, let us fix $\tilde{R}_1$, x, and $\tilde{d}_2$, and calculate $\partial\tilde{A}/\partial\tilde{d}_1$. Using the chain rule and the definitions of α and β, we find,

$$\frac{\partial\tilde{A}}{\partial\tilde{d}_1} = 2\tilde{R}_1^2 \frac{J_1(\tilde{R}_1\sqrt{-t})}{\tilde{R}_1\sqrt{-t}} = \frac{\partial\tilde{A}}{\partial\tilde{L}}\frac{\partial\tilde{L}}{\partial\tilde{d}_1} + \frac{\partial\tilde{A}}{\partial\tilde{G}}\frac{\partial\tilde{G}}{\partial\tilde{d}_1} = -\alpha\tilde{R}_1^2 - 2\beta\tilde{R}_1^2 \frac{I_1(\tilde{R}_1\sqrt{t_1})}{\tilde{R}_1\sqrt{t_1}}$$

Performing the same operation with $\tilde{d}_1 \longleftrightarrow \tilde{d}_2$, we have a similar expression with $\tilde{R}_1 \rightarrow x$. Now, since $D_2 = x$ is not a bonafide drop off point, $\phi(x)$ may not be zero, and is given by the left-hand side of (9) evaluated at $D_2 = x$ and $D_1 = \tilde{R}_1 \neq 0$. But, equation (13) and the analogous one obtained by letting $\tilde{R}_1 \rightarrow x$ together imply that the left-hand side of (9) is zero. Therefore $\phi(x) = 0$. Since

$\phi(b) \geq 0$, $\dot{\phi}(x) \leq 0$. (Since we are at the edge of the domain over which our space is defined, we cannot argue that $\dot{\phi}(x) = 0$.) That is, the left-hand side of (7) evaluated at x is non-positive. These conditions on $\phi(x)$ together with the facts that (i) $b = \tilde{R}_1$ is a bonafide drop off point and (ii) $\tilde{R}_1 < \tilde{R}$ (by the lemma proved above) guarantee that an argument analogous to the one used above for $D_2 \neq x$ based on conditions (7) and (8) applies here also. Consequently, this profile cannot satisfy the variational equations.

We have shown, then, that for $-(y_3/\tilde{R})^2 < -(z/\tilde{R})^2 \leq t \leq 0$ there is no function in our space that satisfies the variational equations other than the one plateau profile, and so this profile generates a unique local minimum.

Finally, it is interesting to see how we can check the consistency of our solution with the variational equations. Using the chain rule of differentiation, and the definition of the multipliers in terms of derivatives of the lower bound, we find for our solution

$$\frac{\partial \tilde{A}}{\partial \tilde{c}} = 2\tilde{R}^2 \frac{J_1(\tilde{R}\sqrt{-t})}{\tilde{R}\sqrt{-t}} = \frac{\partial \tilde{A}}{\partial \tilde{L}}\frac{\partial \tilde{L}}{\partial \tilde{c}} + \frac{\partial \tilde{A}}{\partial \tilde{G}}\frac{\partial \tilde{G}}{\partial \tilde{c}} = -\alpha\tilde{R}^2 - 2\beta\tilde{R}^2 \frac{I_1(\tilde{R}\sqrt{t_1})}{\tilde{R}\sqrt{t_1}} \tag{14}$$

which is the same as (9) with $\phi(0) = 0$. Using (7) and (14) we can solve for α and β and with this value of β consistency requires that inequality (8) be obeyed. This inequality will be satisfied for the region of t we are interested in if

$$\frac{2J_1(y)}{y} - J_0(y) \geq \frac{yJ_1(y)}{4}, \qquad z \geq y > 0$$

and

$$I_0(y) - \frac{2I_1(y)}{y} \leq \frac{yI_1(y)}{4}, \qquad \text{for all } y > 0 .$$

We have explicitly checked these conditions and find that they are satisfied.

Having proved Theorem 0, we now proceed to establish our bound by proving Theorem 1.

Proof of Theorem 1:

Let a(b) be a monotonically decreasing function in $[0, \infty)$, such that $0 \leq a(b) \leq 1$, and

$$L = \int_0^\infty 2b\, a(b)\, db$$

$$G = \int_0^\infty 2b\, a(b)\, I_0(b\sqrt{t_1})\, db \quad .$$

We first prove three inequalities which will be used later. If $w > 0$, the monotonicity of $a(b)$, positivity, and the properties of I_0 imply

$$G \geq \int_0^w 2b\, a(b)\, I_0(b\sqrt{t_1})\, db \geq a(w) \int_0^w 2b\, I_0(b\sqrt{t_1})\, db$$

$$\geq a(w) \int_0^w 2b \frac{b^2 t_1}{4}\, db = a(w)\, t_1 \frac{w^4}{8}$$

so that

$$a(w) \leq \frac{8G}{t_1} \frac{1}{w^4} \quad . \tag{15}$$

Consider now

$$\int_w^\infty 2b\, a(b)\, db \leq \int_w^\infty 2b \frac{8G}{t_1} \frac{1}{b^4}\, db = \frac{8G}{t_1} \frac{1}{w^2} \tag{16}$$

and also

$$\left| \int_w^\infty 2b\, a(b)\, J_0(b\sqrt{-t})\, db \right| \leq \int_w^\infty 2b\, a(b) \left| J_0(b\sqrt{-t}) \right| db \leq \int_w^\infty 2b\, a(b)\, db \leq \frac{8G}{t_1} \frac{1}{w^2} \tag{17}$$

Let $\epsilon > 0$ be given. Choose x large enough so that

$$\frac{8G}{t_1} \frac{1}{x^2} < \frac{\epsilon}{2} \tag{18}$$

and so that

$$I_0(x\sqrt{t_1}) > \frac{G}{L} \quad . \tag{19}$$

Consider the profile $\tilde{a}(b)$ defined over $[0, x]$ and being equal to a(b) for $0 \le b \le x$. Define:

$$\begin{aligned} \tilde{L} &= \int_0^x 2b\, \tilde{a}(b)\, db \quad , & \rho_1 &= \int_x^\infty 2b\, a(b)\, db \quad , \\ \tilde{G} &= \int_0^x 2b\, \tilde{a}(b)\, I_0(b\sqrt{t_1})\, db \quad , & \rho_2 &= \int_x^\infty 2b\, a(b)\, I_0(b\sqrt{t_1})\, db \quad . \end{aligned} \tag{20}$$

Consider the difference

$$\frac{\tilde{G}}{\tilde{L}} - \frac{G}{L} = \frac{G - \rho_2}{L - \rho_1} - \frac{G}{L} = \frac{-\rho_2 L + \rho_1 G}{(L - \rho_1)L} \quad .$$

But

$$\rho_2 \ge I_0(x\sqrt{t_1})\, \rho_1$$

so that

$$\frac{\tilde{G}}{\tilde{L}} - \frac{G}{L} \le \frac{(G - I_0(x\sqrt{t_1})L)\,\rho_1}{(L - \rho_1)L} \le 0 \quad .$$

Define now $\tilde{R}$ and R by

$$2\, \frac{I_1(\tilde{R}\sqrt{t_1})}{\tilde{R}\sqrt{t_1}} = \frac{\tilde{G}}{\tilde{L}} \quad , \qquad 2\, \frac{I_1(R\sqrt{t_1})}{R\sqrt{t_1}} = \frac{G}{L}$$

then from the monotonicity of $I_1(y)/y$ it follows that $\tilde{R} \le R$, and if $R\sqrt{-t} \le z$, also $\tilde{R}\sqrt{-t} \le z$.

The profile $\tilde{a}(b)$ satisfies the conditions of theorem 0, and therefore

$$\int_0^x 2b\,\tilde{a}(b)\,J_0(b\sqrt{-t})\,db \geq \tilde{L}\,\frac{2J_1(\tilde{R}\sqrt{-t})}{\tilde{R}\sqrt{-t}} \geq \tilde{L}\,\frac{2J_1(R\sqrt{-t})}{R\sqrt{-t}} = (L-\rho_1)\frac{2J_1(R\sqrt{-t})}{R\sqrt{-t}}$$

in the range of t considered. Also from (17) and (20)

$$\int_x^\infty 2b\,a(b)\,J_0(b\sqrt{-t})\,db \geq -\rho_1$$

and we find that

$$\int_0^\infty 2b\,a(b)\,J_0(b\sqrt{-t})\,db \geq L\,\frac{2J_1(R\sqrt{-t})}{R\sqrt{-t}} - \rho_1\left(\frac{2J_1(R\sqrt{-t})}{R\sqrt{-t}} + 1\right)$$

But by (16), (18) and (20), $\rho_1 < \epsilon$, and since ϵ can be chosen as small as one wishes, we end up with the result claimed in theorem 1:

$$\int_0^\infty 2b\,a(b)\,J_0(b\sqrt{-t})\,db \geq L\,\frac{2J_1(R\sqrt{-t})}{R\sqrt{-t}} = A(0)\,\frac{2J_1(R\sqrt{-t})}{R\sqrt{-t}}$$

Since the lower bound is positive for $\sqrt{-t} < \frac{z}{R}$, the first zero of the imaginary part of the scattering amplitude must occur at $\sqrt{-t} \geq \frac{z}{R}$.

A bound similar to the one derived above, which does not depend on extrapolations outside the physical region can be stated as follows:

<u>Theorem 2</u>: If at some energy, values of σ_T and

$$A'(0) \equiv \frac{d}{dt}\,A\Big|_{t=0} = \frac{1}{2}\int_0^\infty b^3\,a(b)\,db$$

are given, and if unitarity is obeyed and a(b) is a non-increasing function of b, then for $-z^2/R^2 \leq t \leq 0$, A(t), the absorptive part of the scattering amplitude is bounded from below by

$$A(t) \geq 2c\,R^2\,\frac{J_1(R\sqrt{-t})}{R\sqrt{-t}},$$

where c and R are determined by

$$\frac{\sigma_T}{2\pi} = A(0) = cR^2 \quad ,$$

and

$$A'(0) = \frac{1}{8}\, cR^4 \quad .$$

This theorem can be proved by arguments exactly analogous to those used to prove theorem 1. The comparative utility of these two bounds will be discussed in the next section.

III. Discussion

In the previous section we proved that the imaginary part of any nonflip elastic scattering amplitude, which satisfies unitarity, corresponds to a monotonically decreasing profile in b space, has a given value A(0) at t=0, and has a given slope $S = A'(t=0)/A(t=0)$ (or a given value $G = A(t=t_1)$, $t_1 > 0$), is bounded from below by

$$A(t) \geq A(0)\; \frac{2J_1(R\sqrt{-t})}{R\sqrt{-t}}$$

as long as $R\sqrt{-t}$ is smaller than the first zero of J_1. R is given by either $R_S^2 = 8S$ for the S-constraint or

$$G = A(0)\; \frac{2I_1(R_G\sqrt{t_1})}{R_G\sqrt{t_1}}$$

for the G-constraint.

The interpretation of the equality constraints is clear. A(0) defines the overall strength of the interaction while S or G/A(0) defines the range of the interaction; the natural unit in b space (or, equivalently, in t space).

In the t range with which we are concerned, the minimizing profiles, and hence the functional forms of the two bounds is the same. However, the values of the bounds may be different because R_S may be different from R_G, depending on the numerical values of S and G used in the two problems. We have not treated the problem for larger values of $|t|$, and it is quite possible that in those regions the bounds will be entirely different in their functional form.

We can gain some insight into the relationship between these two bounds by noticing the following: For small t_1, we can approximate $G = A(t{=}t_1)$ by $A(0) + A'(0)t_1$. Then, using the first order expansion of

$$I_1(R_G\sqrt{t_1}) \simeq R_G\sqrt{t_1}\left(\frac{1}{2} + \frac{1}{16} R_G^2 t_1\right) ,$$

we find $R_G{=}R_S$, and the two bounds are identical.

Now, since t_1 is outside the physical region, $A(t_1)$ may be difficult to measure directly. Obviously, if we know the value of $A(t_1)$ (perhaps from theoretical considerations) we can use the G-constraint to derive a bound. However, even in the absence of such information, we can evaluate $A(t_1)$ by extrapolating from the physical region using a Taylor's expansion of A(t) about t=0. Of course, if we keep only the first order term in the expansion, (and the linear approximation of I_1), the two bounds become identical as described above.

Since the lower bound which we have obtained is positive for $0 \leq R\sqrt{-t} \leq z$, where z is the first positive zero of J_1, it follows that A(t) cannot vanish in this range of t. At intermediate to high energies typical values of the slope of pomeron exchange amplitudes correspond to radii, R, of approximately 1 fermi, and therefore A(t) cannot vanish for $|t| \leq 0.6$ BeV^2. We shall show below that the monotonicity requirement is a strong constraint in our problem, and therefore the phenomenological lack of zeroes in diffractive amplitudes in this range may be attributed to the monotonic behavior of such amplitudes in b or ℓ space.

At higher values of $|t|$, zeroes may occur,[3] and present data[4] indicate that a zero may exist near $|t| = 0.8 - 1.2$ BeV^2 in the diffractive amplitudes of the process pp→pp. (We shall comment below on bounds which can improve our results, and may give a lower bound on the position of the zero quite close to the "observed value.")

Of course, since our bounds are derived at a fixed value of s, they can say nothing about the allowed positions of the zeroes as a function of energy without additional information. However, we would like to remind the reader of the following situation: If the interaction radius grows as a function of energy, then the first zero of A(t) can move towards t=0 at higher energies. It has been shown that S cannot grow faster than $(\ln s)^2$,[9] and therefore R_S cannot grow faster than $\ln s$, and the first zero of A(t) cannot approach t=0 faster than $(\ln s)^{-2}$. There is phenomenological evidence[12] that S grows definitely slower than $\ln s$, in which case the motion of any zero of A(t) towards t=0 is slower than $(\ln s)^{-1}$.

We turn now to a more specific discussion of the role of the monotonicity constraint in our results. That this is indeed a strong restriction can be seen by comparing our result with similar bounds derived without the assumption of monotonicity.

Singh and Vengurlekar[9] derive a lower bound on A(t) assuming all the conditions of theorem 1 except monotonicity. Our bound increases by roughly a factor of two the region in t derived by Singh and Vengurlekar over which A(t) must be positive definite. In fact, expanding our result in theorem 2 for small $|t|$, and keeping only the first two terms reproduces the result of Ref. 9. Hahn and Hodgkinson[10] add a value for the elastic cross section to the assumptions of Singh and Vengurlekar. Their lower bound passes through zero at $t \sim -0.4$ BeV^2. Hence, replacing the elastic cross section constraint by the monotonicity constraint again significantly improves the lower bound on the position of the first zero. Because of the behavior of J_0, it is clear that the monotonicity requirement should be in general more important in lower bounds on A(t) than in upper bounds, and this seems to be the case.[11]

While our results go a long way towards understanding the absence of zeroes at small $|t|$ in Pomeron dominated reactions, a further improvement in the lower bound of the position of the first zero would be quite welcome. Looking at the profile which generates our lower bound we see that the physically most unrealistic feature is the existence of a sharp discontinuity. One might attempt to remedy this problem by explicitly requiring the profile function to be smooth. Such an attempt, however, would fail since there exist smooth functions which are arbitrarily close to discontinuous functions. A fermi function with a very narrow transition region, for example, is a smooth function which can be made arbitrarily close to our one plateau, discontinuous profile. In other words, if we try to modify our problem by adding a smoothness assumption on a(b), we will find that the lower bound on A(t) will not be obtained as a minimum in the space. It is possible, however, to implicitly enforce a smoothness assumption, and thus circumvent this problem. For example, adding a fixed value of the elastic cross section to the constraints of theorems 1 or 2 will, for realistic ratios of σ_{el}/σ_T ($\sigma_{el}/\sigma_T \leq 1/4$) significantly alter the profile which gives the lower bound. We believe that the space of functions defined in this way will not have the pathology discussed above. Furthermore, the minimizing profile may well be smooth in the relevant t range, and in any case will substantially improve some of the results presented here.

Acknowledgements

We wish to thank R. Blankenbecler for useful suggestions and comments, and M. Davier and P. Berg for interesting discussions.

REFERENCES

1. M. J. Sopkovich, Nuovo Cimento 26, 186 (1962);

   A. Dar, M. Kugler, Y. Dothan, and S. Nussinov, Phys. Rev. Letters 12, 83 (1964);

   A. Dar and W. Tobocman, Phys. Rev. Letters 12, 511 (1964);

   K. Gottfried and J. D. Jackson, Nuovo Cimento 34, 735 (1964);

   L. Durand and Y. T. Chiu, Phys. Rev. Letters 12, 399 (1964).

2. H. Harari, Ann. Phys. 63, 432 (1971);

   H. Harari, Phys. Rev. Letters 26, 1400 (1971).

3. Y. Avni, Lettere al Nuovo Cimento 4, 725 (1972).

4. J. V. Allaby et al., Phys. Letters 34B, 431 (1970).

5. G. L. Kane, Phys. Letters 40B, 363 (1972);

   V. Barger, R. J. N. Phillips, and K. Geer, Rutherford Laboratory preprint no. RPP/T/12 (1972).

6. M. B. Einhorn and R. Blankenbecler, Ann. Phys. (N.Y.) 67, 480 (1971).

7. R. Savit, R. Blankenbecler, and M. B. Einhorn, J. Math. Phys. 12, 2092 (1971). Our result is physically applicable at high energies and small t, where the approximation of P_ℓ by J_0 is good anyway.

8. Note that this does not entail the assumption that the real part of f(b) is zero. The inequality which we use is always true, and dropping the real part of f(b) simply means that we do not use any information about it.

9. V. Singh and A. S. Vengurlekar, Phys. Rev. D 5, 2310 (1972). These authors (as well as those of Ref. 10), use the formalism of discrete partial wave amplitudes. However, as pointed out in Ref. 7, this difference is not expected to significantly affect our result.

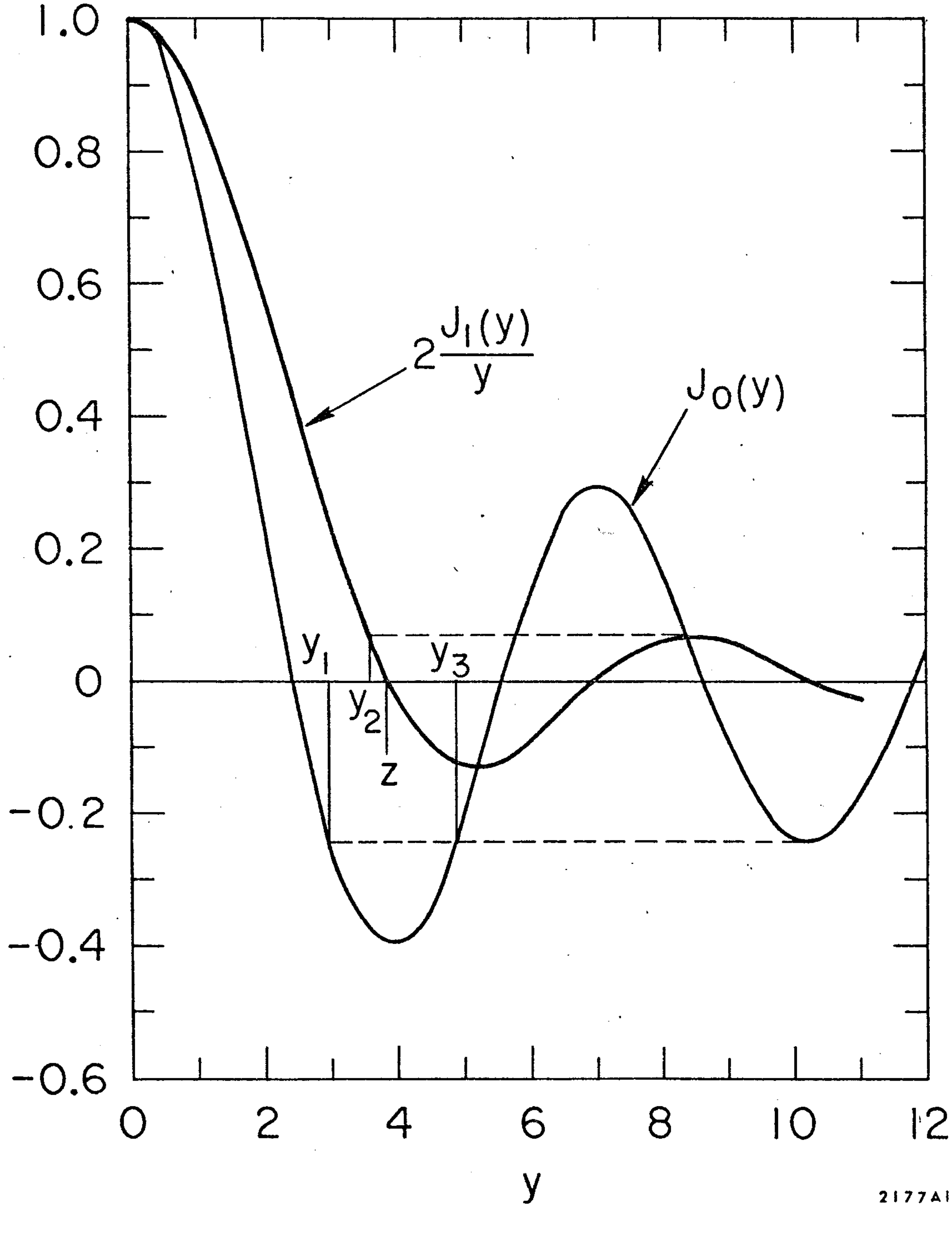
10. B. D. Hahn and D. P. Hodgkinson, University of Cambridge report no. HEP 72-5 (1972).

11. For example, the upper bound on A(t) derived in Ref. 10 corresponds to monotonically decreasing profile although this is not explicitly required in their auxilliary function. For a direct comparison of upper bounds derived with and without monotonicity, see Ref. 7.

12. M. Davier, Phys. Letters 40B, 369 (1972).

FIGURE CAPTION

1. This figure illustrates the definition of the points y_1, y_2, y_3 and z used in the text. z is the first positive zero of $J_1(y)$ or of $J_1(y)/y$. y_1 and y_3 are such that if $u \in (y_1, y_3)$ and $v \notin (y_1, y_3)$, $J_1(u)/u < J_1(v)/v$. y_2 is such that if $u < y_2$ and $v \geq y_2$, $J_0(u) > J_0(v)$.



1.0
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
0
2
4
6
8
10
12
y
$2\frac{J_1(y)}{y}$
$J_0(y)$
y_1
y_2
y_3
z
2177A1

Fig. 1