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ATOMIC PHYSICS AND QUANTUM ELECTRODYNAMICS IN THE INFINITE MOMENTUM FRAME*

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I. INTRODUCTION

Over the past few years it has been shown that the use of an "infinite momentum" Lorentz frame[1] has remarkable advantages for calculations in elementary particle physics and field theory, especially in the areas of current algebra sum rules,[2] parton models,[3,4] and eikonal scattering.[5,6] One important advantage is that it allows a straightforward application of the impulse and incoherence approximations familiar in nonrelativistic atomic and nuclear physics to relativistic field theory and bound state problems.

The central idea is this: Suppose we choose a Lorentz frame such that a bound system has momentum $\vec{P}$ in the z-direction. We shall assume that for P chosen large enough, $(P \to \infty)$ all of its constituents will be moving in the positive z-direction; more specifically, we assume the existence of a wave function in the infinite momentum frame:

$$\lim_{P\to\infty} \psi_P(\vec{p}_i) = \psi(\vec{k}_{i\perp}, x_i) \qquad i = 1, \ldots, N$$

where

$$\vec{p}_i \equiv x_i \vec{P} + \vec{k}_{i\perp}$$

$$\sum_{i=1}^{N} \vec{k}_{i\perp} = 0 \quad , \qquad \sum_{i=1}^{N} x_i = 1$$

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For example, for a bound state with momentum $\vec{P}$, mass M and N constituents, the characteristic energy denominator of the wave function is

$$\frac{1}{2E}\,\frac{1}{E-E_i+i\epsilon}\underset{P\to\infty}{=}\frac{1}{2P}\,\frac{1}{P+\frac{M^2}{2P}-\sum_{i=1}^{N}\left[|x_i|\,P+\frac{\vec{k}_{i\perp}^2+m_i^2}{2|x_i|P}\right]+i\epsilon}$$

$$=\frac{1}{M^2-\sum_{i=1}^{N}\frac{\vec{k}_{i\perp}^2+m_i^2}{x_i}+i\epsilon}\qquad \text{for all } x_i>0$$

$$=0\left[\frac{1}{P^2}\right]\qquad \text{otherwise.}$$

Since $\sum_{i=1}^{N} x_i=1$, finite contributions are obtained only if $0<x_i<1$ for all i. Note that the $P\to\infty$ limit also has the effect of linearizing relativistic square-root phase-space factors. Similarly, one finds that in time-ordered perturbation theory, all diagrams in which intermediate particles are moving backward ($x_i<0$) can be effectively set to zero, leaving only the relatively few diagrams with forward moving intermediate particles to be considered.[1] [See Section II for examples.] The structure of the $P\to\infty$ wavefunction is formally very similar to nonrelativistic theory; the quantity $k_{i\perp}^2/x_i$ plays the role of the kinetic energy. (More generally the relativistic wavefunction contains arbitrary numbers of constituents, but we may treat each N-particle component state as above.)

Thus the intuition and approximation procedures used in the nonrelativistic problems now becomes applicable to high energy physics and rigorous methods for bound states other than the Bethe-Salpeter formalism now present themselves. Conversely, these techniques indicate a new systematic procedure for handling the relativistic and recoil correction to atomic and nuclear physics problems.

In Section II, we discuss the application of the infinite momentum method to quantum electrodynamics, and the implementation of the renormalization procedure in old-fashioned perturbation theory. In Section III, several applications to problems in atomic physics are outlined. These include inelastic electron-atom scattering, high energy scattering, and rearrangement collisions in atom-atom scattering.

II. QUANTUM ELECTRODYNAMICS AND RENORMALIZATION THEORY IN THE INFINITE MOMENTUM FRAME

(The manuscript for this section was prepared in collaboration with Ralph Roskies)

Recently Ralph Roskies, Roberto Suaya and I have found that time-ordered perturbation theory for quantum electrodynamics evaluated in an infinite momentum reference frame represents a viable, instructive, and frequently advantageous calculational alternative to the usual Feynman diagram approach. The renormalization procedure can be implemented in a straightforward manner. We have calculated the electron anomalous magnetic moment through fourth order in agreement with the Sommerfield-Petermann results,[7] and have calculated representative contributions to the sixth order moment. Our results agree with those of Levine and Wright[8] and represent the first independent confirmation of their result for these contributions.

An outline of our techniques follows; a more complete discussion will be published separately.[9]

The electron vertex in quantum electrodynamics may be computed in perturbation theory using the standard time-ordered momentum space expansion of the S-matrix. Although the final results are independent of the choice of Lorentz frame, it is very convenient to choose a limiting reference frame in which the incident electron momentum P is large.[1] In a general frame, a Feynman amplitude of order e^n requires the evaluation of n! time-ordered contributions, but in a frame chosen such that

$$p = \left(\sqrt{P^2+m^2},\ \vec{O}_\perp, P\right) \to \left(P + \frac{m^2}{2P},\ \vec{O}_\perp, P\right) \tag{1a}$$

$$q = \left(\frac{q\cdot p}{P},\ \vec{q}_\perp, O\right) \tag{1b}$$

($2q\cdot p = -q^2 = \vec{q}_\perp^{\,2}$) only the relatively few time-ordered graphs, in which the momenta of all the internal (on-mass-shell) particles $\vec{p}_i = x_i\vec{P} + \vec{k}_{i\perp}$ have positive components along $P(0 < x_i < 1)$, have a surviving contribution in the limit $P \to \infty$. In general, the limit $P \to \infty$ is uniform with respect to the

$$\frac{d^3p_i}{2E_i} = \frac{d^2k_{i\perp}dx_i}{2x_i}$$

phase space integrations for all renormalized amplitudes. Thus the order α correction to the anomalous moment $a = F_2(0)$ is obtained from only one forward-moving time-ordered graph[5,6,10] (see Fig. 1), up to 3 time-ordered graphs yield the Feynman amplitude for the order α^2 corrections;

between 1 and 15 forward-moving time-ordered graphs contribute to various Feynman amplitudes at order α^3.

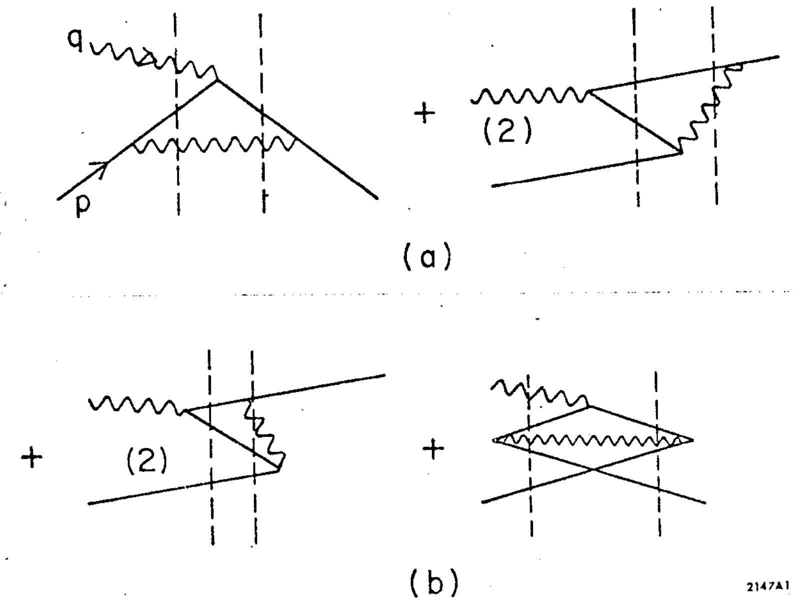

Fig. 1--The six time-ordered contributions of the Feynman amplitude for the proper electron vertex Γ_μ in order α. For the components μ=0 or μ=3, only the contribution of the diagram (a) survives in the infinite momentum limit $P \to \infty$ of Eq. (1). In addition, the "Z-graph" contribution for the μ=1, 2 components which arises from diagram (b) is automatically included by using the modification of the spinor sum for diagram (a) given in Eq. (2).

As emphasized by Drell, Levy, and Yan,[4] time-ordered graphs with backward-moving ($x_i < 0$) internal fermion lines can give surviving P^2/P^2 contributions in the $P \to \infty$ limit if the line extends over only one time interval. These additional contributions (which correspond to contact or "seagull" interactions analogous to the $e^2\phi^+\phi A^2$ interactions in boson electrodynamics) can be automatically included by making a simple modification in the forward-moving contribution: if a forward-moving electron ($x_i > 0$) extends over a single interval I then instead of the usual spin sum

$$\sum_{\text{spin}} u(p_i)\, \bar{u}(p_i) = \not{p}_i + m \,, \qquad p_i^2 = m_i^2 \tag{2a}$$

we write

$$\not{p}_i + \gamma_0(E_0 - E_I) + m \tag{2b}$$

where E_0 is the total incident energy and E_I is the sum of the energies of all of the particles occurring in the intermediate state I. It is easy to check that this replacement (which corresponds to using energy conservation between the initial and intermediate energies to determine p_i^0 rather than the mass-shell condition) automatically accounts for the contribution of the corresponding negative moving ($x_i < 0$) positron line. A similar modification for the energy of a forward-moving positron (spanning one time interval) accounts for the corresponding negative moving electron line. With these changes all "Z-graph" contributions are accounted for, and one need only consider time-ordered diagrams where all internal lines have $x_i > 0$.

The renormalization procedure for quantum electrodynamics using old-fashioned perturbation theory closely parallels the explicitly covariant Feynman-Dyson procedure. Reducible amplitudes with self-energy and vertex insertions may be renormalized using subtraction terms corresponding to δm, Z_2 and Z_1 counter terms. The integrand for the subtraction term is similar in form to the integrand for the unrenormalized amplitude, except that the external energy used for the denominator for the subgraph insertion is not the external (initial) energy E_0 of the entire diagram but is the energy external to the self-energy or vertex subgraph only. For example, the renormalization of the scattering amplitude shown in Fig. 2a requires δm and Z_2 subtractions (Fig. 2b and Fig. 2c).

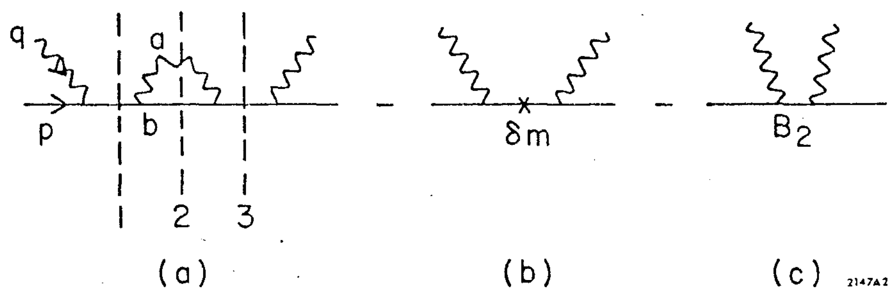

Fig. 2--Illustration of the renormalization procedure in old-fashioned perturbation theory. (a) A representative time-ordered diagram for the self-energy modification of the Compton amplitude. (b) and (c) The corresponding δm and Z_2 counterterms. The integrand for the δm term is proportional to $(E_1-E_2)^{-1}$.

The integrand of the renormalized amplitude for ϕ^3 theory is constructed from

$$\frac{1}{(E_0-E_1)(E_0-E_2)(E_0-E_3)} - \frac{1}{(E_0-E_1)(E_1-E_2)(E_0-E_3)} + \frac{1}{(E_0-E_1)(E_1-E_2)(E_1-E_2)} \quad (3)$$

where E_i is the total energy of the on-shell particles occurring at interval i. Upon integration over the loop momentum variables (x_i, $\vec{k}_{i\perp}$), the second and third terms yield, by definition, the correct δm and Z_2 counter terms (assuming covariant regularization). On the other hand, if scaled variables

$$\vec{p}_a = x(\vec{p} + \vec{q}) + \vec{k}_\perp$$
$$\vec{p}_b = (1 - x)(\vec{p} + \vec{q}) - \vec{k}_\perp \qquad (4)$$

are chosen to parametrize the momenta of the internal particles, then $\vec{k}_\perp \cdot \vec{q}$ cross terms are eliminated and the integration for the renormalized amplitude from the sum of the three terms is point-wise convergent. In the QED case, the appropriate Dirac numerator must also be constructed such that the (covariantly-regulated) subgraph integration defines the correct counter terms. This procedure leads to finite, renormalized pointwise-convergent (and numerically integrable) amplitudes for the case of all self-energy or vertex insertions.[9, 11] The analysis of infrared divergences (via a photon mass regulator) may be carried out in parallel with standard treatments.

In general, we have found that the $P \to \infty$ limit is uniform (i.e., can be taken before the $d^2k_\perp dx$ loop integrations) for the renormalized amplitudes, and there are no subtleties involved at the boundaries of the x_i integration. On the other hand, the evaluation of the (divergent) renormalization constants themselves requires caution. Since covariance is not explicit in this approach, one must be careful to regularize using a covariant procedure, such as the Pauli-Villars method or spectral conditions. The standard covariant expressions for the renormalization constants are obtained if regularization is performed before the $P \to \infty$ limit is taken.[9]

With the above considerations, it is straightforward to calculate renormalized amplitudes for quantum electrodynamics directly from time-ordered perturbation theory and the interaction density $e{:}\bar{\psi}\gamma_\mu\psi A^\mu{:}$. The covariant Feynman amplitude is obtained from the corresponding (forward-moving) time-ordered graphs with the same topology. The Dirac numerator algebra is the same for each of the time-ordered amplitudes and is identical to the corresponding Feynman calculation. Our techniques also show that quantum electrodynamics may be calculated on the light-cone in the Feynman gauge, rather than the Coulomb gauge.

For the calculation of the lepton vertex, the F_1 and F_2 amplitudes can be obtained simply from standard trace projection operators.[12] The integrand in the variables x_i, $\vec{k}_{i\perp}$ is then obtained from the product of phase space, the numerator trace, and the energy denominators characteristic of old-fashioned perturbation theory.[13] One important feature of this method, besides providing a new and independent calculational technique, lies in the fact that the resulting integrand appears to be much smoother function of the variables x_i, $\vec{k}_{i\perp}$ than the corresponding Feynman parametric integrand obtained by the usual techniques. As a result, the numerical integrations (which are often the most difficult part of higher order calculations in quantum electrodynamics) converge considerably faster.

As an indication, the numerical integration of the contribution of the sixth order ladder graph (Fig. 3a) to the electron's anomalous magnetic moment from old-fashioned perturbation theory required 10^5 evaluations of a smooth well-behaved six-dimensional integrand to obtain a 1% level of accuracy.[14] In contrast, the standard Feynman technique, which involves a five-dimensional integral, required 2×10^6 evaluations of the integrand for comparable accuracy. Our result is

$$\left(\frac{g-2}{2}\right)_{\text{Fig. 2a}} = (1.77 \pm 0.01)\,\frac{\alpha^3}{\pi^3}$$

in precise agreement with the result of Levine and Wright.[8] Our results for the fourth order magnetic moment using $P\to\infty$ techniques agree with the Sommerfield and Petermann calculations;[7] again, the integrands were found to be smooth and rapidly integrable by numerical techniques.

The sixth order ladder graph is a highly reducible graph requiring several vertex renormalization counter terms, but only one time-order survives in the infinite momentum limit. We have also calculated a representative irreducible graph, Fig. 3b, which has eight surviving

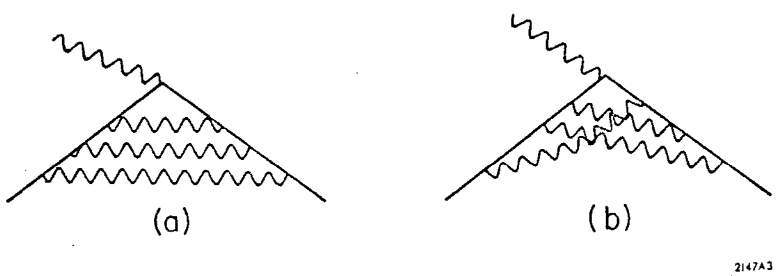

Fig. 3--Representative reducible and irreducible contributions to the sixth order magnetic moment of the electron or muon. The ladder graph (a) is obtained from a single time-ordered contribution at infinite momentum (out of a possible 7!), but requires renormalization of the fourth order and second order vertex insertions. The Feynman amplitude for irreducible graph (b) receives contributions from the eight time-ordered graphs with positive moving internal lines.

time orders. In this case there is an eight-dimensional nontrivial integration to be performed and the algebraic work is much more complex. Our result for this graph is $2(1.11 \pm 0.23)\ \alpha^3/\pi^3$ which is consistent with Levine and Wright's result $2(0.90 \pm 0.02)\ \alpha^3/\pi^3$ obtained from a seven-dimensional Feynman parametric integration. Work is continuing to improve the accuracy of our result.

The validity of the infinite momentum reference frame method as a renormalizable calculational procedure in quantum electrodynamics gives field-theoretical parton model calculations a rigorous basis provided that a covariant regularization procedure is used. Our work also demonstrates that the infinite momentum method provides a useful calculational alternative to standard covariant techniques. The $P \to \infty$ method is closely related to field theory quantized on the light cone.[6, 15] Our method shows how to renormalize the theory and work in the Feynman gauge.[15]

III. THE ATOM IN THE INFINITE MOMENTUM FRAME

Although the infinite momentum method was developed to treat highly relativistic problems, there are interesting applications to problems of the atom.

An important quantity is the normalized probability distribution

$$f_e(x) = \frac{1}{16\pi^3} \int \frac{d^2k}{x(1-x)} \, |\psi(\vec{k}_\perp, x)|^2$$

$$\int_0^1 f_e(x)\, dx = 1$$

which is the probability for finding an electron moving with momentum xP along the $\vec{P}$-direction in a reference frame in which the atom is moving with momentum $P \to \infty$. The electron wavefunction $\psi(\vec{k}_\perp, x)$ may be found from the solution of the wave-equation of the atom in the infinite momentum frame (see Weinberg[1] and Feldman, Fulton, and Townsend[16]) or by a Lorentz boost of the center-of-mass wavefunction. [For corrections in α, higher particle number (photon, electron-positron pair) states must be included, as in the QED case, see Section II.] Note that $f_e(x)$ is peaked at the value $x = E_e/M_T$ where M_T is the total atomic mass and E_e is the bound electron energy, and that $f_e(x) \to \delta(x - m_e/M_T)$ if the binding energy is taken to zero.

A standard result, derived from parton — constituent field theoretic models[4] — is that the bound electron contribution to deep inelastic wide-angle electron-atom scattering (i.e., $\nu = E_{Lab} - E'_{Lab} >> B.E.$, $\vec{q}^{\,2}R^2 >> 1$) is given simply by the Mott cross section (for elastic e-e collisions) times $f_e(x)$, with x taken at the value

$$x = Q^2/2M_T\nu \ , \qquad Q^2 = \vec{q}^{\,2} - \nu^2 \ .$$

Derivations and formulae are given in Ref. 4. This result extends the validity of the impulse and incoherence approximations to the relativistic domain.

A surprisingly simple result can also be obtained for the bound electron contribution to high energy ($\nu >>$ B.E.) forward photon-atom scattering. One finds[17] that the (spin-averaged) forward Compton amplitude $f(\nu)$ is asymptotically constant and real:

$$f(\nu) \xrightarrow[\nu >> \mathrm{B.E.}]{} - Z \frac{e^2}{M_T} \int_0^1 \frac{f_e(x)}{x} dx = \frac{-Ze^2}{m_{eff}}$$

Note that xM_T plays the role of the effective electron mass; m_{eff} contains corrections from atomic binding and finite nuclear mass corrections. The above result is derived in field theory from the electron-positron z-graph contribution to the electron Compton amplitude, which is effectively a "seagull" diagram in the infinite momentum frame. This result may be compared with the beautiful treatment of high energy photon scattering from an electron bound in a potential that has been given by M. Goldberger and F. Low.[18]

Given the infinite momentum wavefunction we may also determine the electron current contribution to atomic (elastic or inelastic) form factors. Ignoring spin complications, one obtains

$$F(q^2) = \frac{1}{16\pi^3} \int \frac{d^2k_\perp}{x(1-x)} \psi^* (\vec{k}_\perp + (1-x)\, \vec{q}_\perp, x)\, \psi(\vec{k}_\perp, x)$$

where $\vec{q}_\perp$ is a vector transverse to $\vec{P}$ with magnitude $\vec{q}_\perp^{\,2} = |q^2|$. Drell and Yan (Ref. 4) have shown that the large q^2 behavior of the elastic form factor $F(q^2)$ is related to the x near one behavior of $f_e(x)$.

A very simple expression may also be given for rearrangement (interchange) collisions in elastic or inelastic atom-atom scattering.[19] The scattering amplitude for elastic H - H rearrangements collisions is proportional to

$$m(t,u) = \int d^2k \int_0^1 \frac{dx}{x^2(1-x)^2} \,\Delta$$

$$\left[\psi(\vec{k}_\perp)\, \psi(\vec{k}_\perp + (1-x)\, \vec{q})\, \psi(\vec{k}_\perp + (1-x)\, \vec{q}_\perp - x\vec{r}_\perp)\, \psi(\vec{k}_\perp - x\vec{r}_\perp)\right]$$

where

$$\Delta = E_0 - E_{\text{intermediate}}$$

$$= 2M_T^2 - \frac{(\vec{k}_\perp - x\vec{r}_\perp)^2 + (\vec{k}_\perp + (1-x)\,\vec{q}_\perp)^2 + 2(xM_p^2 + (1-x)\,m_e^2)}{x(1-x)}$$

The vectors $\vec{q}_\perp$ and $\vec{r}_\perp$ are chosen transverse to $\vec{P}$, with magnitudes

$$\vec{q}_\perp^{\,2} = 2\vec{p}^{\,2}_{c.m.}\ (1 - \cos\theta_{c.m.}) = -t$$

$$\vec{r}_\perp^{\,2} = 2\vec{p}^{\,2}_{c.m.}\ (1 + \cos\theta_{c.m.}) = -u$$

$$\vec{q}_\perp \cdot \vec{r}_\perp = 0 \quad .$$

This result ignores the Coulomb interactions between the electrons and between the atoms, but includes the binding forces correctly (including all recoil and relativistic terms). Spin corrections are discussed in Ref. 19.

The corresponding parton-interchange contribution has been shown to agree well with measurements of high energy, large angle, proton-proton scattering (where the proton is regarded as a quark bound state).[19] It would be interesting to measure hard, large angle atom-atom (elastic or inelastic) in the region where the electron exchange contribution is dominant.

Finally, we note that a very hopeful area of application of the infinite momentum method is the spectra of bound states, especially that of positronium and muonium; the infinite momentum old-fashioned perturbation theory approach provides a rigorous alternative to the Bethe-Salpeter formalism, and does have calculational advantages. The work of Feldman, Fulton, and Townsend,[16] who have treated the spin zero bound state problem in the infinite momentum frame, is an important step in this direction.

Acknowledgements

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