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IMPROVED WEIZSACKER-WILLIAMS METHOD AND ITS APPLICATION TO LEPTON AND W-BOSON PAIR PRODUCTION*

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Abstract

A Weizsäcker-Williams method is derived which handles the elastic and inelastic target form factors properly. The method is applied to calculate energy angle distributions of photoproduced lepton pairs: electrons, muons, and heavy leptons, using target form factors appropriate to each case. The agreement with the exact result is found to be excellent. Simple formulas for pair production of spin 0 and spin 1 particles are also given.

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I. Introduction

Some of the cross sections which involve one photon exchange can be quite complicated. The best example is the calculation of the W pair production, $\gamma + Z \longrightarrow W^+ + W^- +$ anything, which involves threefold integration of roughly 3000 terms. With the advancement in the modern computer technique, even such a complicated calculation can be handled easily. However, it is often desirable to have a simple expression which shows all the gross features of the problem, such as the dependence of the cross section on the incident energy, outgoing energy, angle, mass, magnetic moment, radius of the target, etc. The way in which one can do this quickly was originally suggested by Fermi[1] in 1924, who noted the similarity between the electromagnetic fields of a rapidly moving charged particle and a pulse of radiation. Based on this observation, Weizsäcker[2] and Williams[2] showed independently in 1934 that an incident particle with charge Ze, mass M, energy $E = \gamma M$ would produce the same effect as a beam of photons with a spectrum $\rho(\omega)$ given by[3]

$$\rho(\omega) = \frac{Z^2\alpha}{\pi\omega}\left\{2xK_0(x)K_1(x) - x^2\left[K_1^2(x) - K_0^2(x)\right]\right\}$$

$$\underset{x \ll 1}{\sim} \frac{2Z^2\alpha}{\pi\omega}\left[\ell n\left(\frac{1.123\gamma}{\omega b_{min}}\right) - \frac{1}{2}\right], \qquad (I.1)$$

where ω is the photon energy, $x = \omega b_{min}/\gamma$, b_{min} is the minimum impact parameter, and K_0 and K_1 are the usual Bessel functions. The second expression is valid when $x \ll 1$, which is the usual case when γ is large.

The above formula, which is known as the pseudo-photon flux of the classical Weizsäcker-Williams (W.W.) method, has enjoyed wide applications in processes involving one photon exchange in the past because of its conceptual and mathematical simplicity. However, it can sometimes lead to a numerical value which deviates considerably from the correct one, mainly because it does not take into account the effect due to the rapid variations of the form factors properly. As an example, let us consider the pseudo-photon flux of a nucleus with a form factor $G_e^2 = Z^2/(1 + t/d)^2$ to be used for the pair production of particles of mass m (see Appendix C, Eq. (C.6) and (C.9) and also Eq. (III.23)).

$$\rho(\omega) = \frac{2Z^2\alpha}{\pi\omega}\left[\left(1+2\left(\frac{\omega}{\gamma}\right)^2\frac{1}{d}\right)\ell n\left(\frac{1+d(\gamma/\omega)^2}{1+d/t_{up}}\right) - \left(1+\frac{(\omega/\gamma)^2}{t_{up}}\right)\frac{1+2(t_{up}/d)}{1+(t_{up}/d)}\right], \quad (I.2)$$

where $t_{up} = m^2(1+\ell)^2$. In the limit $d \longrightarrow \infty$, i.e., the case of point particle, we recover (I.1) if we identify

$$b_{min} = 1.123/\left(t_{up}\right)^{\frac{1}{2}}. \quad (I.3)$$

When d is not infinity, (I.2) can be quite different from (I.1).

In an arbitrary one-photon-exchange process, if the initial target is unpolarized and if the final states of the target system are not measured, the cross section can be written in terms of $W_1(t, M_f^2)$ and $W_2(t, M_f^2)$ of Drell and Walecka.[4] Thus we must be able to write the pseudo-photon flux in terms of these two form factors. This was done in Appendix C. The result can be stated in the following sentence: "In the one-photon-exchange process, the target particle, viewed in the frame where it is moving with a great velocity opposite to the incident particle, is equivalent to a beam of real photons produced by an electron after it passes through a target of thickness $(3/4)(\alpha/\pi)X$ radiation lengths, where X is related to W_1 and W_2 by Eq. (II.19)." It should be noted that our treatment of the pseudo-photon flux will be covariant, but the simple physical interpretation given above is possible only in the infinite momentum frame. Feynman's parton theory[5] of hadrons has its genesis in the Weizsäcker-Williams method. Like pseudo-photon flux, the Feynman parton flux has a simple physical interpretation only in the infinite momentum frame. In Appendix C, we also show that only the relatively soft component of the pseudo-photon beam is relevant in the W. W. calculation. Borrowing Feynman's colorful parton terminology,[5] we may say that the W. W. method deals with the "wee photon" of the pseudo-photon flux.

The object of this paper is to give a coherent derivation of this "improved W. W. method" and discuss its applications and accuracies in detail.[6] In Section II, we present the derivation of the W. W. method which takes into account the target form factors. The method is then applied, in Section III, to derive simple formulas for the pair production of particles with spin 0, $\frac{1}{2}$, and 1. Also in Section III, we present numerical comparisons of calculations for pair production of spin $\frac{1}{2}$ particles using our W. W. method and the Born approximation. It is interesting to observe that for calculation of $d\sigma/d\Omega\,dp$, our W. W. method works better for targets with form factors which go to zero as $t \longrightarrow \infty$ (such as nuclear form factors) than those with form factors which go to constant as $t \longrightarrow \infty$ (such as atomic form factors). The reason is that in the W. W. approximation for the process $\gamma + Z \longrightarrow \ell^+ + \ell^-$ + anything, the cross section is a product of pseudo-photon flux mentioned above times the cross section for $\gamma + \gamma \longrightarrow \ell^+ + \ell^-$, with both initial γ's on the mass shell. The small correction due to one photon being off the mass shell is suppressed if the target form factor decreases rapidly with t. As a result, the W. W. approximation works better for the pair production of muons and heavy leptons than for electrons when calculating $d\sigma/d\Omega\,dp$ near the forward angle. Even for electron pair production the result of the W. W. method does not differ greatly from the correct value (see Fig. 5) in estimating $d\sigma/d\Omega\,dp$. This slight difference disappears completely after the angular integration. Thus we obtain an expression for $d\sigma/dp$ which is <u>identical</u> to the expression obtained by Bethe-Heitler[7] for production in the screened nuclear field and the one obtained by

Wheeler-Lamb[8] for production in the screened electron field. In Appendix A, the kinematics near minimum momentum transfer are discussed. In Appendix B, we use an infinite momentum frame to derive a key formula, (II.15), which was used in the derivation of the W. W. approximation. Appendix C derives the formula for the pseudo-photon beam intensity and discusses that concept. In Appendix D, we generalize the results of Section II to treat the energy-angle distribution of a particle b in an arbitrary one-photon-exchange process $a + P_i \longrightarrow b + c + P_f$ in terms of W_1 and W_2 of the target and the differential cross section for the real process $a + \gamma \longrightarrow b + c$. This generalization enables one to use our W. W. method to calculate processes such as $\nu + Z \longrightarrow \mu + \omega +$ anything, $\ell + Z \longrightarrow \ell + Z + \gamma +$ anything, etc.

Let us now give in the following a brief survey of the literature on this subject. The derivation of the W. W. method using the covariant perturbation theory was first given by Dalitz and Yennie,[9] but their main interest was the pseudo-photon flux of the electron in the electroproduction experiment. Later, Pomeranchuk and Schmuskevitch[10] considered a more general problem similar to ours but they did not investigate in detail the effect of the form factors of the target. Also, in their derivation, the rest frame of the incident particle a was used, hence it is inapplicable to the case where a is massless unless one derives the bremsstrahlung formula first and then uses the substitution rule. Gribov et al.[11] also gave the covariant deriviation of the W. W. formula, but their derivation applies only to the calculation of the total cross section, not of the energy angle distribution. They also ignored the effects of the target form factors. Überall[12] applied this method to calculate $\nu + P_i \longrightarrow W + \mu + P_f$. Next Jurisic and Stodolsky[13] derived this method in the coordinate system where the sum of the momenta of the produced particles is zero, thus generalizing the Pomeranchuk[10] et al.'s results to the case where the mass of the incoming particle is zero.

None of the above authors considered the effect of the form factors except Überall[12] who calculated the coherent production from the nucleus using form factors with polynomial approximation. Also, a detailed estimate of errors involved in the W. W. calculations was so far lacking in the literature. In order to do this, one has to know the exact value of the cross sections and this was done[14] in 1972. With this result at hand, Kim and Tsai[6] generalized the method of Gribov[11] et al. to include the case of the differential cross sections and also the effect of the form factors. It turned out that the resulting formula reproduces the exact values quite well. Since then we have applied the method to the pair productions of the W bosons and also rederived our results with a simpler method which exhibits more clearly the physics involved and which, we believe, puts earlier derivations in a better perspective. In this paper, we are mainly concerned with these recent development.

When one or more of a, b, and c in the reaction $a+P_i \rightarrow b+c+P_f$ are the strongly interacting particles, e.g., pions, one can relate the pion decay rates or pion-pion phase shift to the Coulomb production from the nucleus.[10,13] This so-called "Primakoff effect" will not be discussed here. Recently the W.W. method has been applied to the calculation[15] of $e^{\pm}+e^{-} \rightarrow e^{\pm}+e^{-}+$anything. We shall not consider this process here either.

As far as the W. W. method is concerned, Eq. (D. 1) in Appendix D is the main result of our paper. However, there are many other useful formulas in this paper, which can be listed as follows:

1. The energy-angle distribution of an electron in electron pair production near the forward direction is given by Eq. (III. 17) with X given by Eq. (III. 19). This formula is not as good as the more carefully derived formula given by Eq. (III. 1) of Paper A, but the discrepancy is small.
2. The energy-angle distribution of a muon in the muon pair production near the forward direction is given by Eq. (III. 17) with X given by Eq. (III. 23) if the

target is a nucleus other than a proton. If the target is a proton, X given by Eq. (III.27) must be used. Since the minimum momentum transfer involved t_{min}, given by Eq. (A.5) for muon production, is not very large compared with the inverse of the nuclear radius squared, we need to consider only the elastic form factors. Therefore, these formulas are very good approximations except at angles much larger than one characteristic angle.

3. Equation (III.17) can also be used to calculate the production of heavy leptons or the production of an electron or muon at angles much larger than one characteristic angle, provided the inequalities given by Eq. (A.2) are satisfied and appropriate form factors given in Section II of Paper A[14] are employed to calculate X defined by Eq. (II.19).

4. Equation (III.16) gives the pair production cross section of spin 0 particles. Since there are no nonstrongly interacting particles with zero spin, this equation is of academic interest only.

5. Equation (III.18) gives the W. W. version of the pair production of $W^{\pm}$ bosons with an arbitrary magnetic moment. This equation is much simpler than the result[16] obtained from the Born approximation which involves triple integrations of an integrand consisting of several thousand terms. We have not made numerical comparisons of this formula with the Born approximation, but Eq. (III.18) shows very transparently the gross features of the cross section, especially its dependence on the magnetic moment, angle, energy, and mass.

II. Derivation of W.W. Formula

A. Kinematics and Gauge Invariance

For concreteness let us consider pair production of charged particles by a photon shown in Fig. 1. The treatment is generalized to other processes in Appendix D. In the Born approximation the cross section in Fig. 1 can be written as

$$d\sigma = \frac{e^6}{(2\pi)^5} \quad \frac{1}{4k} \quad \frac{d^3p}{E} \int \frac{d^3p_+}{E_+} \quad \frac{1}{t^2} \; L^{\mu\nu} \; W_{\mu\nu} \tag{II. 1}$$

where the tensor $W_{\mu\nu}$ can be written in terms of the usual W_1 and W_2 in the inelastic electron scattering as:

$$\begin{aligned} W_{\mu\nu} &= M_i^{-2}(P_{i\mu} - q_\mu(q \cdot P_i)/q^2)\,(P_{i\nu} - q_\nu\,(q \cdot P_i)/q^2)\,W_2 \\ &\quad - (g_{\mu\nu} - q_\mu q_\nu/q^2)\,W_1. \end{aligned} \tag{II. 2}$$

$L_{\mu\nu}$ is a gauge invariant tensor. In our case it is symmetric in μ and ν. Four such independent tensors can be constructed out of three independent vectors p, p_+ and q and $g_{\mu\nu}$. Because we are interested in the limit $t \rightarrow 0$ and we know that $L_{\mu\nu}$ is not singular there, it is convenient to choose each of them free of kinematical singularity at $t = 0$. We will choose the following set:

$$L_{1\mu\nu} = (q^2/p \cdot q)\, p_\mu\, p_\nu + (p \cdot q) g_{\mu\nu} - p_\mu q_\nu - p_\nu q_\mu,$$

$$L_{2\mu\nu} = (q^2/p_+ \cdot q)\, p_{+\mu} p_{+\nu} + (p_+ \cdot q) g_{\mu\nu} - p_{+\mu} q_\nu - p_{+\nu} q_\mu,$$

$$L_{3\mu\nu} = [k \cdot p \;\, p_\mu - k \cdot p_+ p_{+\mu} + \frac{q_\mu}{2} (k \cdot p - k \cdot p_+)] \;\, [\mu \rightarrow \nu]$$

and

$$L_{4\mu\nu} = q^2 g_{\mu\nu} - q_\mu q_\nu\,. \tag{II. 3}$$

$L_{\mu\nu}$ can then be written as

$$L_{\mu\nu} = \sum_{j=1}^{4} T_j L_{j\,\mu\nu}, \tag{II.4}$$

where the T_j's are invariant functions which depend on the spins of the particles produced. For the production of a pair of spin $\frac{1}{2}$ particles with no anomalous magnetic moment, they are

$$T_1 = \frac{k\cdot p_+ - \frac{1}{2}t}{(k\cdot p)\ (k\cdot p_+)}, \qquad T_2 = \frac{k\cdot p - \frac{1}{2}t}{(k\cdot p)\ (k\cdot p_+)}, \qquad T_3 = \frac{2m^2}{(k\cdot p)^2 (k\cdot p_+)^2}$$

and

$$T_4 = \frac{m^2}{2}\left(\frac{1}{k\cdot p} + \frac{1}{k\cdot p_+}\right)^2. \tag{II.5}$$

The integration with respect to the phase space of the unobserved particle p_+ can be carried out in the rest frame of $u = p_+ + P_f$ as shown in Fig. 2. In this frame the integration can be cast into a convenient form[17]

$$I \equiv \int \frac{d^3p_+}{E_+}\,\frac{1}{t^2}\left(L^{\mu\nu}W_{\mu\nu}\right) = \frac{1}{4M_i\,|\vec{k}-\vec{p}|}\int_{t_{min}}^{t_{max}} \frac{dt}{t^2}\int_{M_i^2}^{(u-m)^2} dM_f^2 \int_0^{2\pi} d\phi\left(L^{\mu\nu}W_{\mu\nu}\right), \tag{II.6}$$

where (see paper A)

$$u = [k+P_i - p)^2]^{\frac{1}{2}} = [M_i^{\ 2} + m^2 + 2M_i(k-E) - 2k\cdot p]^{\frac{1}{2}}, \tag{II.7}$$

and[4]

$$t_{min} = -(k-p)^2 - m^2 + 2E_{+s}(k_s - E_s) - 2p_{+s}P_{is}, \qquad \text{(II.8)}$$

$$E_{+s} = (p_+ \cdot u)/u = (u^2 + m^2 - M_f^2)/(2u),$$

$$k_s = (k \cdot u)/u = (kM_i - k \cdot p)/u,$$

$$E_s = (p \cdot u)/u = (k \cdot p + M_i E - m^2)/u,$$

$$p_{+s} = (E_{+s}^2 - m^2)^{\frac{1}{2}}$$

$$P_{is} = M_i(k^2 + p^2 - 2kp\cos\theta)^{\frac{1}{2}}/u$$

and

$$t_{max} \approx 4p_{+s}P_{is}. \qquad \text{(II.9)}$$

The above expression for t_{min} is extremely complicated. We need an approximate formula valid when the energy is large and the angle is small. In the rest frame of u(u-frame), the minimum momentum transfer occurs when $\theta_+ = 0$. Since the laboratory system can be obtained by a Lorentz transformation along $\vec{P}_{is}$ from this frame, we conclude that the minimum momentum transfer also occurs in the laboratory system when $\vec{p}_+$ is parallel to $\vec{k} - \vec{p}$. Using this fact we show in Appendix A, that t_{min} is given approximately by

$$t_{min} \approx t'_{min} + 2\Delta(t'_{min})^{\frac{1}{2}}, \qquad \text{(II.10)}$$

where

$$\Delta = (M_f^2 - M_i^2)/(2M_i)$$

and

$$(t'_{min})^{\frac{1}{2}} = k \cdot p/(k - E). \qquad \text{(II.11)}$$

It is also shown that at $t = t_{min}$, we have

$$(q_z - q_0) \approx (t'_{min})^{\frac{1}{2}} \qquad \text{and} \qquad q_0 \approx \Delta \ . \tag{II.12}$$

Near $t = t_{min}$, the increase of t is given by the purely transverse part:

$$t - t_{min} \approx q_x^{\ 2} + q_y^{\ 2} \equiv q_{\perp}^2 \ . \tag{II.13}$$

Comparing Eq. (II.8) with the approximate expression (II.10), we see one of the reasons why it is sometimes impossible to use the computer to calculate the cross sections. The largest terms in Eq. (II.8) is of order k^2 while (II.10) gives an expression of order m^4/k^2 when $\Delta = 0$. Hence there occurs a tremendous number of cancellations among different terms each of which is numerically large. However, there is another even more important kind of cancellation taking place. This is due to the gauge invariance. Let us proceed to study this effect.

Because of gauge invariance, i.e., $q_\mu L_{\mu\nu} = 0$, we may write the term proportional to W_2 as

$$L^{\mu\nu} P_{i\mu} P_{i\nu} / M_i^{\ 2} = L^{\mu\nu} [P_{i\mu} + q_\mu M_i/(q_z - q_0)] \, [P_{i\nu} + q_\nu M_i/(q_z - q_0)] / M_i^{\ 2} \tag{II.14}$$

$$= \frac{q_z^{\ 2}}{(q_z - q_0)^2} \, (L_{00} + L_{zz} - 2 L_{0z}) + \frac{q_{\perp}^{\ 2}}{(q_z - q_0)^2} \, (\cos^2\varphi \, L_{xx} + \sin^2\varphi \, L_{yy}) . \tag{II.15}$$

Here we have used the fact that $P_i = M_i$ in lab. and $q_x = q_{\perp} \cos\varphi$, $q_y = q_{\perp} \sin\varphi$ with φ = azimuthal angle. Because of the t^2 term in the denominator of (II.6) (the photon propagator), most of the contribution to the integral comes from the region $t \approx t_{min}$. Hence, we can use (II. 12) and (II. 13) to get

$$L^{\mu\nu} P_{i\mu} P_{i\nu} / M_i^2 \approx \frac{t_{min} + \Delta^2}{t'_{min}} (L_{00} + L_{zz} - 2 L_{0z})$$

$$+ \frac{t - t_{min}}{t'_{min}} (\cos^2 \varphi \, L_{xx} + \sin^2 \varphi \, L_{yy}). \tag{II. 16}$$

The identity (II. 14) is true but mysterious. In Appendix B a more physical way of deriving this identity is given.

The Weizsacker-Williams approximation consists of two further approximation of (II. 16): First the term $L_{00} + L_{zz} - 2 L_{0z}$ is ignored and second L_{xx} and L_{yy} are evaluated at $t = t_{min}$, in which case L_{xx} and L_{yy} become independent of φ. Hence the resulting W.W. approximation is

$$\frac{1}{2\pi} \int_0^{2\pi} d\varphi \; L^{\mu\nu} P_{i\mu} P_{i\nu} / M_i^2 \approx \left(\frac{t - t_{min}}{t'_{min}}\right) \frac{1}{2} (L_{xx} + L_{yy})_{t=t_{min}}$$

$$\equiv \left(\frac{t - t_{min}}{t'_{min}}\right) \left(- \frac{1}{2} g_{\mu\nu} L^{\mu\nu}\right)_{t=t_{min}}, \tag{II. 17}$$

which is obviously covariant.[18] The validity of the above mentioned approximation will be discussed in detail in the next subsection but first let us observe that L_{00}, L_{0z} and L_{zz} are all of order E^2, so there must be huge cancellations taking place in order for us to neglect the term $L_{00} + L_{zz} - 2 L_{0z}$. This fact, combined with the purely kinematical cancellations mentioned before, makes the computer calculation very dangerous if proper care is not exercised when dealing with the Born approximations.

B. W.W. Approximation

Substituting (II.17) into (II.6) and (II.1), we obtain the desired energy-angle distribution of the particle p in the W.W. approximation:[19]

$$\left(\frac{d\sigma}{d\Omega\, dp}\right)_{W.W.} = \frac{\alpha^3}{2\pi} \frac{E}{(k - E)\, k\, t'_{min}} \left(- \frac{1}{2} g_{\mu\nu} L^{\mu\nu}\right)_{t=t_{min}} X, \qquad (II.18)$$

where

$$X = \frac{1}{2 M_i} \int_{t_{min}}^{t_{up}} \frac{dt}{t^2} \int_{M_i^2}^{(u-m)^2} dM_f^2 \,[(t - t_{min}) W_2 + 2 t'_{min} W_1]. \qquad (II.19)$$

In practice the condition $t = t_{min}$ in Eq. (II.18) is replaced by $t = 0$, in which case $g_{\mu\nu} L^{\mu\nu}$ is related to the $\gamma + \gamma \rightarrow p + p_+$ process (see Appendix C). Notice, however, that the former condition ($t = t_{min}$) specifies the kinematics, $\vec{p}_+ // \vec{k} - \vec{p}$ (see the discussion preceding Eq. (II.10)), whereas the latter ($t = 0$) does not.

In the following, we shall discuss the approximations leading to (II.18) more carefully. We shall also discuss the upper limit t_{up} of the t integral toward the end of this section.

i) Cancellation of Longitudinal and Scalar Parts

Let us understand why the longitudinal-scalar contribution $D \equiv L_{00} + L_{zz} - 2 L_{0z}$ is negligible compared with the transverse contributions $T \equiv \frac{1}{2}(L_{xx} + L_{yy})$. This can be done easily with the aid of the explicit tensor decomposition given in Eq. (II.3) and consider D and T in the rest frame of $s \equiv p + p_+$. The rest frame of s can be obtained from the laboratory system by boosting along the direction of the incident photon. In the laboratory frame, we have used the direction of $\vec{k} - \vec{p}$ as the z axis, which is different from the incident photon direction. However the angle between them is small by assumption, hence it can be ignored in the Lorentz transformation. With this approximation we have $L_{xx} \approx L'_{xx}$, $L_{yy} \approx L'_{yy}$ and $L_{00} + L_{zz} - 2L_{0z} \approx L'_{00}/\gamma^2$, where γ is the usual γ for the Lorentz transformation and is given by $\gamma = (E + E_+)/s \approx k/s$. Now L'_{xx}, L'_{yy} and L'_{00} are roughly of the same magnitude because $L'_{\mu\nu}$ is constructed from the vectors p', p'_+ and q' together with $g_{\mu\nu}$. The components of all these vectors have magnitudes equal to $s/2$ or less and the coefficients of $g_{\mu\nu}$ have magnitude roughly equal to s^2. Thus we conclude that $L_{00} + L_{zz} - 2L_{0z}$ becomes negligible compared with $\frac{1}{2}(L_{xx} + L_{yy})$ when k^2/s^2 is large. $L_{00} + L_{zz} - 2L_{0z}$ is equal to the square of the difference between the scalar and the longitudinal matrix elements. The argument given above shows that these two matrix elements almost cancel each other when the laboratory energy of the incident particle is much larger than the invariant mass s of the produced particles. From Eq. (II.16) we see that when $\Delta^2 \gtrsim t'_{min}$, we need one extra condition for our approximation to be good, namely $\Delta \ll k$. This inequality comes about because the range of the t integration is roughly from t_{min} to s^2 as will be shown later. Hence in order to drop the term $L_{00} + L_{zz} - 2L_{0z}$ in Eq. (II.16) we must have $\Delta^2/\gamma^2 \ll s^2$ which is equivalent to $\Delta \ll k$. In summary, the necessary and sufficient conditions for ignoring the longitudinal-scalar term in Eq. (II.16) are

$$k \gg (s_{t=t_{min}} \text{ and } \Delta) \; . \tag{II.20}$$

These conditions are weaker than the conditions required to obtain the approximate expression for t_{min} given by (A.2).

ii) We give here another method which also leads to Eq. (II.17) without using the trick employed in Eq. (II.14). This method is more straightforward but tedius. Hence we only sketch the outline. Let us first define

$$A_j(t) \equiv \frac{1}{2\pi} \int_0^{2\pi} d\varphi \, L_{j00}(t, \varphi), \tag{II.21}$$

where L_{j00} is the 00 component of the tensors defined in Eq. (II.3). Since this quantity is nonsingular near $t = t_{min}$, it must have a Taylor series expansion:

$$A_j(t) = A_j(t_{min}) + \left(\frac{\partial A_j}{\partial t}\right)_{t=t_{min}} (t - t_{min}) + \cdots . \tag{II.22}$$

With the help of the expressions for $L_{j\mu\nu}$ given in Eq. (II.3) one can show that $A_j(t_{min}) \equiv L_{j00}(t_{min})$ is indeed negligible and also

$$\left(\frac{\partial A_j}{\partial t}\right)_{t=t_{min}} = \left(- \frac{g_{\mu\nu}}{2} L_j^{\mu\nu}\right)_{t=t_{min}} / t'_{min} \ . \tag{II.23}$$

The advantage of this method is that it enables us to see why the expression[18] $(t - t_{min})$ occurs in Eq. (II.17) from the analyticity of functions. One is tempted to ask whether a better approximation can be obtained by making the next order expansion and include the term

$$\frac{1}{2!} \left(\frac{\partial^2 A_j}{\partial t^2}\right)_{t=t_{min}} (t - t_{min})^2 \tag{II.24}$$

in Eq. (II.17). However in order to be consistent we also have to expand T_j in power series:

$$T_j(t) = T_j(t_{min}) + \left(\frac{\partial T_j}{\partial t}\right)_{t=t_{min}} (t - t_{min}) , \tag{II.25}$$

and keep the second term as well as the first. However keeping the second term is against the spirit of the W.W. approximation because this term depends upon the off-shell behavior of the interaction $\gamma + \gamma \rightarrow p + p_+$.

iii) Cutoff Function

To complete our derivation, we must discuss the upper limit t_{up} of the t integration in (II. 19). We need a t_{up} which is usually much smaller than t_{max} because if one examines carefully the exact expressions, the integrand is much smaller than the approximate integrand in (II. 19) when $t \gtrsim m^2$. Therefore, one must multiply the integrand by a cutoff function C(t) which is unity for t near t_{min} and a rapidly decreasing function for $t \gtrsim m^2$. Two more complications arise for the function C(t). First its form depends upon the spins of the produced particles. Second and worse, it is actually different for different j's in (II. 4). For the case of the photoproduction of lepton pairs (see Eq. (II. 5)), C(t) is roughly $[1 + t/(m^2(1 + \ell))]^{-1}$ or $[1 + t/(m^2(1 + \ell)^2)]^{-1}$ depending upon j. Notice that this gives a cutoff of order m^2 which agrees with the classical estimate based on the uncertainty principle.

We shall take the point of view that C(t) is an empirical function to be determined by the criteria that it gives the best numerical results. The form of C(t) is expected to be most crucial when the form factor does not fall off rapidly at large t. The atomic screening form factor to be discussed in the next section is such a case. It was found that if we take $C(t) = \theta(m^2(1+\ell)^2 - t)$, i.e., $t_{up} = m^2(1 + \ell)^2$, then our result reproduces the Bethe-Heitler's formula[7] for $d\sigma/dp$. For nuclear or nucleon form factors which die off rapidly at large t, whether t_{up} is chosen to be $m^2(1 + \ell)^2$ or ∞, it hardly makes any numerical difference. In the application to be discussed in the following section, we use $t_{up} = m^2(1 + \ell)^2$. As for the meson production part of W_1 and W_2, the form of C(t) is crucial because the form factor is slowly varying. The choice $C(t) = (1 + t/(m^2(1 + \ell)))^{-1}$ seems to give the best result for this case.

The concept of a pseudo photon beam is discussed in Appendix C where we have also put Eq. (II. 18) in a covariant and more compact form.

III. Applications

In this section, we apply our W.W. formula (II. 18) to the pair productions of particles of spin 0, $\frac{1}{2}$ and 1. We also discuss the accuracy of the W. W. method by comparing the numerical results of the W.W. approximations with those obtained by the Born approximation given in Paper A for the spin $\frac{1}{2}$ case. From Eq. (II. 18) we see that the problem can be divided into the evaluation of two quantities:

1. The quantity χ which involves only the properties of the target except in the upper and lower limits of the integrations and

2. The quantity

$$F_s \equiv \left(-\tfrac{1}{2} L_{\mu\nu} g^{\mu\nu}\right)_{t=t_{min}} \qquad \text{(III. 1)}$$

which involves only the particles produced.

The subscript s refers to the spin of the produced particles. $-L_{\mu\nu} g^{\mu\nu}$ is in general a function of t, $k \cdot p$ and $k \cdot p_+$, besides the mass, magnetic moment and quadrupole moment etc., of the particles produced. As explained previously, the subscript $t=t_{min}$ is just our convention that in evaluating $(-L_{\mu\nu} g^{\mu\nu})$ not only t should be set equal to zero, but also $k \cdot p_+$ should be evaluated at $t=t_{min}$, yielding the relation (A. 12). In general F_s is symmetric with respect to the interchange $p \longleftrightarrow p_+$, hence it is symmetric with respect to $k \cdot p \longleftrightarrow k \cdot p_+$. Let us therefore define two new variables A and B:

$$A \equiv k \cdot p + k \cdot p_+ \approx m^2(1+\ell)/[2x(1-x)] \qquad \text{(III. 2)}$$

and

$$B \equiv (k \cdot p)(k \cdot p_+) \approx m^4(1+\ell)^2/[4x(1-x)], \qquad \text{(III. 3)}$$

where $\ell = (\theta p/m)^2$ and $x = E/k$.

The expressions for F_s corresponding to spin 0, spin $\frac{1}{2}$ and spin 1 particles can be written in terms of A and B as follows:

Spin 0 Case

$$F_0 = m^4A^2/B^2 - 2m^2A/B + 2 \qquad \text{(III. 4)}$$

Spin $\frac{1}{2}$ Case (no anomalous magnetic moment)

From Eq. (II. 3) and (II. 4) we obtain,

$$F_{\frac{1}{2}} = -m^4A^2/B^2 + 2m^2A/B - 2 + A^2/B. \qquad \text{(III. 5)}$$

Spin 1 Case

Using the Feynman rules given by Lee and Yang,[20] we obtain

$$\begin{aligned} F_1 = {} & m^4A^2/B^2 - 2m^2A/B + 2 + \frac{1}{48}(\mu^4 + 8\mu^3 + 8\mu^2 - 32\mu + 16)A^4/B^2 \\ & - \frac{1}{24}(-\mu^4 + 40\mu^2 - 48\mu + 16)A^2/B \\ & - \frac{(\mu-2)^2}{24m^2}\left\{(-3\mu^2 - 8\mu + 4)A^3/B + 2(\mu^2 + 12\mu + 4)A\right\} \\ & - \frac{(\mu-2)^2}{24m^4}\left\{(-7\mu^2 + 12\mu - 12)A^2 + 2(\mu-2)^2B\right\}, \end{aligned} \qquad \text{(III. 6)}$$

where μ is the magnetic moment of the vector boson in unit of $e/(2m)$. μ is equal to $1 + \kappa$ of Lee and Yang.[20] In Lee and Yang's version of the quantum electrodynamics of the vector boson the electric quadrupole moment is not arbitrary but is given by $-e\kappa/m^2$. In Weinberg's theory[21] of electromagnetic and weak interactions, $\gamma W^- W^+$ has the Yang-Mill type[22] of coupling which corresponds to $\kappa = 1$, hence $\mu = 2$. When $\mu = 2$, terms proportional to m^{-2} and m^{-4} disappear from Eq. (III. 6). Equation (III. 6) was derived with the aid of the algebraic computer program "Reduce" written by A. C. Hearn.[23] It should be mentioned that the expression $-\frac{1}{2} g_{\mu\nu} L^{\mu\nu}$ with $t \neq 0$ is about two orders of magnitude more complicated than Eq. (III.6).

We note that in the limit $k \to 0$, the three expressions for F_s given above coincide, except that the sign for $F_{\frac{1}{2}}$ is different from that of F_0 and F_1. This is due to the fact that in the Compton scattering the cross section must be given by the Thomson cross section irrespective of the spin of the target when $k \to 0$. The minus sign for $F_{\frac{1}{2}}$ is associated with $\bar{v}v = -1$ for the hole state. The angular distributions of $\gamma + \gamma \to p_- + p_+$ in the center of mass system can be obtained from Eqs. (C.2) and (II.4, 5 and 6):

$$\left(\frac{d\sigma^s_{\gamma\gamma \to p_+ + p_-}}{d \cos\theta}\right)_{cm} = \frac{\pi\alpha^2\beta}{4k^2_{cm}} F_s , \qquad \text{(III.7)}$$

where $\beta^2 = 1 - m^2/k^2_{cm}$, and A and B in F_s are given by $A = 2k^2_{cm}$ and $B = k^4_{cm}(1 - \beta^2 \cos^2\theta_{cm})$. In Fig. 3 we show these angular distributions at $k_{cm} = 3$ GeV and $m = 2$ GeV. The angular distribution must be symmetric with respect to $\theta_{cm} = \pi/2$, therefore only the distributions from $\cos\theta = 1$ to $\cos\theta = 0$ are shown. 0^o has more cross sections than 90^o in all cases. This is especially true for the case of spin 1 with $\mu = 2$, whereas for spin 0 and spin 1 with $\mu = 0$ they are almost flat.

Integrating Eq. (III.7) with respect to $\cos\theta$, we obtain the total cross section for $\gamma + \gamma \to p_+ + p$ as follows:

Spin 0

$$\sigma^o = \frac{\pi\alpha^2\beta}{k^2_{cm}} \left[\frac{m^2}{k^2_{cm}} + 1 + \frac{m^2}{k^2_{cm}\beta}\left(\frac{1}{2}\ \frac{m^2}{k^2} - 1\right) \ln \frac{1+\beta}{1-\beta}\right]. \qquad \text{(III.8)}$$

Spin $\frac{1}{2}$

$$\sigma^{\frac{1}{2}} = -\sigma^{o} + \frac{\pi \alpha^2}{k_{cm}^2} \ln \frac{1+\beta}{1-\beta} . \tag{III.9}$$

Spin 1

$$\sigma^{1} = \sigma^{o} + \frac{\pi \alpha^2 \beta}{12 k_{cm}^2} \left[(\mu^4 + 4\mu^3 - 16\mu^2 + 8\mu) \frac{1}{\beta} \ln \frac{1+\beta}{1-\beta} \right.$$
$$+ (\mu^4 + 8\mu^3 + 8\mu^2 - 32\mu + 16) \frac{k_{cm}^2}{m^2}$$
$$- \frac{(\mu-2)^2 k_{cm}^2}{m^2} \left\{ (-3\mu^2 - 8\mu + 4) \frac{1}{\beta} \ln \frac{1+\beta}{1-\beta} + (\mu^2 + 12\mu + 4) \right.$$
$$\left.\left. + (-7\mu^2 + 12\mu - 12) \frac{k_{cm}^2}{m^2} + (\mu-2)^2 (1 - \frac{1}{3}\beta^2) \frac{k_{cm}^2}{2m^2} \right\} \right] . \tag{III.10}$$

The total cross sections are plotted in Fig. 4. m is fixed at 2 GeV. σ^{o} and $\sigma^{\frac{1}{2}}$ decrease when k is increased except near the threshold. For the spin 1 particles, the cross sections increase as the energy is increased. The rate of increase of the cross section with energy is more pronounced for $\mu = 0$ and $\mu = 1$ than that for $\mu = 2$. Let us consider the range of k_{cm} relevant to pair production from a nuclear target. From $A = 2k_{cm}^2$ and Eq. (III.2) we have

$$k_{cm} \approx m(1+\ell)^{\frac{1}{2}} / [4x(1-x)]^{\frac{1}{2}} . \tag{III.11}$$

Now most of the pair production of heavy particles occurs within $\ell < 1$. We are also not particularly interested in $x \to 0$ or $(1-x) \to 0$. Hence the range of k_{cm} of interest to us is not very large, roughly $m < k_{cm} < 3$ or 4 m. From the ratios of the cross sections given in Fig. 4 in this range of k_{cm}, we can estimate the cross sections for pair production of particles with spin 0 and spin 1 from the total cross sections for the production of spin $\frac{1}{2}$ particles given in Paper A. Qualitatively we obtain:

$$\sigma^{0} \approx \sigma^{\frac{1}{2}}/(3 \text{ to } 4), \tag{III. 12}$$

$$\sigma^{1}\ (\mu = 0) = \sigma^{\frac{1}{2}}(10 \text{ to } 20), \tag{III. 13}$$

$$\sigma^{1}\ (\mu = 1) = \sigma^{\frac{1}{2}}(3 \text{ to } 5), \tag{III. 14}$$

and

$$\sigma^{1}(\mu = 2) = \sigma^{\frac{1}{2}}(8 \text{ to } 10). \tag{III. 15}$$

Substituting Eqs. (III. 4, 5 and 6) into Eq. (II. 18), we obtain $d\sigma/d\Omega dp$ for the reaction $\gamma + Z \rightarrow p + p_{+}$ + anything for particles with spin 0, $\frac{1}{2}$ and 1 as follows:

$$\left(\frac{d\sigma}{d\Omega dp}\right)_{\substack{WW \\ \text{spin } 0}} = \frac{2\alpha^3}{\pi k}\left(\frac{E^2}{m^4}\right)\left[\frac{2x(1-x)}{(1+\ell)^2} - \frac{4x(1-x)\ell}{(1+\ell)^4}\right] \chi\ , \tag{III. 16}$$

$$\left(\frac{d\sigma}{d\Omega dp}\right)_{\substack{WW \\ \text{spin } \frac{1}{2}}} = \frac{2\alpha^3}{\pi k}\left(\frac{E^2}{m^4}\right)\left[\frac{2x^2 - 2x + 1}{(1+\ell)^2} + \frac{4x(1-x)\ell}{(1+\ell)^4}\right] \chi\ , \tag{III. 17}$$

and

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega dp}\right)_{\substack{WW \\ \text{spin } 1}} = \frac{2\alpha^3}{\pi k}\left(\frac{E^2}{m^4}\right)\Bigg[& \frac{2x(1-x)}{(1+\ell)^2} - \frac{4x(1-x)\ell}{(1+\ell)^4} \\ & + \frac{1}{48}(\mu^4 + 8\mu^3 + 8\mu^2 - 32\mu + 16)\ \frac{1}{(1+\ell)^2\, x(1-x)} \\ & - \frac{1}{24}(-\mu^4 + 40\mu^2 - 48\mu + 16)\ \frac{1}{(1+\ell)^2} \\ & - \frac{(\mu-2)^2}{48x(1-x)(1+\ell)}\left\{(-3\mu^2 - 8\mu + 4) + 2(\mu^2 + 12\mu + 4)\ x(1-x)\right\} \\ & - \frac{(\mu-2)^2}{96\, x(1-x)}\left\{(-7\mu^2 + 12\mu - 12) + 2(\mu-2)^2 x(1-x)\right\}\Bigg]\ \chi\ . \end{aligned} \tag{III. 18}$$

We observe that $(d\sigma/d\Omega\, dp)$ for spin 0 and spin $\frac{1}{2}$ become independent of m when the transverse momentum $p\theta$ is much larger than m, i.e., $\ell = \gamma^2\theta^2 >> 1$. For the spin 1 case this would be true only when $\mu = 2$. When $\mu \neq 2$, the terms in the last curly bracket are independent of angle except through t_{min} in X. The origin of this peculiar angular behavior comes from the energy dependence of the cross section for $\gamma + \gamma \rightarrow W^+ + W^-$ given by Eq. (III.10). When k_{cm} is large, $\sigma(\gamma + \gamma \rightarrow W^+ + W^-)$ is proportional to k^2_{cm}/m^4 if $\mu \neq 2$ and it is proportional to $1/m^2$ if $\mu = 2$. In order to see whether unitarity is violated at high energy for the interaction $\gamma + \gamma \rightarrow W^+ + W^-$, we have to decompose the matrix element into helicity amplitudes and project out each partial wave amplitude. This question is under investigation. It is very likely that the unitary bounds for partial waves are exceeded as $k_{cm} \rightarrow \infty$ similar to the interaction[24] $e^+ + e^- \rightarrow W^+ + W^-$. We conclude that if $W^{\pm}$ bosons exist and if they are produced electromagnetically via $\gamma + z \rightarrow W_+ + W_- +$ anything, both the total cross section and the angular distribution will be strongly dependent upon the magnetic moment.[16] This can be used to determine μ.

X involves the integration of W_1 and W_2 with respect to t and M_f^2. For many simple form factors given in Section II of Paper A, this integration can be carried out explicitly. In the following we give expressions for X corresponding to atomic, nuclear, nucleon and meson production form factors and discuss the accuracy of the W.W. approximations by comparing with the results obtained in Paper A for spin $\frac{1}{2}$ particles.

(i) Electron Pair Production (and Bremsstrahlung)

Most of the electrons and positrons are produced within a very small angle (a few units of m/E). For the production of electrons at such a small angle, the nuclear form factors can be ignored, but the atomic form factors must be taken into account. The atomic form factors have two parts: elastic and inelastic. The elastic form factor is a function of t only, but the inelastic form factor is a function of t and M_f^2. However, one can sum over all the excited states of the atom using the sum rule and obtain an inelastic form factor which is a function of t only. Since we are interested in the range of t much small compared with m_e^2, W_1 can be ignored. Thus we shall use the form factor of the form

$$W_2 = W_2^{el} + W_2^{inel} ,$$

where

$$W_2^{el}\left(q^2,\, M_f^2\right) = 2\, M_i \delta\left(M_f^2 - M_i^2\right) G_2^{el}(t)$$

$$W_2^{inel}\left(q^2,\, M_f^2\right) = 2\, M_i \delta\left(M_f^2 - M_i^2\right) G_2^{inel}(t)$$

where $G_2^{el}(t)$ and $G_2^{inel}(t)$ are normalized such that

$$G_2^{el}(\infty) = Z^2,\ G_2^{inel}(\infty) = Z,\ \text{and } G_2^{el}(0) = G_2^{inel}(0) = 0 .$$

We notice that both elastic and inelastic atomic form factors have t dependence which is exactly opposite to that of nuclear form factors. The elastic and inelastic form factors for a hydrogen atom are known exactly; they are

$$G_2^{el}(t) = \left(1 - F(t)\right)^2 ,$$

$$G_2^{inel}(t) = 1 - \left|F(t)\right|^2 ,$$

where

$$F = \left(\frac{t}{4\alpha^2 m_e^2} + 1\right)^{-2} .$$

For other light Z elements, the atomic form factors can be obtained from the Hartree-Foch method or some improved version of it. For high Z elements Thomas Fermi model of the atom is adequate.

It is shown in paper A that the correct energy angle distribution from an arbitrary atomic form factor can be written as

$$\frac{d\sigma}{d\Omega dp} = \frac{2\alpha^3}{\pi k}\left(\frac{E^2}{m^4}\right)\left\{\left[\frac{2x(1-x)}{(1+\ell)^2} - \frac{12\ell x(1-x)}{(1+\ell)^4}\right] G_2(\infty) + \left[\frac{2x^2-2x+1}{(1+\ell)^2} + \frac{4\ell x(1-x)}{(1+\ell)^4}\right]\left[\chi - 2Z^2 f((\alpha Z)^2)\right]\right\} , \qquad (III.19)$$

where $G_2(\infty) = G_2^{el}(\infty) + G_2^{inel}(\infty) = Z^2 + Z$,

$$\chi = \chi^{el} + \chi^{inel} = \int_{t_{min}}^{m^2(1+\ell)^2} \left[G_2^{el}(+) + G_2^{inel}(+)\right] \frac{t-t_{min}}{t^2}\, dt , \qquad (III.20)$$

and $f((\alpha Z)^2)$ is the Coulomb correction due to Bethe and Maximon[25]:

$$f(z) = z \sum_{n=1} \frac{1}{n(n^2+z)} \approx 1.202\, z - 1.0369\, z^2 + 1.008\, z^3/(1+z)$$

where $z = (Z/137)^2$. The terms proportional to $G_2(\infty)$ and $f((\alpha Z)^2)$ can be regarded as the correction to the W.W. approximation which yields exactly the term proportional to χ. The terms proportional to $G_2(\infty)$ are due to the off-the-mass shell correction to the reaction $\gamma+\gamma \rightarrow e^+ + e^-$. This statement is based on the following observations:

1. We have also obtained a similar formula using a nuclear form factor for production of heavy particles and in this case we found that this term is missing if $m^2(1+\ell)^2$ is much larger than the inverse square of the nuclear radius. Thus when the form factor cuts off the high t events this term disappears.

2. According to the argument given in Section II, the contribution due to the longitudinal photon diminishes as the incident photon energy increases. But the term proportional to $G_2(\infty)$ does not have this property, hence it is not due to the longitudinal photon.

Equations (III. 19) and (III. 20) are probably the best justification for choosing the upper limit of t integration for χ to be $t_{up} = m^2(1+\ell)^2$. It is interesting to observe that after integration with respect to the solid angle, the term proportional to $G_2(\infty)$ in (III. 19) disappears. Thus the expression for $d\sigma/dp$ obtained from the W. W. approximation is <u>exactly</u> the same as that obtained from the lowest order Born approximation for <u>both nuclear and atomic</u> form factors.

The energy-angle distribution of bremsstrahlung can be obtained from that of pair production by the following substitution rule:

$$\frac{d\sigma_b}{d\Omega_k dk} = -\left(\frac{d\sigma_{pair}}{d\Omega dp}\right)_{\substack{k_\mu \to -k_\mu \\ p_\mu \to -p_\mu}} \frac{k^2 E}{p^3} \tag{III. 21}$$

$$= \frac{2\alpha^3}{k}\left(\frac{E^2}{m^4}\right)\left\{\left[\frac{2(y-1)}{(1+\ell)^2} - \frac{12\ell(y-1)}{(1+\ell)^4}\right] G_2(\infty) + \left[\frac{2-2y+y^2}{(1+\ell)^2} + \frac{4\ell(y-1)}{(1+\ell)^4}\right]\left[\chi - 2Z^2 f((\alpha Z)^2)\right]\right\} \tag{III. 21'}$$

where E is the incident electron, $y = k/E$, $\gamma = E/m$, $\ell = \gamma^2\theta_k^2$. χ, $G_2(\infty)$ and $f((\alpha Z)^2)$ are the same expressions as those in (III. 19). The minus sign in the right-hand side of (III. 21) comes from the fact that in the pair production there is an odd number of antiparticles, whose states are normalized such that $\bar{v}v = -1$. Integrating (III. 21') with respect to the solid angle, we see again that the term proportional to $G_2(\infty)$ vanishes, hence we conclude also that for the bremsstrahlung the expression for $d\sigma_b/dk$ obtained from the W. W. approximation is <u>exactly</u>

the same as that obtained from the lowest order Born approximation. The term proportional to $G_2(\infty)$ in (III.21') has an opposite sign from that in (III.19). For the pair production the coefficient of $G_2(\infty)$ is negative when $2+3^{1/2} > \ell > 2-3^{1/2}$ and positive otherwise, whereas for bremsstrahlung the opposite is true. In Fig. 5, we compare $d\sigma/d\Omega dp$ for pair production from Be atom using the W.W. approximation and the Born approximation. The difference is all due to the term proportional to $G_2(\infty)$. Since for the pair production of heavy particles ($m \geq 0.5$ GeV) we do not have term proportional to $G_2(\infty)$, the W.W. approximation for $d\sigma/d\Omega dp$ or $d\sigma/d\Omega_k dk$ actually works better for heavy particles than for electrons. For muons it is debatable whether $G_2(\infty)$ terms should be kept or not, because $m^2(1+\ell)^2$ is not in general much larger than the inverse square of the nuclear radius. Numerically if we replace $G_2(\infty)$ by $G_2(m^2(1+\ell)^2)$, we would get a better approximation.

For a hydrogen atom one can obtain analytical expressions for $d\sigma/d\Omega dp$, $d\sigma/dp$, $d\sigma/d\Omega_k dk$ and $d\sigma/dk$ using the form factors given previously. The results are given in Paper A. For other atoms, the results can be presented only numerically, except in the limits of complete screening and no screening. All these are reviewed in great detail in Paper A. In order to understand the qualitative features of the pair production and bremsstrahlung, let us use the simplest parameterizations of $G_2^{el}(t)$ and $G_2^{inel}(t)$:

$$G_2^{el}(t) = Z^2\left(\frac{a^2 t}{a^2 t+1}\right)^2$$

and

$$G_2^{inel}(t) = Z\left(\frac{a'^2 t}{a'^2 t+1}\right)^2 ,$$

where a (or a') is a parameter obtained by comparing the resultant expression for χ^{el} (or χ^{inel}) with that obtained from a more accurate form factor in the limit of complete screening as shown below. From these simple form factors, we obtain

$$\chi^{el} = Z^2 \left(\ell n \frac{a^2 m^2 (1+\ell)^2}{a^2 t'_{min} + 1} - 1 \right) \quad ,$$

$$\chi^{inel} = Z \left(\ell n \frac{a'^2 m^2 (1+\ell)^2}{a'^2 t'_{min} + 1} - 1 \right) \quad .$$

These two expressions reduce to

$$\chi^{el} = Z^2 \left[2 \ell n \frac{2k x(1-x)}{m} - 1 \right]$$

and

$$\chi^{inel} = Z \left[2 \ell n \frac{2k x(1-x)}{m} - 1 \right]$$

in the no screening limit ($a^2 t'_{min} >> 1$ and $a'^2 t'_{min} >> 1$) and

$$\chi^{el} = Z^2 \left[2 \ell n \, a m (1+\ell) - 1 \right] \quad ,$$

$$\chi^{inel} = Z \left[2 \ell n \, a' m (1+\ell) - 1 \right] \quad ,$$

in the complete screening limit ($a^2 t_{min} \ll 1$ and $a'^2 t'_{min} \ll 1$). The no screening limit corresponds to using the constant form factors $G_2^{el} = Z^2$, and $G_2^{inel} = Z$, hence any atomic form factor must give identical expressions for χ^{el}/Z^2 and χ^{inel}/Z in the no screening limit. In the complete screening limit, χ^{el} and χ^{inel} obtained from more complicated atomic form factors also have the functional forms given above. Thus, we can determine the constants a and a'. The results[14] are given below:

H(Z=1)	$a = 122.8/m_e$	$a' = 282.4/m_e$
He(Z=2)	$a = 90.8\ Z^{-1/3}/m_e$	$a' = 268.6\ Z^{-2/3}/m_e$
Thomas Fermi	$a = 111\ Z^{-1/3}/m_e$	$a' = 773\ Z^{-2/3}/m_e$

The simple expressions for χ^{el} and χ^{inel} are constructed so that they coincide with the correct expression in both the complete screening and no screening cases. In the intermediate screening case these expressions are 3% larger than the correct ones at most. The simple expressions for χ^{el} and χ^{inel} obtained above tell us the following: (1) the parameters "a and a' " roughly represent the radius of the atom and the distance between two neighboring electrons respectively. These two parameters represent the atomic dimensions and they appear in χ only as arguments of logarithms, therefore we can understand why χ^{el} and χ^{inel} obtained are so close to the real ones even though the form factors used are quite different from the correct ones. (2) We also notice that the angle, energy etc., of the particles appear also only inside the logarithm, hence χ is only mildly dependent upon the values of these kinematical parameters. In other words, the dependence on angles and energies in $d\sigma/d\Omega dp$ and $d\sigma/d\Omega_k dk$ are mostly determined by the cross sections of $\gamma+\gamma \rightarrow e^+ + e^-$ and $e+\gamma \rightarrow e+\gamma$ respectively. The use of simple form factors above is a generalization of work of Schiff[26] who first used this kind of form factor to simulate the elastic part of the Thomas-Fermi form factors and obtained an approximate expression for $d\sigma/d\Omega_k dk$. What we have shown then are essentially that (1) the method can be used for any atom, not just the Thomas-Fermi atom; (2) the method can be applied to inelastic as well as elastic; and (3) it can also be used in the calculation of the pair production. The accuracy of this generalized Schiff's approximation is not worse[14] than the case treated by Schiff.[26]

ii) Coherent Production from a Nucleus

When we produce muons or particles with heavier masses, the magnitude of t_{min} is such that the presence of the atomic electrons can be completely ignored but we have to take into account both the elastic and inelastic nuclear form factors. For muon production within a few characteristic angles (characteristic angle is defined as $\theta_c = m/p$), we need to consider only the elastic form factors of the nucleus. This part of the production is usually called the coherent production. A simple expression for χ can be obtained if we use the simple elastic form factor given by

$$W_2(\text{coherent}) = 2M_i\,\delta(M_f^2 - M_i^2)\,Z^2/(1 + t/d)^2 \quad , \tag{III.22}$$

where $d = 6/(1.2 \text{ fermi} \times A^{1/3})^2 = 0.164\,A^{-2/3}\,\text{GeV}^2$. Substituting these form factors into Eq. (II.19), we obtain

$$\chi(\text{coherent}) = Z^2 \left[(1 + 2b)\,\ell n\,\frac{1 + b^{-1}}{1 + c^{-1}} - (1 + \frac{b}{c})\,\frac{1 + 2c}{1 + c}\right], \tag{III.23}$$

where $b = t_{min}/d$ and $c = m^2(1 + \ell)^2/d$.

In Table I, we give the numerical results of the W.W. approximation using Eqs. (III.23) and (III.17) for a Be nucleus and various masses of leptons. The results from Paper A are also shown for comparison. We see that our W.W. formula reproduces the exact results very well in a wide kinematical region. One should note, however, that all of the entries of Table I satisfy the conditions of our W.W. approximations given by Eq. (A.2). How does the form factor affect the energy-angle distribution for the production of different heavy lepton masses? We show in Fig.6 $d\sigma^{W.W.}/d\sigma^{W.W.}_{d=\infty}$ as a function of $\gamma\theta$ for various lepton mass m. $d\sigma^{W.W.}$ is the cross section for the coherent production from Be using the formula just discussed and $d\sigma^{W.W.}_{d=\infty}$ is the same cross section with a unit form factor. We see that the effect of the form factor is less than 30% for the muon production whereas for the production of heavier particles the effect is larger.

Elastic Production from a Proton

The integration with respect to t can also be carried out when the form factors involved are the so called dipole form, popularly employed to describe the form factors of the proton and the neutron. Approximating $(1+\tau)^{-1}$ by $(1-\tau)$ and assuming $G_{mp} = G_{ep} \times 2.79$, we obtain:

$$W_{2p} = 2M_p\, \delta\,(M_f^2 - M_p^2)\, \frac{G_{ep}^2 + \tau G_{mp}^2}{1+\tau}$$

$$\approx 2M_p\, \delta\, (M_f^2 - M_p^2)\, G_{ep}^2 \left\{1 + \tau(\mu_p^2 - 1)\right\} \tag{III. 24}$$

and

$$W_{1p} = 2M_p\, \delta\,(M_f^2 - M_p^2)\tau\mu_p^2\, G_{ep}^2\,, \tag{III. 25}$$

where $\tau = t/(4M_p^2)$, $\mu_p = 2.79$ and $G_{ep} = (1 + t/.71)^{-2}$. Substituting W_{2p} and W_{1p} into Eq. (II. 19) we obtain

$$X = \int_{t_{min}}^{t_{up}} \frac{dt}{t^2}\, \frac{(t - t_{min})\left\{1 + \tau(\mu_p^2 - 1)\right\} + 2t'_{min}\, \tau\mu_p^2}{(1 + t/.71)^4} \tag{III. 26}$$

This integration can be done easily if we change the variable: $(1 + t/.71)^{-1} = 1 - z$. The result is

$$\begin{aligned} X = {} & (A + 4X_{min})\, \ln \frac{z_{max}}{z_{min}} + X_{min}\left(\frac{1}{z_{max}} - \frac{1}{z_{min}}\right) \\ & + (B - 3A - 6X_{min})(z_{max} - z_{min}) \\ & + (-B + \tfrac{3}{2}A + 2X_{min})(z_{max}^2 - z_{min}^2) \\ & + \tfrac{1}{3}(B - A - X_{min})(z_{max}^3 - z_{min}^3), \end{aligned} \tag{III. 27}$$

where

$$X_{min} = t_{min}/.71,$$

$$X_{max} = m^2(1 + \ell)^2/.71,$$

$$z_{min} = X_{min}/(1 + X_{min}),$$

$$z_{max} = X_{max}/(1 + X_{max}),$$

$$A = 1 + t_{min}\,(1 + \mu_p^2)/(4\,M_p^2),$$

and

$$B = (\mu_p^2 - 1)\,0.71/(4M_p^2).$$

This expression for χ is substituted into Eq. (III.17) to obtain $d\sigma/dpd\Omega$ from a proton. The comparison with the Born approximation is given in Table II. The agreement with the exact result is good up to $m = 4$ and $k = 200$. Even at $m = 6$ and $k = 200$, the result seems to be correct within a factor of two.

Production Accompanied by Meson Production

So far, all of our discussions were limited to elastic cases $M_f^2 = M_i^2$. We now consider the meson production case where $\Delta \neq 0$. Let us immediately remark that our formula does not work as well as for the other cases because t_{min} is not so small as can be seen from Eq. (II.10). Furthermore, the form factors do not decrease fast enough when t is increased. Accordingly, the form of the cutoff function C(t) is important as we discussed in the subsection (II.B.iii). We have tried the cutoff function

$$C_N(t) = \frac{1}{(1 + t/[(1 + \ell)^N m^2])}, \qquad \text{(III.28)}$$

which was indicated by the analysis of the exact formula. As for the form factors W_2 and W_1, we have used the parameterization of suri-Yennie.[27] We have tried $C_1(t)$ and $C_2(t)$, both of which give correct order of magnitude but $C_1(t)$ was somewhat better. In Table III, we compare our formula (with $C_1(t)$) with the exact results. Considering all the ambiguities of our formula for this case, the numerical agreement is quite encouraging.

IV. Concluding Remarks

From the numerical results given in this paper, we see that the W.W. method does give amazingly reliable answers. For the spin $\frac{1}{2}$ case, the W.W. method is about two orders of magnitude simpler to handle than the Born approximation. For the spin 1 case the W.W. method is about three orders of magnitude simpler. Therefore the W.W. method can be used to determine whether certain experiments are worth doing or whether certain complicated calculations are worth performing. Even after the experiment is done and more respectable calculations are performed, the results of the W.W. approximation are often useful because they show the gross features of the cross section much more transparently than the more complicated calculations. Besides the derivation of our version of the W.W. method, we have also derived several useful formulas which are very convenient to have in the laboratory. For example, Eq. (III.17) with X given by Eq. (III.23) can be used to estimate the muon flux from an electron machine. [See also Eqs. (IV.1 and IV.5) of Paper A.] The information on the muon flux is important because it is a source of the muon secondary beam, a background to many experiments and also an important health hazard. We hope that various formulas given in this paper for the production of heavy leptons and W bosons are useful to the experimentalists who are trying to discover these particles. The reader is referred to the review paper[28] by Martin L. Perl for the summary of experimental works concerning heavy leptons. We finally mention that many interesting processes in the

"Positron-Electron-Proton Colliding Beam Machine" being proposed by people[29] at LBL and SLAC can be calculated by combining the pseudo photon flux from the electron[1](or positron) and that from the proton given in this paper.

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Appendix A

Kinematics near the Minimum Momentum Transfer

As we discussed in Section II, t is minimum when $\vec{p}_+ // \vec{k} - \vec{p}$, therefore

$$t_{min} = (|\vec{k} - \vec{p}| - |\vec{p}_+|)^2 - q_0^2 . \tag{A.1}$$

Now we shall assume

$$[E \text{ and } k - E] >> [m,\ k \cdot p/M_i,\ (k \cdot p)^{\frac{1}{2}} \text{ and } \Delta] . \tag{A.2}$$

Under these conditions it is easy to calculate

$$|\vec{k} - \vec{p}| = [(k - E)^2 + 2k \cdot p - m^2]^{\frac{1}{2}} \approx (k - E)\ [1 + (k \cdot p - \tfrac{1}{2} m^2)/(k - E)^2]$$

and

$$|\vec{p}_+| = [(k - E - q_0)^2 - m^2]^{\frac{1}{2}} \approx (k - E)[1 - q_0/(k - E) - \tfrac{1}{2} m^2/(k - E)^2] .$$

Therefore

$$(q_z^2)_{min} = (|\vec{k} - \vec{p}| - p_+)^2 \approx \left(\frac{k \cdot p}{k - E} + q_0\right)^2 \tag{A.3}$$

and

$$t_{min} \approx \left(\frac{k \cdot p}{k - E}\right)^2 + 2 q_0 \left(\frac{k \cdot p}{k - E}\right) . \tag{A.4}$$

Substituting $q_0 = \Delta + t/(2 M_i)$ into Eq. (A.4) we obtain

$$t_{min} \approx \left[\left(\frac{k \cdot p}{k - E}\right)^2 + 2\Delta\ \frac{k \cdot p}{k - E}\right] \Big/ \left[1 - \frac{k \cdot p}{M_i(k - E)}\right]$$

$$\approx \left(\frac{k \cdot p}{k - E}\right)^2 + 2\Delta\ \frac{k \cdot p}{k - E} \equiv t'_{min} + 2\Delta\, (t'_{min})^{1/2} . \tag{A.5}$$

The conditions (A.2) are necessary and sufficient conditions to obtain this expression for t_{min}. On the other hand under these conditions, t_{min} given by

Eq. (A.5) is necessarily much less than $k \cdot p$, which is precisely the condition required for the derivation of the W.W. approximation. Since $k \cdot p \sim m^2(1+\ell)/(2x)$, one can express t_{min} in terms of ℓ and x:

$$t_{min} = \left(\frac{m^2(1+\ell)}{2kx(1-x)}\right)^2 + \Delta \frac{m^2(1+\ell)}{kx(1-x)} \quad . \tag{A.6}$$

Again choosing the Z-axis along $\vec{k} - \vec{p}$ in the laboratory system, we can obtain the expression for t in the vicinity of t_{min} in terms of the laboratory quantities as follows:

$$t = q_z^2 + q_\perp^2 - q_0^2 = (\vec{k} - \vec{p} - \vec{p}_+)^2 - q_0^2 , \tag{A.7}$$

where

$$q_\perp^2 = p_{+x}^2 + p_{+y}^2$$

and

$$q_z = (|\vec{k} - \vec{p}| - (p_+^2 - q_\perp^2)^{\frac{1}{2}})$$

$$\approx \frac{k \cdot p}{k - E} + q_0 + \frac{1}{2} q_\perp^2/p_+ . \tag{A.8}$$

Since we are interested only in the region where $k \cdot p << q_\perp^2$, the term $\frac{1}{2} q_\perp^2/p_+$ can be dropped. We obtain

$$q_z^2 \approx t_{min} + q_0^2 , \tag{A.9}$$

$$t = t_{min} + q_\perp^2 , \tag{A.10}$$

and

$$(q_z - q_0)^2 \approx (k \cdot p)^2/(k - E)^2 = t'_{min} . \tag{A.11}$$

Equations (A.10) and (A.11) are used in the alternative derivation of the W.W. approximation in an infinite momentum frame (see Appendix B).

The expression $k \cdot p_+$ at $t = t_{min}$ can be written as

$$(k \cdot p_+)_{t = t_{min}} = k(E_+ - p_+ \cos\theta_k) \approx \frac{k\,m^2}{2\,E_+} (1 + p_+^2\, \theta_k^2/m^2).$$

where θ_k is the angle between $\vec{k}$ and $\vec{p} - \vec{k}$. From the relation,

$$\frac{\sin\theta_k}{p} = \frac{\sin\theta}{|\vec{k} - \vec{p}|} \approx \frac{\theta}{p_+} ,$$

we obtain readily the desired relation:

$$(k \cdot p_+)_{t=t_{min}} \approx \frac{E}{k - E} (p \cdot k) . \tag{A. 12}$$

It should be noted that this relation is invariant as long as the energies E and k - E are very relativistic and both angle θ and θ_+ are much smaller than unity. For example this relation is true also in the center of mass system. As mentioned in the text, this relation can not be obtained by vaguely stating that $g_{\mu\nu} L^{\mu\nu}$ is evaluated at $t = 0$.

Appendix B

Derivation in Infinite Momentum Frame

Here we shall derive (II.15) using a frame where the target $\vec{P}_i$ is moving with a great velocity in the direction opposite to the direction of $(\vec{k} - \vec{p})_{lab}$. Our infinite momentum frame is obtained from the laboratory frame (or the u frame) by boosting in the direction of $(\vec{k} - \vec{p})_{lab}$ so that the direction of the initial target momentum lies along $-\hat{e}_z$ with $E'_i/M_i = \gamma >> 1$. The prime is used to denote the quantities in the infinite momentum frame whereas the unprimed quantities refer to those in the laboratory system. Let us consider the current j_μ which is related to the tensor $L_{\mu\nu}$ of Section II by

$$L_{\mu\nu} \propto \sum_{\substack{\text{spins of}\\ k,\, p \text{ and } p_+}} j_\mu j_\nu \; .$$

For the coefficient of W_2, we are interested in the quantity:

$$j_\mu P_i^\mu / M_i = j_0 = j'_0 E'_i/M_i + j'_z p'_i/M_i \equiv \gamma j'_0 + \beta\gamma j'_z$$

$$\approx \gamma(j'_0 + j'_z) - j'_z/(2\gamma), \qquad \text{(B.1)}$$

where we have used the approximation:

$$\beta \equiv (1 - 1/\gamma^2)^{\frac{1}{2}} \sim 1 - 1/(2\gamma^2).$$

Current conservation, $q^\mu j_\mu = 0$, yields

$$q'_0 j'_0 - q'_z j'_z = q_x j_x + q_y j_y = \vec{q}_\perp \cdot \vec{j}_\perp \; , \qquad \text{(B.2)}$$

where we have used the fact that the transverse components of the vectors q_μ and j_μ are not affected by the Lorentz transformation. q'_0 and q'_z are related to

q_0 and q_z by

$$q_0' = \gamma q_0 - \beta\gamma q_z \approx -\gamma(q_z - q_0) + q_z/(2\gamma) \qquad \text{(B.3)}$$

and

$$q_z' = -\beta\gamma q_0 + \gamma q_z \approx \gamma(q_z - q_0) + q_0/(2\gamma) \ . \qquad \text{(B.4)}$$

Substituting Eqs. (B.3) and (B.4) into Eq. (B.2), we obtain

$$\gamma(j_0' + j_z') \approx \frac{1}{(q_z - q_0)}\left[\frac{1}{2\gamma}\,(q_z j_0' - q_0 j_z') - \vec{q}_\perp \cdot \vec{j}_\perp\right] \ . \qquad \text{(B.5)}$$

Substituting Eq. (B.5) into Eq. (B.1), we obtain

$$j_0 = \frac{q_z}{q_z - q_0}\ \frac{1}{2\gamma}\ (j_0' - j_z') - \frac{1}{q_z - q_0}\ \vec{q}_\perp \cdot \vec{j}_\perp \qquad \text{(B.6)}$$

Substituting [see Eqs. (B.3) and (B.4)]

$$j_0' \approx \gamma(j_z - j_0) + j_z/(2\gamma)$$

and

$$j_z' \approx \gamma(j_z - j_0) + j_0/(2\gamma)$$

into Eq. (B.6), we finally obtain

$$j_0 \approx \frac{-q_z}{q_z - q_0}\ (j_z - j_0) - \frac{1}{q_z - q_0}\ \vec{q}_\perp \cdot \vec{j}_\perp \ . \qquad \text{(B.7)}$$

Now this relation is actually exact because we can make γ as large as we wish. Eq. (B.7) is obviously equivalent to Eq. (II.15), which we derived in the text by employing a somewhat mysterious gauge transformation (II.14).

Appendix C

Concept of a Pseudo Photon Beam

In our W.W. approximation the cross section for $\gamma + Z \to \ell^+ + \ell^- +$ anything is proportional to $(-\frac{1}{2} g_{\mu\nu} L^{\mu\nu})_{t = t_{min}}$. From $k = p + p_+ + q$, this factor can be written as a function of four variables: m^2, $k \cdot p$, $k \cdot p_+$ and $t = -q^2$. Under our approximation the statement $t = t_{min}$ is equivalent to setting $t = 0$ and $k \cdot p_+ = k \cdot p\, E/(k - E)$ when evaluating this factor. Equation (II.18) can be written covariantly as

$$\frac{d\sigma}{d(p \cdot k)\, d(p \cdot P_i)} = \frac{d\sigma_{\gamma\gamma \to \ell^+ + \ell^-}}{d(p \cdot k)} \frac{\alpha}{\pi} \frac{\chi}{P_i \cdot (k - p)} , \tag{C.1}$$

where

$$\frac{d\sigma_{\gamma\gamma \to \ell^+ + \ell^-}}{d(p \cdot k)} = \frac{\pi \alpha^2}{(k \cdot q)^2} \left(- \frac{q_{\mu\nu} L^{\mu\nu}}{2} \right)_{, t = t_{min}} . \tag{C.2}$$

This equation says that if we know the cross section for the process $\gamma + \gamma \to \ell^+ + \ell^-$, we can obtain $d\sigma/dpd\Omega$ provided that we set $k \cdot p^+ = k \cdot p\; E/(k - E)$. From $k - q = p + p_+$, we obtain

$$k \cdot q = -p \cdot k - p_+ \cdot k = -p \cdot k \quad k/(k - E). \tag{C.3}$$

Differentiating Eq. (C.3) with respect to E, we have

$$d(p \cdot P_i) = - (P_i \cdot k)(p \cdot k) \frac{d(k \cdot q)}{(k \cdot q)^2} . \tag{C.4}$$

Hence Eq. (C.1) can also be written as

$$\frac{d\sigma}{d(p \cdot k)\, d(k \cdot q)} = \frac{d\sigma_{\gamma\gamma \to \ell^+ + \ell^-}}{d(p \cdot k)} \rho , \tag{C.5}$$

where ρ is the pseudo photon flux with the covariant spectral distribution given by

$$\rho\, d(k \cdot q) = (\alpha/\pi) \times \frac{d(k \cdot q)}{(k \cdot q)} . \qquad (C.6)$$

Let us consider the expression $d(k \cdot q)/(k \cdot q)$ in some detail. If the photon represented by $k' = -q$ were truly on the mass shell, then

$$k \cdot q = -k \cdot k' = -k_0 k'_0 (1 - \cos\theta_{kk'})$$

and we would have

$$d(k \cdot q)/(k \cdot q) = dk'_0/k'_0 \qquad (C.7)$$

in any coordinate system where k'_0 is the energy of the pseudo photon. However when the photon is slightly off the mass shell, as in our situation, this is no longer the case. For example in the laboratory system this relation is obviously not true. In the infinite momentum frame where the initial target is moving with a great velocity ($\beta \to 1$) opposite to the incident photon, this relation is true even if $q^2 = -t_{min} \neq 0$. Writing $k \cdot q = k_0(q'_0 - q' \cos\theta'_k)$, we have

$$k_0 = k(\gamma - \beta\gamma) \approx k/2\gamma,$$

$$q'_0 = q_0\gamma - \beta\gamma q_z \cos\theta_k \to \gamma(q_0 - q_z \cos\theta_k), \qquad (C.8)$$

and

$$q' \cos\theta'_k = -\beta\gamma q_0 + \gamma q_z \cos\theta_k \to -\gamma(q_0 - q_z \cos\theta_k) = -q'_0 .$$

Hence $k \cdot q = 2k_0 q'_0 = -2k_0 k'_0$ and we obtain the relation (C.7). The direction of the pseudo photon flux is parallel to the momentum of the target particle in the infinite momentum frame. $\cos\theta_k$ can be approximated by unity because at $t = t_{min}$, $\cos\theta_k \approx 1 - \frac{1}{2}\theta_k^2 \approx 1 - \frac{1}{2}\theta^2 x^2/(1-x)^2 \approx 1$. Using Eqs. (A.11) and (C.8) we obtain the energy of the pseudo photon in this infinite momentum frame

$$k'_0 \equiv -q'_0 = \gamma\, k \cdot p/(k - E) \simeq \gamma (t'_{min})^{1/2} . \qquad (C.9)$$

The maximum energy of the photon which can be emitted by a particle with an ultra-relativistic energy γM_i is roughly γM_i. Thus

$$k'_0/k'_{0\ max} = (t'_{min})^{1/2}/M_i \ . \tag{C.10}$$

This quantity is very small compared with unity under the condition (A.2). Hence we conclude that only the relatively soft component of the pseudo photon beam is relevant in the W.W. calculation. Using the concept of the equivalent radiator introduced in Section IV of Paper A, we may summarize the contents of this appendix as follows: "In the one photon exchange process, the target particle, viewed in the frame where it is moving with a great velocity opposite to the incident particle, is equivalent to a beam of real photons produced by <u>an electron</u> after it passes through a target of thickness $(3/4)(\alpha/\pi)\,X$ radiation lengths."

Appendix D

Generalized W.W. Formula

Our results can be generalized to the treatment of an arbitrary one photon exchange process,

$$a + p_i \rightarrow b + c + p_f,$$

in terms of W_1 and W_2 of the target and the differential cross section for the process $a + \gamma \rightarrow b + c$, where particle a is no longer massless, $m_a \neq 0$, and the particle c is no longer the antiparticle of b, $m_b \neq m_c$. As far as the derivation of the W.W. formula is concerned, all of our arguments go through if we replace k_μ by a_μ, p_μ by b_μ and $p_{+\mu}$ by c_μ and use a different $L_{3\mu\nu}$

$$L_{3\mu\nu} = (q \cdot b\, c_\mu - q \cdot c\, b_\mu)(q \cdot b\, c_\nu - q \cdot c\, b_\nu).$$

The generalized version of Eq. (C.1) is

$$\left[\frac{d\sigma(a + p_i \rightarrow b + c + p_f)}{d(a \cdot b)\, d(b \cdot P_i)}\right]_{W.W.} = \left[\frac{d\sigma(a + \gamma \rightarrow b + c)}{d(a \cdot b)}\right]_{t = t_{min}} \frac{\alpha}{\pi} \quad \frac{\chi}{c \cdot P_i} \qquad \text{(D. 1)}$$

Equation (D. 1) can be derived under the conditions

$$E_a^2 >> [\, (a + b)^2 \text{ and } m_a^2],$$

$$(E_a - E_b) >> \left[\Delta,\ m_c,\ \left\{(a - b)^2\right\}^{\frac{1}{2}} \text{ and } \frac{m_c^2 - (a - b)^2}{M_i}\right],$$

and

$$E_b >> m_b\ .$$

Also under these conditions t_{min} can be written approximately as

$$t_{min} = t'_{min} + 2\Delta(t'_{min})^{\frac{1}{2}}, \qquad \text{(D. 2)}$$

where

$$\Delta = (M_f^2 - M_i^2)/(2\,M_i),$$

and

$$(t'_{min})^{\frac{1}{2}} = [a \cdot b - \frac{1}{2}(m_a^2 + m_b^2 - m_c^2)]/(E_a - E_b)\ . \tag{D.3}$$

The subscript $t = t_{min}$ in Eq. (D. 1) means that when evaluating the cross section $d\sigma(a + \gamma \rightarrow b + c)/d(a \cdot b)$, not only the photon is put on the mass shell $(t = 0)$ but also the momentum of particle c has to be set parallel to $\vec{a} - \vec{b}$ in the laboratory system. The consequence of this condition is the approximate relation:

$$q \cdot a/E_a = q \cdot b/E_b = q \cdot c/(E_a - E_b) = (t'_{min})^{\frac{1}{2}}\ . \tag{D.4}$$

Equations (D. 3) and (D. 4) enable us to write $d\sigma_\gamma/d(a \cdot b)$ in terms of the variables E_a, E_b and $a \cdot b$.

Table I

Coherent Production $d\sigma/dpd\Omega$ from Be

	m = 0.1056		m = 0.5		m = 1.0		m = 2.0		m = 4.0		m = 6.0	
	k = 20		k = 100		k = 100		k = 200		k = 200		k = 200	
	p = 8		p = 40		p = 40		p = 80		p = 80		p = 80	
	10^{-31} Cm^2/GeV/sr		10^{-33} Cm^2/GeV/sr		10^{-35} Cm^2/GeV/sr		10^{-36} Cm^2/GeV/sr		10^{-38} Cm^2/GeV/sr		10^{-40} Cm^2/GeV/sr	
$\gamma\theta$	Born	W.W.	Born	W.W.	Born	W.W.	Born	W.W.	Born	W.W.	Born	W.W.
0.0	1584	1543	1068	1069	3274	3280	2228	2235	1136	1162	1530	1682
0.6	1032	1088	686	695	1882	1900	1126	1135	337	348	360	387
1.2	310	317	174	173	345	345	143	144	15	15	11	12
1.8	84	31	38	37	46	46	12	12				

Table II

Elastic Production, $d\sigma/dpd\Omega$, from a Proton

	10^{-32} Cm^2/GeV/sr		10^{-34} Cm^2/GeV/sr		10^{-36} Cm^2/GeV/sr		10^{-37} Cm^2/GeV/sr		10^{-39} Cm^2/GeV/sr		10^{-40} Cm^2/GeV/sr	
$\gamma\theta$	Born	W.W.	Born	W.W.	Born	W.W.	Born	W.W.	Born	W.W.	Born	W.W.
0.0	1116	1022	950	947	3845	3978	3457	3659	6485	8130	2805	4957
0.6	728	748	619	658	2401	2585	2075	2262	3062	4204	586	1382
1.2	231	242	181	186	600	638	438	497	271	499		
1.8	70	70	47	46	124	135	73	90	2	7		

Table III

Continuum Production, $d\sigma/d\Omega dp$, from a Proton Target

	m = 0.1056 k = 20 p = 8 10^{-31} Cm^2/GeV/sr		m = 0.5 k = 100 p = 40 10^{-33} Cm^2/GeV/sr		m = 4.0 k = 200 p = 80 10^{-36} Cm^2/GeV/sr		m = 6.0 k = 200 p = 80 10^{-37} Cm^2/GeV/sr	
$\gamma\theta$	Born	W.W.	Born	W.W.	Born	W.W.	Born	W.W.
0.0	2.758	1.813	9.734	6.834	2.509	2.428	1.141	1.250
0.6	1.725	1.621	5.920	5.509	1.202	1.263	0.222	0.254
1.2	0.800	0.755	2.529	2.064	0.118	0.115	0	0
1.8	0.420	0.306	1.096	0.663	0.001	0.001	0	0

FIGURE CAPTIONS

1. Feynman diagrams for pair production. The segal diagram (the last diagram) needs to be considered only for the pair production of particles with an integer spin.
2. The coordinate system used in the integration over the unobserved particle p_+. The subscript s refers to the rest frame of $u = p_+ + P_f$.
3. Comparison of angular distributions in the center-of-mass system for the reactions $\gamma + \gamma \rightarrow p_+ + p_-$ for various values of spin and magnetic moment of the final particles.
4. The total cross sections for the reactions $\gamma + \gamma \rightarrow p_+ + p_-$ for various values of spin and magnetic moment of the final particles.
5. The energy angle distribution $d\sigma/d\Omega\, dE$ of the electron in pair production from Be, $x = E/k$.
6. The effects of the elastic Be form factor on the energy angle distribution in the production of a lepton pair.

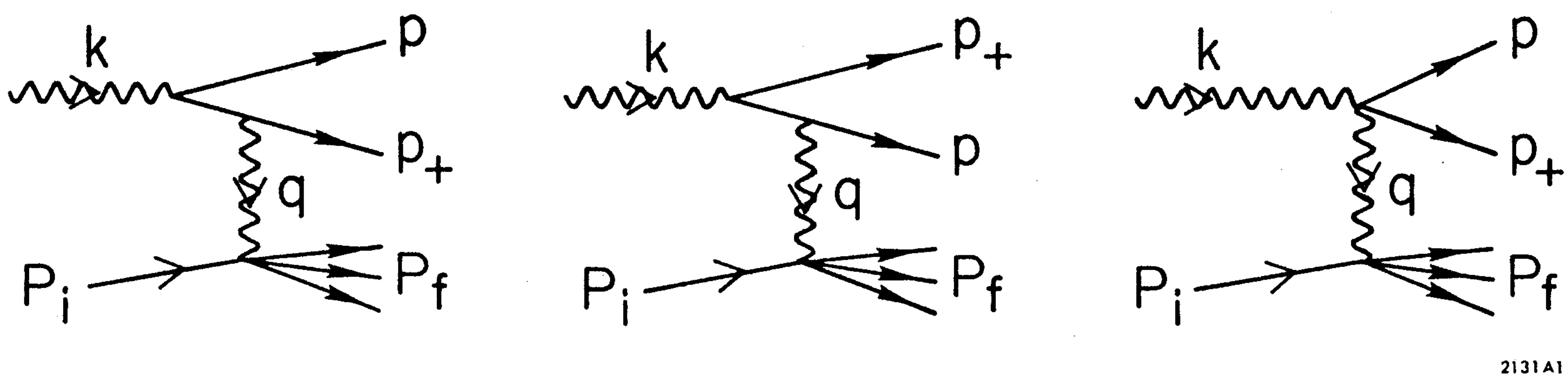
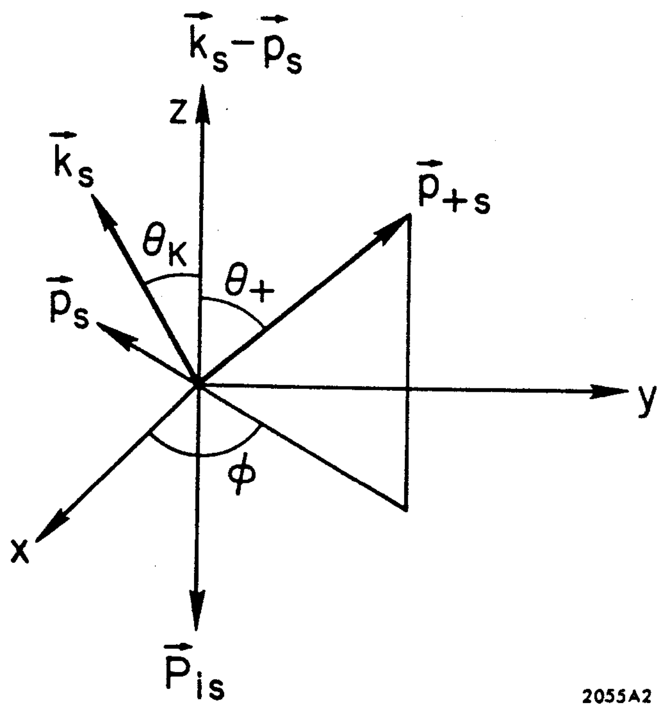
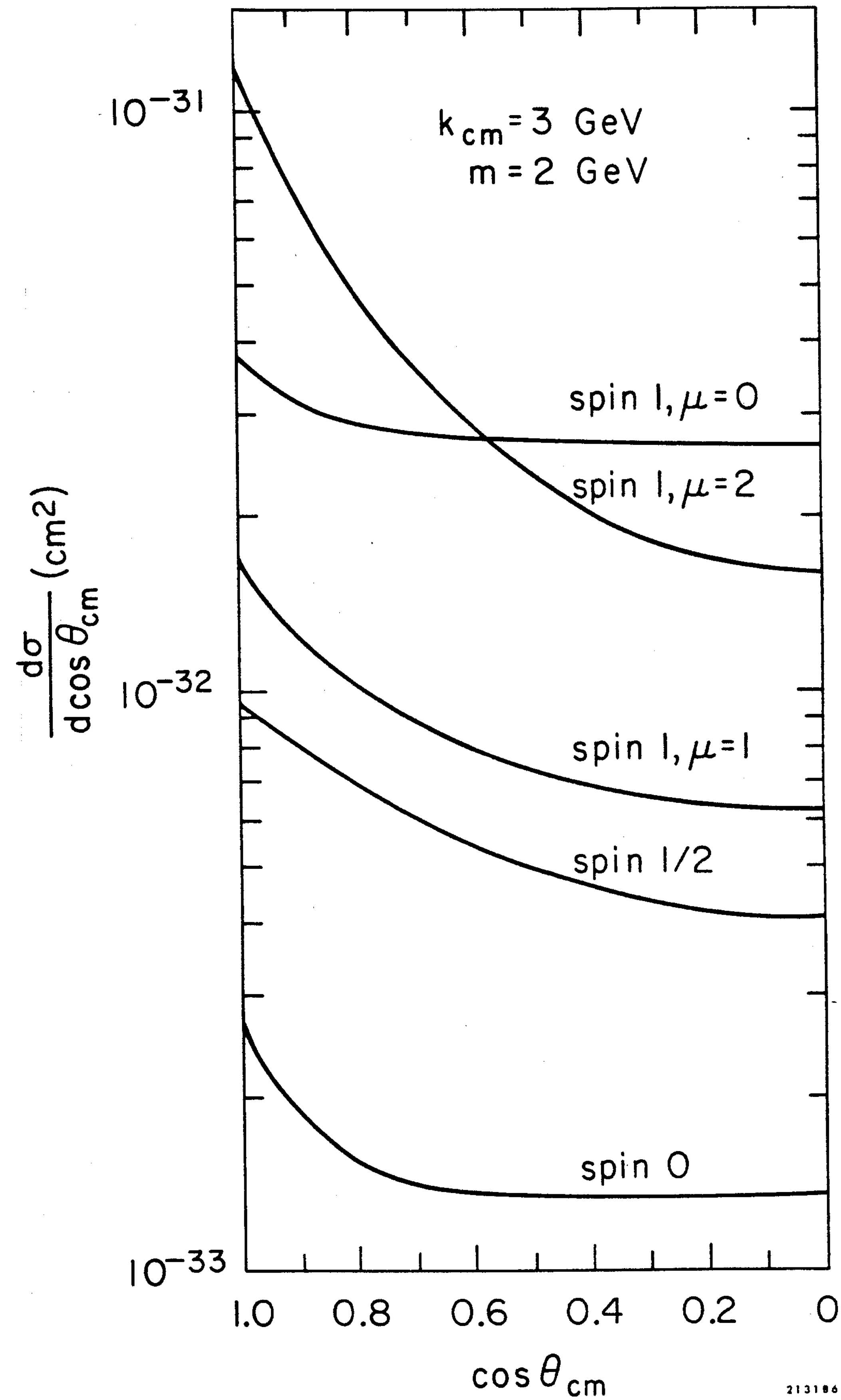


Fig. 1



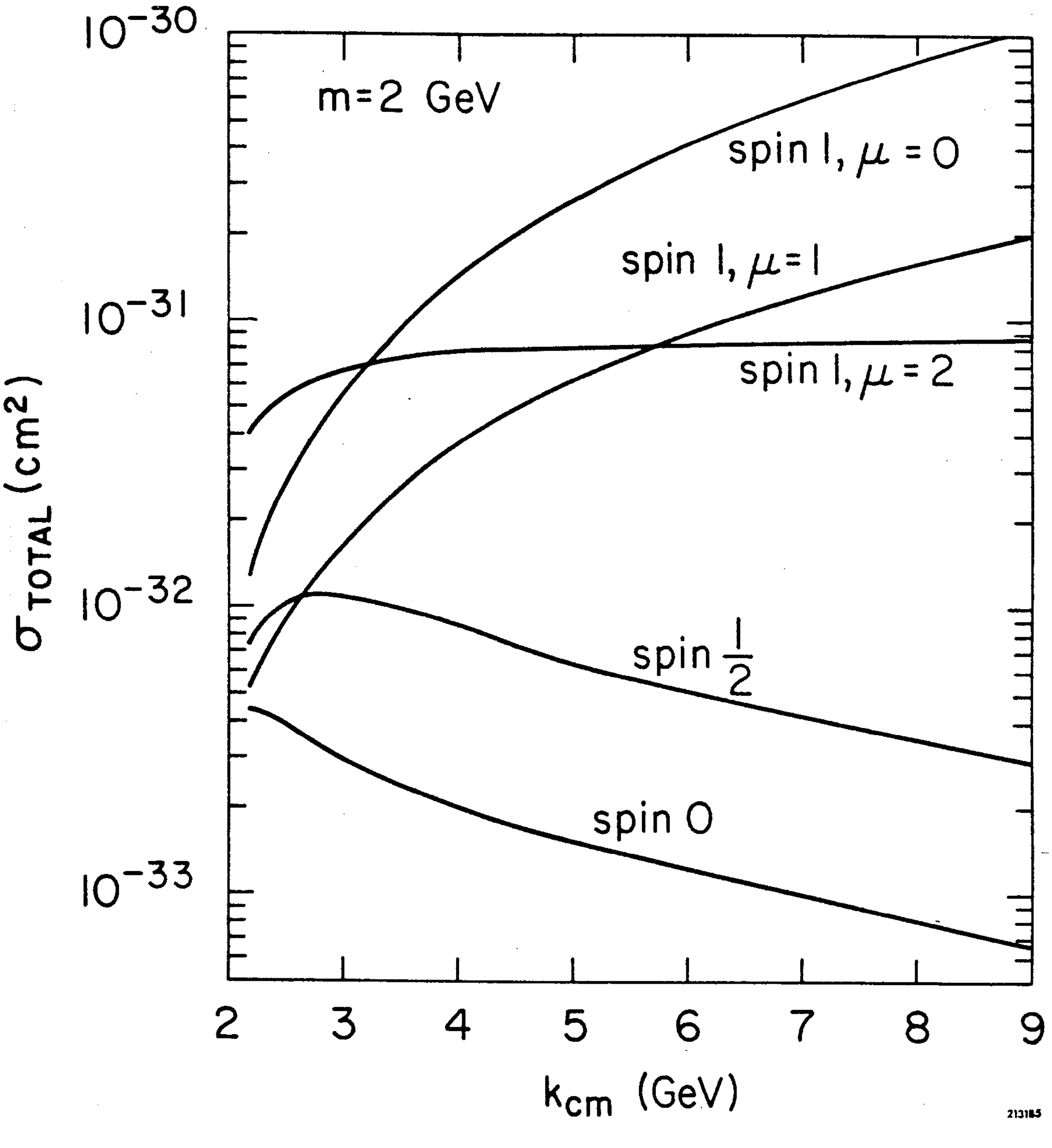
$\vec{k}_s - \vec{p}_s$
z
$\vec{k}_s$
$\vec{p}_{+s}$
θ_K
θ_+
$\vec{p}_s$
y
ϕ
x
$\vec{P}_{is}$
2055A2

Fig. 2



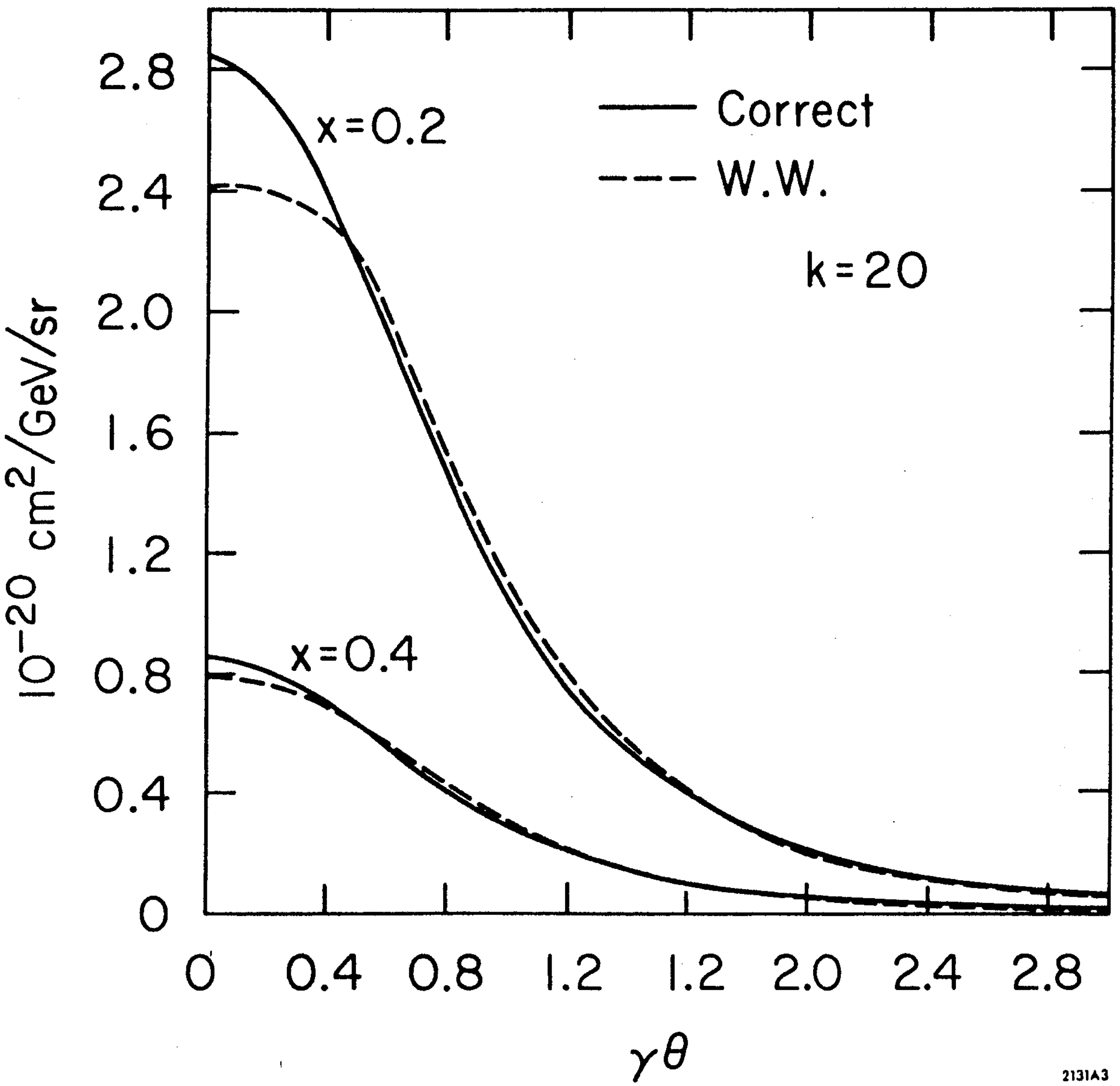
10^{-31}
10^{-32}
10^{-33}
k_{cm} = 3 GeV
m = 2 GeV
spin 1, μ = 0
spin 1, μ = 2
spin 1, μ = 1
spin 1/2
spin 0
1.0
0.8
0.6
0.4
0.2
0
$\cos\theta_{cm}$
$\frac{d\sigma}{d\cos\theta_{cm}}$ (cm^2)
213186

Fig. 3



m=2 GeV
spin 1, μ = 0
spin 1, μ = 1
spin 1, μ = 2
spin 1/2
spin 0
10^{-30}
10^{-31}
10^{-32}
10^{-33}
σ_{TOTAL} (cm²)
2
3
4
5
6
7
8
9
k_{cm} (GeV)
213185

Fig. 4



Correct
W.W.
k=20
x=0.2
x=0.4
10^{-20} cm^2/GeV/sr
0
0.4
0.8
1.2
1.6
2.0
2.4
2.8
0
0.4
0.8
1.2
1.2
2.0
2.4
2.8
$\gamma\theta$
2131A3

Fig. 5

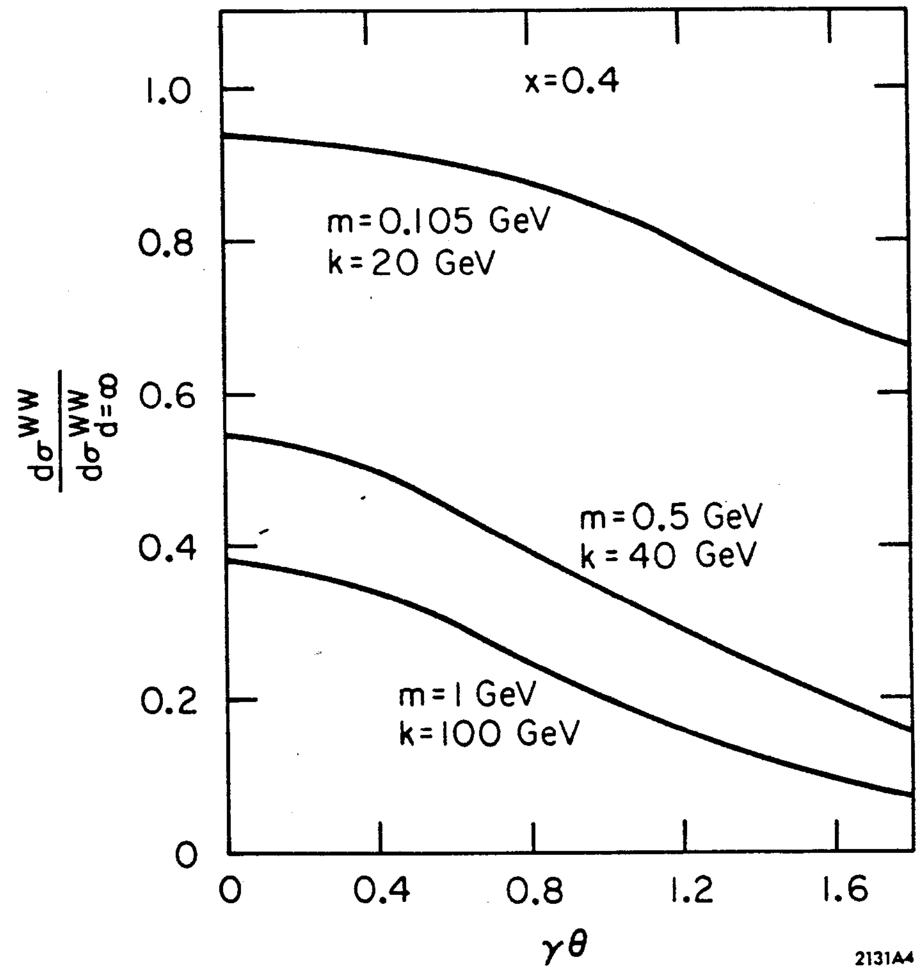

x=0.4
m=0.105 GeV
k=20 GeV
m=0.5 GeV
k=40 GeV
m=1 GeV
k=100 GeV
1.0
0.8
0.6
0.4
0.2
0
0
0.4
0.8
1.2
1.6
$\frac{d\sigma^{WW}}{d\sigma^{WW}_{d=\infty}}$
$\gamma\theta$
2131A4

Fig. 6