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ANOMALY OF THE AXIAL-CURRENT IN ONE SPACE AND ONE TIME DIMENSION*

Sun-Sheng Shei

Laboratory of Nuclear Studies
Cornell University, Ithaca, New York 14850

and

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

ABSTRACT

We use Wilson's theory of broken scale invariance to study the anomaly of axial current in a world with one space and one time dimension. It is shown that in the Schwinger and Thirring models, Wilson's approach and perturbative approach yield similar results for the PCAC anomaly.

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I. Introduction

In recent years the problem of PCAC anomaly in the presence of electromagnetism has been studied extensively in the framework of renormalized perturbation theory.[1] It was found that the anomaly is related to the breakdown of the naive Ward identity caused by the presence of triangle graph in the renormalized perturbation theory. The anomaly also leads to many low energy theorems for the electromagnetic decays of neutral pseudoscalar mesons[2] and other electromagnetic processes.[3]

Another approach to the problem of PCAC anomaly was proposed by Wilson.[4] He applies his formulation of broken scale invariance and operator product expansion to this problem. He shows qualitatively that the anomaly is related to the short-distance behavior of the product of currents.

Recently Crewther,[5] following the suggestion of Wilson, has proved that in fact the anomaly can be explained by the short-distance behavior of the product of currents. He also relates the anomaly constant to other physical constants in high-energy electroproduction and electron-positron annihilation processes.

So far, the anomaly has been studied either entirely in the framework of renormalized perturbation theory or in the framework of Wilson's theory of broken scale invariance. However, the connection between these two different approaches has not been examined.[7] In the perturbation theory one can treat anomaly successfully. Yet it is not at all clear whether Wilson's theory of broken scale invariance can be applied. On the other hand, in Wilson's approach although we have interesting results relating the anomaly to other physical quantities, we don't know how to calculate these quantities in strong interaction. It is therefore very desirable to find models in which both the perturbation theory and Wilson's theory of broken scale invariance can be applied.

It is the purpose of this note to study the PCAC anomaly in some solvable models. The models we discuss are Schwinger model[8] and Thirring model.[9] Both are field-theoretic models in one space and one time dimension. These models have been very useful to provide testing grounds for theoretical ideas. Although they are very special models, nevertheless, any general feature of quantum field theory should remain true. We will show that in these models the anomaly is related to the short-distance behavior of the product of two currents, and the results so obtained for the anomaly are the same as those obtained by perturbation theory.

In Section II we discuss the anomaly in one space and one time dimension using Wilson's theory of broken scale invariance and operator product expansion. The PCAC anomaly will also be related to the fictitious decay $\pi^0 \to \gamma$ in one space and one time dimension.

In Section III we discuss the Schwinger model which is the quantum electrodynamics in one space and one time dimension. The model has been solved explicitly by Schwinger. Our intention here is to use the explicit solution for studying the problem of anomaly. We shall obtain PCAC anomaly as an operator equation. We show further that the anomaly is given by the formula we derived in Section II.

The Thirring model is considered in Section IV. The solution of the Thirring model both with or without coupling to external classical electromagnetic field are known explicitly.[10,11] In this section we use the Thirring model as our skeleton theory in the sense of Wilson.[4] Johnson's solution[10] will be used to obtain the solution we need when we couple fermion in Thirring model to the radiation field. The PCAC anomaly as an operator equation can be derived. We then use the method described in Section II to the anomaly. We find that both results agree to all order of coupling constant. The conclusions of perturbation theory will also be mentioned.

In the last section we make some pertinent remarks. First remark concerns the nonvanishing mass of the photon in one space and one time dimension and its effect on the anomaly. We also comment on the intimate connection between anomaly constant and Schwinger terms in the models considered in Section III and IV.

II. Wilson's Theory of Broken Scale Invariance and PCAC Anomaly

One of the applications of the theory of broken scale invariance is the resolution of difficulties with naive current algebra calculation[12] of $\pi^o \to 2\gamma$. Wilson explains nonvanishing $\pi^o \to 2\gamma$ decay in terms of the short-distance behavior of the product of currents.[4]

We sketch Wilson's explanation below. The invariant amplitude for the $\pi^o \to 2\gamma$ decay is

$$T_{\mu\nu}(p, k) = \epsilon_{\mu\nu\alpha\beta} p^\alpha k^\beta T(k^2) .$$

The value of $T(k^2)$ when $k = 0$, $T(0)$, can be expressed as

$$\epsilon_{\mu\nu\alpha\beta} T(0) = \frac{1}{F_\pi} \int\int dxdy \; x_\alpha y_\beta <0 | T j_\mu(x) j_\nu(0) \; \partial^\lambda j^5_\lambda(y) | 0> ,$$

where F_π is the pion leptonic decay amplitude, and j_μ, j^5_λ are hadronic current and axial current respectively. Using the technique of Ward identity and carefully analyzing various limiting processes, Wilson shows that the amplitude T(0) (or equivalently the anomaly constant S) is determined uniquely by the leading short-distance behavior of

$$T < 0 | j_\mu(x) \; j_\nu(0) \; j^5_\lambda(y) | 0> .$$

From the theory of broken scale invariance, the leading behavior of the above expression scales as $(\epsilon)^{-9}$ for x and y of the order of ϵ. Wilson's analysis also shows that in general one should not expect T(0) to be zero.

The leading singularity of $T\langle 0|j_\mu(x)\, j_\nu(0)\, j^5_\lambda(y)|0\rangle$ is determined by Schreier[6] using the arguments of conformal invariance. Crewther[5] uses this explicit form to calculate the anomalies. He is able to relate T(0) to the short-distance behavior of the q — number part of $T(j_\mu(x)\, j_\nu(0))$ and the c — number part of $T(j^5_\mu(x)\, j^5_\nu(0))$ which are related to various high energy inelastic processes and are in principle measurable.

In what follows we shall apply Wilson's idea to the world with one space and one time dimension. Our purpose is to study the PCAC anomaly or equivalently the decay $\pi^0 \to \gamma$ from this point of view.

Let us imagine a fictitious decay $\pi^0 \to \gamma$ in one space and one time dimension. The matrix element describing this decay is[13]

$$S_{fi} = (\mathrm{K.F.})\, e_0\, \epsilon^*_\mu\, T^\mu(k)\,, \tag{2.1}$$

where (K. F.) denotes kinematic factors, e_0 is the unrenormalized coupling constant and

$$\begin{aligned} T^\mu(k) &= \epsilon^{\mu\lambda} k_\lambda T(k^2) \\ &= \left(\frac{m^2 - k^2}{F_\pi m^2}\right) \frac{1}{\sqrt{Z_3}} \int d^2x (e^{-ikx} - 1) T^* \langle 0|\, \partial^\gamma j^5_\gamma(x)\, j^\mu(0)|0\rangle\,, \end{aligned} \tag{2.2}$$

where Z_3 is the wave function renormalization constant for the photon. The form $T^\mu(k) = \epsilon^{\mu\lambda} k_\lambda\, T(k^2)$ follows from invariance under parity and is consistent with gauge invariance.

For the definition of the Fourier transform of time-ordered product, we follow the prescriptions given by Wilson.[14] Here we also make one subtraction in the Fourier transform to make it well defined and gauge invariant.[14] Explicitly, the Fourier transform in Eq. (2.2) is defined as

$$\int d^2x(e^{-ikx} - 1)\ T^* < 0 | \partial^\gamma j_\gamma^5(x)\, j^\mu(0) | 0 >$$

$$\equiv \lim_{\eta \to 0^+} \left\{ \int_\eta^\infty dx^o + \int_{-\infty}^{-\eta} dx^o \right\} \int dx^1 (e^{-ikx} - 1) T < 0 | \partial^\gamma j_\gamma^5(x)\, j^\mu(0) | 0 >, \tag{2.3}$$

where the symbol T stands for the usual time ordered-product and $\eta \to 0^+$ means the limit as η approaches zero from $\eta > 0$.

By differentiating with respect to k and setting k equal to zero on both sides of Eq. (2.2), we find

$$T(0) = -\frac{i}{2F_\pi} \frac{1}{\sqrt{Z_3}} \int d^2x\, \epsilon^{\lambda\mu} x_\lambda\, T^* < 0 | \partial^\gamma j_\gamma^5(x)\, j_\mu(0) | 0 > \tag{2.4}$$

Our main goal in this section is to relate T(0) (or equivalently the anomaly constant S) to the short-distance behavior of the product of currents. For convenience, we define

$$I = \int d^2x\, \epsilon^{\lambda\mu}\, x_\lambda\, T^* < 0 | \partial^\gamma j_\gamma^5(x)\, j_\mu(0) | 0 >, \tag{2.5}$$

and

$$X_\gamma = \epsilon^{\lambda\mu} x_\lambda\, T^* < 0 | j_\mu(0) j_\gamma^5(x) + j_\gamma(0)\, j_\mu^5(x) | 0 > \tag{2.6}$$

In virtue of the current conservation and symmetry of $j_\mu(0) j_\gamma^5(x) + j_\gamma(0) j_\mu^5(x)$ under the interchange of indices μ, γ, the divergence of X_γ is

$$\partial^\gamma X_\gamma = \epsilon^{\lambda\mu} x_\lambda\, T^* < 0 | \partial^\gamma j_\gamma^5(x)\, j_\mu(0) | 0 >. \tag{2.7}$$

This is precisely the integrand of I. Note that we do not have to worry about the Schwinger terms because the point $x = 0$ is excluded from the integration region with our definition of T*. Using the Ward identity in Eq. (2.7), one can write I as

$$I = \int d^2x \ \partial^\gamma X_\gamma \ .$$

Were it not for the discontinuity of X_γ at $x^0 = 0$, I would be zero. As we have mentioned before, I is defined precisely as

$$I = \lim_{\eta \to 0^+} \left(\int_{-\infty}^{-\eta} dx^0 + \int_{\eta}^{\infty} dx^0 \right) \int dx^1 \ \partial^\gamma X_\gamma$$

$$= \lim_{\eta \to 0^+} \left(\int_{-\infty}^{-\eta} dx^0 + \int_{\eta}^{\infty} dx^0 \right) \int dx^1 \ \partial^0 X_0$$

$$= \lim_{\eta \to 0^+} \int dx^1 \left[- X_0(x^0 = \eta) + X_0(x^0 = -\eta) \right] . \qquad (2.8)$$

In deriving the second expression of I, we have made use of the fact that the spatial integration of $\partial^1 X_1$ is zero.

From Eq. (2.8), it is clear that I is related to the behavior of X_0 around $x^0 = 0$. As it has been discussed in detail by Wilson,[4] the x^1 integration can be divided into two pieces, with one piece satisfying $|x^1| >> \eta$. It is easy to convince oneself that one can let $\eta \to 0^+$ inside the integral with $|x^1| >> \eta$ without making errors and the integral becomes equal-time commutator which is zero in this region. Only the x^1 integration with x^1 of the order η or smaller will contribute to integral I. This argument suffices to show that I depends uniquely on the short-distance behavior of X_0 or the product of two currents.

We turn now to the short-distance behavior of the product of currents. According to Wilson's theory of broken scale invariance, the short-distance behavior of the product of currents is determined by the dimensional argument. In the world with one space and one time dimension, we have

$$T(j^5_\mu(x)\, j_\nu(0)) = \frac{R}{4\pi}\; i\, \epsilon_{\mu\lambda}(g^\lambda_\nu\; \partial^2 - \partial^\lambda\, \partial_\nu)\log(-x^2 + i\,\epsilon)$$

$$+ \cdot\cdot\cdot\cdot\,, \qquad (x \neq 0)\;. \tag{2.9}$$

Here $\cdot\cdot\cdot\cdot$ denotes the less singular terms which break the scale invariance. The dimensionless constant R however is not determined by the dimensional argument. It is model dependent, i.e., R is determined by the skeleton theory.[15]

One peculiarity of the models we will consider in one space and one time dimension is the following: the axial current is related to the current by

$$j^5_\mu = \epsilon_{\mu\nu}\, j^\nu \tag{2.10}$$

Notice that Eq. (2.10) implies that gauge invariance of the axial current is guaranteed by that of the current j^ν. It is easy to derive from Eqs. (2.9), (2.10) the short-distance behavior of X_γ.

$$X_\gamma = -\left(\frac{R}{\pi}\right)\left(\frac{x_\gamma}{-x^2 + i\,\epsilon}\right) + \cdot\cdot\cdot\cdot\,, \qquad \text{with} \quad \epsilon > 0 \tag{2.11}$$

The evaluation of the integral I is straightforward.

$$I = \lim_{\eta \to 0^+} \frac{R}{\pi} \int dx^1 \left\{ \frac{\eta}{x_1^2 - \eta^2 + i\,\epsilon} + \frac{\eta}{x_1^2 - \eta^2 + i\,\epsilon} \right\}$$

$$= -\,2\,i\,R \tag{2.12}$$

Consequently, the constant T(0) in the decay $\pi^0 \to \gamma$ is

$$T(0) = -\,\frac{R}{F_\pi}\;\frac{1}{\sqrt{Z_3}} \tag{2.13}$$

The anomaly in the PCAC equation is related to T(0). If we write

$$\partial^{\mu} j^5_{\mu} = " \partial^{\mu} j^5_{\mu} " + Se\, \epsilon_{\mu\nu}\, \partial^{\mu} A^{\nu}$$

$$= F_{\pi}\, m^2_{\pi}\, \phi + Se\, \epsilon_{\mu\nu}\, \partial^{\mu} A^{\nu} \qquad (2.14)$$

where " $\partial^{\mu} j^5_{\mu}$" is the expression in the absence of electromagnetism, $e = \sqrt{Z_3}\, e_0$ is the renormalized coupling constant and ϕ is the renormalized pion field. Then the anomaly constant S is given by

$$e\, S = - F_{\pi}\, T(0)\, e_0\, Z_3. \qquad (2.15)$$

This together with Eq. (2.13) implies

$$S = R. \qquad (2.16)$$

Thus, we have established that the anomaly constant in the PCAC equation is uniquely determined by the short-distance behavior of the product of currents. Equation (2.16) is our essential result for the anomaly in one space and one time dimension. The corresponding equation in 4-dimensional space time, $S = KR$, was first derived by Crewther.[5]

It is also possible to describe the anomaly in the divergence of axial current without any reference to the decay $\pi^0 \to \gamma$ at all. We begin with

$$S = \frac{i}{2} \int d^2x\, \epsilon^{\lambda\mu}\, x_{\lambda}\, T^* \langle 0 | \partial^{\gamma} j^5_{\gamma}(x) j_{\mu}(0) | 0 \rangle$$

We can proceed as before. The final result relating S to the short-distance behavior of the product of currents is still given by Eq. (2.16).

In Section III and IV, we will check explicitly the validity of Eq. (2.16) in the Schwinger model and the Thirring model.

III. Schwinger Model

The Schwinger model is the quantum electrodynamics with massless Dirac field in one space and one time dimension. It is a rather unique model for which an exact and divergence-free solution exists.

Schwinger has shown that the vacuum expectation value of the gauge invariant current is related to the external field A by

$$\langle j_\mu(x) \rangle = - \frac{e_0}{\pi} A_\mu(x) + \frac{e_0}{\pi} \partial_\mu \int (dx') D_F(x - x') \partial'_\nu A^\nu(x') \qquad (3.1)$$

where D_F is the outgoing-wave Green's function defined by

$$\partial^2 D_F(x, x') = \delta(x, x') \qquad (3.2)$$

and $\langle \ \rangle$ denotes vacuum expectation value. From Eq. (3.1) it follows that the divergence of axial current defined in Eq. (2.10) is

$$\langle \partial^\mu j^5_\mu(x) \rangle = - \frac{e_0}{\pi} \epsilon^{\mu\nu} \partial_\mu A_\nu . \qquad (3.3)$$

In fact starting with Schwinger's general solution for 2n-point fermion Green's functions, it is straightforward to show that Eq. (3.3) holds as an operator equation between fermion fields. In other words

$$\partial^\mu j^5_\mu = - \frac{e_0}{\pi} \epsilon^{\mu\nu} \partial_\mu A_\nu \qquad (3.4)$$

as an operator equation for all external field A. This further implies that Eq. (3.4), holds as an operator equation with both j^5_μ and A being operators.[16] This tells us that there is an anomaly in the divergence equation for the gauge-invariant axial current.

Now we turn to the problem of the short-distance behavior of the product of currents. The most convenient way to proceed is to start with the relation between

$<A_\mu(x) A_\nu(x')>$ and $<j_\mu(x) j_\nu(x')>$.[17] They are related in a very simple way by the field equation. In terms of the spectral representation

$$<A_\mu(x)A_\nu(x')> = \int \frac{(dp)}{(2\pi)} \quad e^{-ip(x-x')} \quad \theta(p^o) \; \delta(p^2 - m^2) dm^2 A_{\mu\nu}(p) , \quad (3.5)$$

where

$$A_{\mu\nu}(p) = B(m^2) \quad [-g_{\mu\nu} + \cdot \circ \circ \cdot] \qquad (3.6)$$

with $\cdot \circ \circ \circ$ denoting terms which depend on the gauge. Also,

$$<j_\mu(x)j_\nu(x')> = \int \frac{(dp)}{2\pi} \quad e^{-ip(x-x')} \; \theta(p^o) \, \delta(p^2 - m^2) dm^2 j_{\mu\nu}(p) . \qquad (3.7)$$

The Maxwell equation supplies the link between $A_{\mu\nu}(p)$ and $j_{\mu\nu}(p)$. One finds that

$$e_0^2 \; j_{\mu\nu}(p) = m^2 B(m^2)(p_\mu p_\nu - g_{\mu\nu} p^2) \qquad (3.8)$$

Schwinger derives the expression for the Green's function of radiation field. It is

$$G_{\mu\nu}(x, x') = \pi_{\mu\nu}(i\,\partial) \; G(i\,\partial) \quad \delta(x - x') \qquad (3.9)$$

where $\pi_{\mu\nu}(p)$ is a gauge-dependent projection matrix and

$$G(p) = \int dm^2 \quad \frac{B(m^2)}{-p^2 + m^2 - i\epsilon} \quad = \quad \frac{1}{-p^2 + e_0^2/\pi - i\epsilon} \qquad (3.10)$$

This indicates that $Z_3 = 1$ in Schwinger model.

With the aid of Eqs. (3.7 - 3.10), it is straightforward to derive the following expression for the time-ordered product of currents

$$T <j_\mu(x)j_\nu(x')> = \frac{i}{\pi} \; (-\partial_\mu \; \partial_\nu + g_{\mu\nu} \, \partial^2) \; \Delta_F\left(\frac{e_0^2}{\pi}, x - x'\right), \text{ for } x \neq x' . \quad (3.11)$$

where Δ_F is the Green's function satisfying

$$(\partial^2 + \mu^2)\, \Delta_F(\mu^2, x - x') = - \delta(x - x'), \tag{3.12}$$

together with out-going boundary conditions. In the limit $x \to 0$, the singular behavior of Δ_F is given by

$$\Delta_F \left(\frac{e_0^2}{\pi}, x\right) \longrightarrow \frac{i}{4\pi} \log(-x^2 + i\epsilon) + \cdots \tag{3.13}$$

where $\cdots$ denotes nonsingular terms which break the scale invariance and are of order $\frac{e_0^2}{\pi}$ or higher. Incidentally, the limit as $e_0 \to 0$ of $\Delta_F \left(\frac{e_0^2}{\pi}, x\right)$ is $\frac{i}{4\pi} \log(-x^2 + i\epsilon)$.

From Eqs. (2.10), (3.11) and (3.13), we obtain

$$T(j_\mu^5(x) j_\nu(0)) = - \frac{1}{4\pi^2} \epsilon_{\mu\lambda} (-\partial^\lambda \partial_\nu + g_\nu^\lambda \partial^2) \log(-x^2 + i\epsilon) + \cdots . \tag{3.14}$$

We see clearly that Wilson's hypothesis agrees with the result obtained by the explicit calculation. Using our result for anomaly in the previous section we obtain

$$S = R = - \frac{1}{\pi} \tag{3.15}$$

This is precisely the coefficient of $e_0 \epsilon^{\mu\nu} \partial_\mu A_\nu$ on the right hand side of Eq. (3.4). We see clearly that the result we derive in Section II agrees with the results obtained through explicit calculation to all order in e_0.

The perturbation calculation of anomaly in the Schwinger model has been studied by Georgi and Rawls.[18] We don't want to repeat their arguments here. It suffices to mention that in perturbation theory the anomaly in PCAC is due to the presence of fermion bubble diagram. The result of Georgi and Rawls is in complete agreement with ours.

Therefore, we conclude that in the Schwinger model both the perturbation theory and Wilson's theory of broken scale invariance can be used. They yield the same result as that of the explicit solution to all orders in coupling constant e_0.

IV. Thirring Model

The Thirring model is a field theory model of massless Dirac field in one space and one time dimension with current-current interaction. The solutions of the Thirring model both with and without coupling to the external electromagnetic field are known.[10,11] In this section we use Johnson's solution to investigate the problem of PCAC anomaly in the Thirring model.

Without electromagnetism, the Thirring model contains two conservation laws: conservation of charge and conservation of axial charge

$$\partial_\mu j^\mu = 0 \tag{4.1}$$

and

$$\partial_\mu \epsilon^{\mu\nu} j_\nu = 0. \tag{4.2}$$

In the presence of external c-number electromagnetic field, Eq. (4.2) is no longer true. Johnson has shown that Eq. (4.2) becomes

$$\partial_\mu \epsilon^{\mu\nu} j_\nu = - \frac{e_0}{\pi} \frac{1}{1 - (\lambda/2\pi)^2} \epsilon^{\mu\nu} \partial_\mu A_\nu \, . \tag{4.3}$$

The axial current $\epsilon^{\mu\nu} j_\nu$ is invariant under the usual gauge transformation. Equation (4.3) is true as an operator equation for all external c-number electromagnetic field. This then further implies that Eq. (4.3) holds as an operator equation with A_μ corresponding to the usual quantum mechanical operator of the electromagnetic potential.[16]

We can consider the Thirring model as a skeleton theory in the sense of Wilson, for the Thirring model without electromagnetism is scale invariant for all values of λ, the coefficient of current-current interaction. In this skeleton theory, the time-ordered product of two currents is (according to Johnson)

$$T\langle 0|j_\mu(\xi)j_\nu(0)|0\rangle = \frac{i}{\pi}\frac{1}{1-(\lambda/2\pi)^2}(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)\,D_F(\xi) \quad \text{for} \quad \xi \neq 0 . \tag{4.4}$$

Now we want to know what will happen if the Dirac field is coupled to the radiation field. To answer this question all we need is the expression for the vacuum expectation value of j^μ in the presence of an external field. According to Johnson, $\langle j^\mu(x)\rangle$ is given by

$$\langle j^\mu(x)\rangle = -\frac{e_0}{\pi}\frac{1}{1-(\lambda/2\pi)^2}\,\epsilon^{\mu\alpha}\partial_\alpha\,\epsilon^{\nu\beta}\partial_\nu\int D_F(x-x')A_\beta(x')dx' \tag{4.5}$$

Once the dependence of $\langle j^\mu(x)\rangle$ on A_β is known, the Green's functional of the radiation field can be solved.[8] We find, in particular, the 2-point Green's function for the radiation field is given by

$$G_{\mu\nu}(x, x') = \pi_{\mu\nu}(i\partial)\;G(i\partial)\,\delta(x-x') \tag{4.6}$$

with

$$G(p) = \frac{1}{-p^2 + \frac{e_0^2}{\pi}\frac{1}{1-(\lambda/2\pi)^2} - i\epsilon} = \frac{1}{-p^2+\mu'^2 - i\epsilon} \tag{4.7}$$

where

$$\mu'^2 = \frac{e_0^2}{\pi}\frac{1}{1-(\lambda/2\pi)^2} \tag{4.8}$$

Equations (4.6-7) indicate that we have $Z_3 = 1$ and $e = e_0$ in this model.

As before, we can relate the expression $<j_\mu j_\nu>$ to $<A_\mu A_\nu>$ by Maxwell equation. Therefore we obtain the expression for $T<0|j_\mu j_\nu|0>$, for the case in which the fermion couples to the radiation field,

$$T<0|j_\mu(x)j_\nu(x')|0> = \frac{i}{\pi} \frac{1}{1-(\lambda/2\pi)^2} (g_{\mu\nu}\partial^2 - \partial_\mu \partial_\nu)\Delta_F(\mu'^2; x-x') \quad (4.9)$$

$$\text{for} \quad x - x' \neq 0.$$

Note that the dominating behavior of Eq. (4.9) is the same as that of Eq. (4.4), which is the expression in the skeleton theory, in agreement with Wilson's hypothesis.

Now the anomaly constant S can be determined by the method of Section II. The result is

$$S = -\frac{1}{\pi} \frac{1}{1-(\lambda/2\pi)^2} \quad (4.10)$$

This is precisely the coefficient of $e_0 \epsilon^{\mu\nu} \partial_\mu A_\nu$ in Eq. (4.3). We see clearly that Wilson's approach gives the correct result for anomaly in the Thirring model.

We would like to mention the result of perturbation calculation in the Thirring model.[19] One can find the expression for anomaly following Adler's treatment. The short-distance behavior of the product of two currents can also be obtained. The relation we obtained (Eq. (2.16)) is consistent with the results of perturbation calculation. Although the precise expressions for R and S in perturbation calculation differ from ours, this is not in contradiction with our conclusion, as it is well known that Thirring model allows an infinite number of solutions. This has been studied carefully by Hagen.[11]

V. Discussion

In the previous sections we have shown that in the Schwinger model and Thirring model the anomaly in the divergence of axial vector current can be studied from perturbation theory as well as from Wilson's theory of broken scale invariance and short-distance behavior of the product of currents. Both approaches give the same result for the anomaly.

To conclude we would like to make several remarks concerning the model in one space and one time dimension.

(1) It is a peculiarity of model in one space and one time dimension that the photon has nonvanishing mass. This is a reflection of the fact that the coupling constant e_0 has the dimension of mass in one space and one time dimension. Historically quantum electrodynamics in one space and one time dimension was used by Schwinger to demonstrate that gauge invariance does not necessarily require the existence of massless photon.[8,17] In view of this fact, it is necessary for us to reconsider the method we use in Section II. There we assume implicitly that the photon is massless. When we take into account the photon mass ($=\mu$), the expression $e_0 T^{\mu}(k)$ in Eq. (2.1) becomes

$$e_0 T^{\mu}(k) = e_0\, \epsilon^{\mu\lambda} k_{\lambda} T(k^2)$$

$$= \left(\frac{m^2 - k^2}{F_{\pi}\, m^2}\right)(\mu^2 - k^2)\,\frac{1}{\sqrt{Z_3}} \int d^2x (e^{-ikx} - 1)\; T^* \langle 0 | \partial^{\gamma} j_{\gamma}^5(x) A^{\mu}(0) | 0 \rangle \tag{5.1}$$

Recall that in Schwinger and Thirring models

$$\partial^{\gamma} j_{\gamma}^5(x) = S\, e\, \epsilon^{\alpha\nu}\, \partial_{\alpha} A_{\nu}$$

Therefore

$$e_0 T^{\mu}(k) = \left(\frac{m^2 - k^2}{F_\pi m^2}\right) (\mu^2 - k^2) \frac{Se}{\sqrt{Z_3}} \epsilon^{\alpha\nu}$$

$$\times \int d^2x (e^{-ikx} - 1) T^* <0 | \partial_\alpha A_\nu(x) A^{\mu}(0) | 0>. \qquad (5.2)$$

Note that since the photon Green's function has the structure of $\frac{Z_3}{k^2 - \mu^2}$ with $Z_3 = 1$, the factor $\mu^2 - k^2$ in front of the integral in Eq. (5.2) is cancelled by the factor of $\frac{1}{\mu^2 - k^2}$ from the integral. We end up with the same result even when we set μ to be zero (that is, pretending the photon to be massless). If we do set $\mu = 0$, then Eq. (5.1) can be cast into the form of Eq. (2.2). This is the reason why the result we derive in Section II is valid to all order in e_0 in Schwinger and Thirring models.

(2) In a recent preprint by Adler, et al.,[7] the problem of the constraints on anomalies is discussed in several models. We find that in the Thirring model, the Schwinger term is given by

$$<0 | [j^o(x), j^1(x')] | 0> = i\partial_1(x - x') \left\{ -\frac{1}{\pi} \frac{1}{1 - (\lambda/2\pi)^2} \right\}$$

Hence the Schwinger term is intimately related to the PCAC anomaly. Same comment applies to the Schwinger model.

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