SLAC-PUB-1099
(TH)
August 1972

TIME OPERATORS, PARTIAL STATIONARITY, AND THE ENERGY-TIME UNCERTAINTY RELATION*

Joseph H. Eberly†
Stanford Linear Accelerator Center, Stanford University
Stanford, California 94305

and

Department of Physics, Stanford University
Stanford, California 94305

and

L. P. S. Singh
Department of Physics and Astronomy, University of Rochester
Rochester, New York 14627

ABSTRACT

The reciprocal time operator which is suggested by the notion of partial stationarity is shown to permit an unambiguous and non-singular statement of the energy-time uncertainty relation.

(Submitted to Phys. Rev. D)

---

*Research partially supported by the U.S. Air Force Office of Scientific Research and the U.S. Atomic Energy Commission

†Permanent address: Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627

The recurring debates[1] over the formulation and meaning of the Heisenberg uncertainty relation for energy and time make clear its unsatisfactory features. Some of these features have counterparts in the ambiguous phase-number and angle-angular momentum uncertainty relations which have only recently been investigated thoroughly.[2]

The most serious questions, however, accompany the notion of a time operator. Such an operator, $\mathcal{T}$ say, may satisfy $[\mathcal{T}, \mathcal{H}] = i\hbar$ and thereby imply the canonical uncertainty relation; but it cannot simultaneously be an operator with a continuous, unbounded and real spectrum (which one would identify with the values taken by the parameter t, physical time).

This has not prevented the introduction[1, 3, 4, 5] of various operators which in restricted circumstances behave very much as a time operator "should" behave. Often subtle arguments are necessary to show that these operators are well-enough behaved, when applied properly. The central unsatisfactory feature of these operators, though, apart from their singularites and ambiguites, is their ad hoc character. Operators designed for free particle wave packets rarely have relevance to the decay of bound systems, and conversely. At the most heuristic level this appears to be widely recognized, but is never remarked upon explicitly.[6]

Thus it is remarkable that there already exists one operator which is unambiguous and non-singular and which serves to define an energy uncertainty time in agreement with the familiar decay lifetimes and packet spreading times. As it happens, this is not a "time" operator but a "reciprocal time" operator. It exists for all quantum systems with a density matrix and a Hamiltonian.

In the following paragraphs we show how the concept of partial stationarity leads to a consideration of this operator. Then the notion of a stationarity time for a quantum statistical system follows naturally. We show that this stationarity

time is precisely the energy uncertainty time of the system. When applied to a free wave packet the stationarity time is indentical with the packet spreading time; and when applied to a decaying excited state the stationarity time is found to be equal to the usual lifetime.

It is well-known that the dynamical variables of a system whose density matrix commutes with the Hamiltonian are statistically stationary, and conversely.[7] Although stationarity, or translation invariance, is also useful as a spatial or angular concept,[8] we are here concerned with it in the sense of time translations alone. For example, if A and B are two Heisenberg operators, stationarity implies that the expectation $<A(t)\, B(t+\tau)>$ depends on the time argument difference τ and not on the origin of time implied by t.

Since $i\hbar\, d\rho(t)/dt = [\mathscr{H}, \rho(t)]$ (in the Schrödinger picture), we see that stationarity implies $d\rho/dt = 0$, and conversely.[7] It is always true that $<\dot{\rho}> = 0$ so it is natural to adopt the dispersion of $\dot{\rho}$ as a measure of the degree of stationarity which pertains in any given physical situation. Clearly $\dot{\rho}$ is, in some sense, a reciprocal time operator. It is Hermitean. Let us define a stationarity time T_s by the relation

$$1/T_s^2 \equiv \Delta\dot{\rho}^2. \tag{1}$$

We prove below that for any quantum system,

$$T_s^2\, \Delta H^2 \geq \hbar^2, \tag{2}$$

where the mean square dispersion of an operator 0 is computed as usual: $\Delta 0^2 = <(0 - <0>)^2>$, and $<(\ldots)>$ means $\mathrm{Tr}[\rho(\ldots)]$. In other words, we prove that the stationarity time T_s can be invoked unambiguously as the true energy uncertainty time of the quantum system.

In some respects our uncertainty time is similar to a class of uncertainty times described first, apparently, by Mandelstam and Tamm.[3, 9] However

T_s has several distinctions not enjoyed by the Mandelstam-Tamm set of times, the most important of which are:

(1) T_s refers to the quantum system as whole, through $\dot{\rho}$, rather than to a single dynamical variable of the system.

(2) While <u>not</u> a member of the Mandelstam-Tamm set of times, T_s appears to bound all of those times from below, thus providing a sharper statement of the energy-time uncertainty relation. This is easily proved for pure-state expectations.

(3) T_s has a physical meaning which is independent of its appearance in the energy uncertainty relation.

As far as the formal uncertainty principle goes, it is of course irrelevant what this physical meaning is and what considerations motivated the study of the parameter T_s. Nevertheless, it seems worth a further brief digression to establish the sense in which T_s is a stationarity time, or an indicator of partial stationarity. Consider two simple examples. The first is a free single-mode radiation field, with Hamiltonian $\mathscr{H} = \hbar\omega a^+ a$. Assume the system is in the coherent state $|v>$, so that $\rho = |v><v|$, where $v = |v| \exp(i\phi)$; and compute the auto-correlation of the real "field" $A(t) = a(t) + a^+(t)$:

$$<A(t)\, A(t+\tau)> = e^{i\omega\tau} + (2/\omega T_s)^2 \cos\ (\omega(t+\tau)+\phi)\cos(\omega t+\phi) \qquad (3)$$

As expected, since $[\rho, \mathscr{H}] \neq 0$, the correlation depends explicitly on t as well as on τ. However an apparently general feature is evident here. The importance of the non-stationary t-dependent part of the correlation function is directly controlled by T_s. That is, if the sationarity time T_s is long, the t-dependence is unimportant, and we can consider the system to be stationary.

As a second simple example, consider a free particle of mass m moving in

one dimension and described at time t = 0 by the wave function $\psi(x, o) = C\, e^{ikx} \exp\left[-\frac{1}{2}\lambda^2 x^2\right]$ where C is the normalization constant. In this case a simple calculation shows that the stationarity time is inversely proportional to λ^2; specficially, $T_s = \frac{8}{5}\,(m/\hbar\lambda^2)$. This is natural since when $\lambda \to 0$ the state is a free particle energy eignenstate and thus completely stationary, requiring an infinite stationarity time. The position variance $F(t, \tau) = <\Delta x(t)\Delta x(t+\tau)>$ is worth examining. In this case we find

$$F(t, \tau) = <\Delta x(o)^2> + \frac{i\hbar\tau}{2m} + \frac{8}{5} <\Delta x(o)^2> \frac{t(t+\tau)}{T_s^2} \quad . \tag{4}$$

Thus, again, for times t short compared with the stationarity time T_s the correlation function effectively depends only on τ and the system is essentially stationary.

We now prove the uncertainty relation $T_s^2 \Delta\mathscr{H}^2 > \hbar^2$. We work in the (necessarily discrete) basis of eigenstates of ρ, and write those eigenstates $|m>$, so that $\rho = \sum_m p_m |m><m|$. We have $1 \geq p_m \geq 0$ in general, and in the special case of a pure state all of the p's are zero except one which is unity.

First we calculate the dispersion in energy. A short calculation leads to

$$\Delta\mathscr{H}^2 = \sum_m p_m (\Delta\mathscr{H}^2)_m + \sum_m p_m \left|\mathscr{H}_{mm} - \sum_\ell p_\ell \mathscr{H}_{\ell\ell}\right|^2 , \tag{5}$$

where $(\Delta\mathscr{H}^2)_m$ is the dispersion calculated in the pure state $|m><m|$ and $\mathscr{H}_{kk} = <k|\mathscr{H}|k>$. Clearly then,

$$\Delta\mathscr{H}^2 \geq \sum_m p_m (\Delta\mathscr{H}^2)_m = \sum_{m, k \neq m} p_m \mathscr{H}_{mk} \mathscr{H}_{km} \quad . \tag{6}$$

Next consider $(1/T_s^2) = \Delta\dot{\rho}^2$. One finds the result:

$$\hbar^2 \Delta\dot{\rho}^2 = \sum_{k, m} p_m (p_m - p_k)^2 \mathscr{H}_{mk}\mathscr{H}_{km} \quad . \tag{7}$$

Since $1 \geq (p_m - p_k)^2$, a comparison of Eqs. (6) and (7) proves the inequality $\Delta\mathscr{H}^2 \geq \hbar^2 \Delta\dot{\rho}^2$ or, what is the same things, $T_s^2 \Delta\mathscr{H}^2 \geq \hbar^2$, the energy-time

uncertainty relation given in (2).

The circumstances under which the equality holds are easily deduced. In the case of a pure state density matrix one always has the minimum energy-time uncertainty product $T_s^2 \Delta\mathscr{H}^2 = \hbar^2$, and conversely.

In this brief note we have not explored all of the implications of the ideas presented here. There seems to be nothing especially quantum mechanical about the underlying notion of stationarity time. Perhaps all statistical theories in which the statistical distribution and the random variables both obey a dynamical law of motion should be expected to have sensible stationarity times. As a final step here, however, we should make clear that the stationarity time T_s is intimately related to all of the commonly accepted measures of an energy uncertainty time.

First, let us show how T_s is related to the lifetime of an unstable excited state. Consider the well-known Weisskopf-Wigner model[10] for excited state decay. In this model the lifetime of the excited state τ_0 is given by

$$\frac{1}{\tau_0} = (2\pi/\hbar^2) |\langle f | H_I | i \rangle|^2 \rho(\omega_f), \tag{8}$$

where $|i\rangle$ and $|f\rangle$ are the initial and final states, H_I is the interaction Hamiltonian, and $\rho(\omega_f)$ is the usual density of final states. For this same excited system it is also possible to compute T_s. Initially $\rho = |i\rangle\langle i|$, and one finds

$$1/T_s^2 = \hbar^{-2} \sum_{\chi}{}' |\langle i | H_I | \chi \rangle|^2 , \tag{9}$$

where the primed sum means that one includes only those intermediate states which H_I connects with the initial state. In the Weisskopf-Wigner spirit, these are of course just the states designated $|f\rangle$ above. The sum includes all of them in a narrow range of final energies $\delta E = \hbar \delta\omega$. Thus we find $1/T_s^2 = \delta\omega/2\pi\tau_0$. However, in the Weisskopf-Wigner approximation the range of final state frequencies which occur with appreciable probability is simply related to the state

lifetime, $\delta\omega \sim 2\pi/\tau_0$. Thus we find the canonical result $T_s \sim \tau_0$.

As a second, much simpler, example let us look at the free particle discussed earlier. In that case we find, up to an unimportant numerical factor, essentially the conventional relation between T_s and the uncertainties in position and velocity: $T_s^2 = (8/5)\frac{\Delta x^2}{\Delta v^2}$. So, in these two very different contexts of free particle and excited bound state, we find that T_s reduces in each ease to an appropriate familiar heuristic "uncertainty time".

The first author wishes to thank a number of colleagues, especially Prof. L. Mandel and Dr. K. Whitney, for the benefit of conversations concerning partial stationarity and the time-energy uncertainty relation; and to express his appreciation to Profs. S.D. Drell and A.L. Schawlow for their hospitality at Stanford during the year 1971-72.

REFERENCES

1. See, for example: Y. Aharonov and D. Bohm, Phys. Rev. 122, 1649 (1961); V.A. Fock, Sov. Phys. JETP 15, 784 (1962); Y. Aharonov and D. Bohm, Phys. Rev. 134, B1417 (1964).
2. A review of the phase-number and angle-angular momentum studies, as well as references to the energy-time controversy, are given by P. Carruthers and M. M. Nieto, Rev. Mod. Phys. 40, 411 (1968).
3. L. Mandelstam and I. Tamm, J. Phys. (U.S.S.R.) 9, 249 (1945). See also A. Messiah, Quantum Mechanics (North-Holland Publishing Co., Amsterdam, 1961), pp. 319-320.
4. F.T. Smith, Phys. Rev. 118, 349 (1960).
5. B.A. Lippmann, Phys. Rev. 151, 1023 (1966).
6. The conventional understanding is essentially dichotomous. That is, the uncertainty times associated with wave packet spreading and with excited state decay are regarded as unrelated consequences of the uncertainty principle. This point of view is apparent in every quantum mechanics text known to the authors.
7. Y. Kano, Annals of Phys. 30, 127 (164).
8. J.H. Eberly and A. Kujawski, Phys. Letters 24A, 426 (1967); see also D. Dialetis and C.L. Mehta, Il Nuovo Cimento 56B, 89 (1968).
9. The closest approach which we have found to the ideas embodied in our discussion is in the text by D. Bohm, Quantum Theory (Prentice-Hall, Englewood Cliffs, N.J., 1951) pp. 227-228.
10. V.F. Weisskopf and E. Wigner, Z. Physik, 63, 54 (1930); G. Källén, Encyclopedia of Physics (Springer-Verlag, Berlin, 1958), Vol. V/1. Although a significant error occurs in these well-known discussions of spontaneous

emission [see J. R. Ackerhalt, P. L. Knight, and J. H. Eberly (to be published)], it affects the predicted frequency shift but not the line width, and so does not alter the present discussion.