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EXPERIMENTAL CONSEQUENCES OF POSITIVITY AND DUALITY IN THE ZERO-WIDTH APPROXIMATION*

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ABSTRACT

A rigorous, model-independent treatment of positivity and duality is reviewed and experimental implications are discussed. Under the assumption that dibaryon resonances do not exist or do not participate in duality, it is shown that the least massive spin J meson which couples to baryons must have I = 0 and $P = C = (-1)^J$ for all $J \geq \alpha(0)$, the leading effective trajectory intercept. When degeneracy occurs, the couplings of the I=0, $P = C = (-1)^J$ mesons must dominate. The mass inequalities $M_\omega \leq M_\rho$, $M_\omega \leq M_{K^*}$, $M_f \leq M_{A2}$, $M_f \leq M_{K^{**}}$ follow immediately. Among the implications for $J > 2$ is some rather strong support for a $J^P = 3^-$ assignment for the $\phi(1675)$ as well as a $K\bar{K}$ decay mode for this resonance. At present, there is very little information about higher mass states with I=0. The identification of any Regge recurrence with $J \geq 4$ among the high mass $I > 0$ states would immediately provide an upper bound on the mass of a required I=0 meson with the same spin and $P = C = (-1)^J$. The experimentally possible J^P assignments of 2^- for $\eta'(958)$, 2^- for A1(1070) and 3^- for the $K_N(1420)$ would be in dramatic disagreement with positivity and a dual role for these resonances. Inequalities involving meson and baryon masses are also obtained. In terms of baryon and leading meson trajectories, the result is that $\alpha_M \geq \alpha_B - 1/2$, which forces the leading meson trajectory to rise at least as fast as the leading baryon trajectory. A possible stronger inequality, $\alpha_M \geq \alpha_B + 1/2$, which is satisfied by the observed low-spin resonances, is discussed. Inequalities are found among couplings of arbitrary spin baryons to recurrences on the leading

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meson trajectory. For ω-nucleon coupling $g\gamma^{\mu} + i(f/2m)\sigma^{\mu\nu} q_{\nu}$ with $M_{\omega} < M_{\rho}$ one finds that $g^2 \geq (fM_{\omega}/2m)^2$. With ω-ρ degeneracy, $g_{\omega}^2 - g_{\rho}^2 \geq (M_{\omega}/2m)^2 \left| f_{\omega}^2 - f_{\rho}^2 \right|$. Stronger inequalities follow from the additional assumptions of nonet ω-ϕ mixing with SU_3 invariant couplings to the baryon octet and decoupling of ϕ from the proton. The strongest inequality for couplings is found to be satisfied as an equality if numerical values given by vector dominance are substituted. Under the same additional assumptions, it is shown that the next lowest mass J = 1, nonstrange meson above the ω and ρ should have P = C = -1 and I = 0, which are just the quantum numbers of the ϕ. Other possible low-mass J = 1 resonances are the A1(1070), B(1235), D(1285) which satisfy $M_{\phi} < M_{A1}, M_B, M_D$. Since the 2^+ nonet also exhibits nonet symmetry, $M_{f'}$ may provide a similar lower bound on the masses of J = 2 nonstrange mesons which lie above the f and the A2. The agreement between presently available experimental data, previous models for couplings, and results obtained for $B\bar{B}$ systems from positivity, duality, and the assumption of no dual dibaryon resonances, tends to indicate that duality does apply to $B\bar{B}$ systems. This in turn provides some general support for the existence of exotic mesons which couple to baryons.

INTRODUCTION

Duality in one form or another has generated many predictions which are relevant to experimental meson spectroscopy.[1] Unfortunately, specific dual models such as the Veneziano model[1,2] frequently disagree with experiment[3] and have well-known problems associated with ghosts.[4] Another troublesome fact is that general arguments[5] involving duality, when applied to particular processes such as $\Delta\bar{\Delta} \to \Delta\bar{\Delta}$, lead to conflicts with the naive quark model which has been useful in understanding hadron spectroscopy. Of course, the naive quark model may be wrong. In any case, there is apparently need for caution regarding predictions of the familiar approaches to duality.

We will describe here a relatively new method[6] for obtaining predictions from model-independent features of duality in the zero-width approximation. This method involves a minimal set of assumptions, has no diseases or difficulties, and conflicts with no well-established experimental data. In this approach, Regge behavior and factorization are replaced by the more general requirements of power boundedness and positivity. The zero-width approximation is used because it leads to rigorous results for resonance poles instead of inferences drawn from imaginary parts of assumed asymptotic forms. Our results are in the form of mass and coupling constant inequalities. There seems to be a tendency for some of the inequalities to be experimentally satisfied as equalities.

We begin by considering the amplitude for elastic scattering of two spinless particles. Duality is associated with the statement that the amplitude can be represented by a sum of s-channel poles

$$M(s,t) = \sum_{i=0}^{\infty} \frac{1}{s-s_i} \sum_{n=0}^{J_i} A_{in} t^n \qquad (1)$$

or by a sum of t-channel poles

$$M(s,t) = \sum_{j=0}^{\infty} \frac{1}{t-t_j} \sum_{m=0}^{J_j} B_{jm} s^m \qquad (2)$$

provided that there are no u-channel poles or other singularities. Subtractions may be needed in Eq. (1) and (2). Due to positivity of residues of elastic partial wave amplitudes and properties of Legendre polynomials, the coefficients A_{in} are all positive for poles above threshold. If the t-channel amplitudes are not elastic, then we cannot make a similar argument for the signs of B_{jm}. However, it happens that because of duality, some of the B_{jm} must be positive. If t_k is the lowest lying resonance pole with spin J, then the coefficient B_{kJ} of the highest power of s in the polynomial residue can be shown to be positive. This implies positivity of residues of those poles which in Regge terminology would be recurrences on the leading t-channel trajectory. A statement of the relevant theorem[7] is as follows: If (a) M(s,t) is an analytic function whose only singularities are simple poles at $s = s_i > 0$ and $t = t_j > 0$, $i, j = 0, 1, 2, \ldots$, where s_i and t_j are real constants, (b) M(s,t) is polynomially bounded away from its poles as $|s| \to \infty$ for fixed t, (c) residues of poles in s are polynomials in t, and (d) with the possible exception of a finite number of these residues, the coefficient of each power of t in each polynomial residue is positive, then (i) residues of poles in t are polynomials in s, and (ii) the coefficient of the highest power of s, say s^J, in the residue of each leading pole (the lowest-lying pole with a polynomial residue of a given order) in t is positive for $J \geq J_0$, where $s^{-J_0} M(s,0) \to 0$ as $s \to \infty$.

Conditions (a), (b), and (c) are satisfied by the zero-width approximation to any kinematic singularity-free scattering amplitude which has no u-channel poles. For the scattering of particles with spin, one can construct certain linear combinations of kinematic singularity-free helicity amplitudes which satisfy condition (d). Polynomial boundedness insures convergence of partial fraction expansions[8] so that M(s,t) can be expressed as a sum of poles in s (with subtractions) without the explicit appearance of poles in t. This is a necessary requirement of duality. Since effective trajectory intercepts fall in the range $1 > \alpha(0) > 0$, our results should apply for $J \geq J_0 = 1$. This would become $J \geq J_0 = 2$ if the Pomeranchukon were included in dual amplitudes and if $\alpha_P(0) = 1$. It should be pointed out that the theorem is nontrivial in two respects. First, in problems with spin and internal symmetries, s-channel poles with positive residues are dual to t-channel poles with both positive and negative residues due to signs of crossing matrix elements. Second, the theorem applies even when the t-channel reactions are not elastic in which case there is no a priori knowledge of the sign of the residues of t-channel poles.

BARYON-ANTIBARYON CHANNELS

We now describe the application[6] of the theorem to the process $B_1\bar{B}_2$ $B_1\bar{B}_2$ with arbitrary spin baryons B_1 and B_2. The purpose of such considerations is to derive mass and coupling constant inequalities for the mesons which can be exchanged in this process. It is undesirable to consider only the spin $\frac{1}{2}$ nucleons because $P = -C = (-1)^J$ mesons do not couple to $N\bar{N}$. Furthermore, restrictions on the baryon spins would leave open the possibility of obtaining inconsistent results for different amplitudes. The first step is to find amplitudes which satisfy requirements (a) - (d) of the theorem. This problem has been solved[6] under the assumption that dibaryon resonances do not exist or do not participate in duality. If we denote s-channel helicity amplitudes[9] for $B_1\bar{B}_2 \rightarrow B_1\bar{B}_2$ by $M^s_{\mu\beta\lambda\alpha} = \langle \mu\beta|M|\lambda\alpha\rangle$ and introduce a set of amplitudes

$$F^s_{\mu\beta\lambda\alpha} \equiv (-1)^{\mu-\beta}\left[s^{1/2}\tan(\theta_s/2)\right]^{\lambda-\alpha+\mu-\beta} M^s_{\mu\beta\lambda\alpha} , \tag{3}$$

then a function which has been shown to satisfy the requirements of the theorem can be written in the form[10]

$$\Lambda \equiv \left[(s-\Delta)(s-s_0)\right]^{j_1+j_2}(s^2t)^{-\min(\lambda-\alpha,\mu-\beta)}\left[a^2F^s_{\mu\beta\mu\beta} + 2bF^s_{\mu\beta\lambda\alpha} + c^2F^s_{\lambda\alpha\lambda\alpha}\right] , \tag{4}$$

where a, b, c are real constants with $|b| < |ac|$, $s_0 \equiv (m_1+m_2)^2$, $\Delta \equiv (m_1-m_2)^2$ and j_1, j_2, m_1, m_2 are the spins and masses of B_1 and B_2. Internal symmetries are easily included since the conditions of the theorem are satisfied if the diagonal helicity amplitudes in Λ are also diagonal in all internal quantum numbers. For simplicity we initially neglect isospin, strangeness and SU(3) and focus our attention on parity and charge conjugation. Λ is free of kinematic singularities and, in general, the highest power of s appearing in the polynomial residue of a spin-J, t-channel pole in Λ is associated with a power of s^J in each of the amplitudes $F^s_{\mu\beta\mu\beta}$, $F^s_{\mu\beta\lambda\alpha}$, and $F^s_{\lambda\alpha\lambda\alpha}$.

Since t-channel resonances with $P = -C$ decouple[11,12] from diagonal s-channel helicity amplitudes, the residue of a $P = -C$, t-channel pole in Λ is proportional to b, the coefficient of the off-diagonal amplitude. Thus, the sign of the residue of any $P = -C$, t-channel pole in Λ can be made negative since b has an arbitrary sign. Consequently, the $P = -C$ poles cannot be the lowest-lying t-channel poles for a given spin without contradicting conclusion (ii) of the theorem. This result is inescapable because we are dealing with the t-channel poles of $F^S_{\mu\beta\lambda\alpha}$ with arbitrary helicities for arbitrary-spin baryons.

A more involved argument can be made to rule out leading $C = (-1)^{J+1}$ resonances. Straightforward application of the helicity crossing matrix[13] reveals that the coefficient of s^J in the residue of a spin-J, $C = (-1)^{J+1}$, t-channel pole in $F^S_{\mu\beta\mu\beta}$ is the negative of the corresponding coefficient of s^J in $F^S_{\mu-\beta\mu-\beta}$. Consequently, if the $C = (-1)^{J+1}$ pole is the lowest-lying pole with spin J, then we have a contradiction with the theorem unless the coefficient of s^J is zero for every diagonal F^S amplitude. However, application of the theorem to Λ shows that the coefficient of s^J must vanish for all off-diagonal amplitudes $F^S_{\mu\beta\lambda\alpha}$ if the coefficients of s^J vanish for diagonal amplitudes. This complete decoupling is impossible for a pole which really corresponds to a spin-J resonance in some $B_1\bar{B}_2$ channel. We conclude that the lowest-lying spin-J meson which couples to baryons cannot have $C=(-1)^{J+1}$. This result, together with the previous exclusion of $P = -C$, implies that in the absence of degeneracy, the leading t-channel poles must have $P=C=(-1)^J$. If degeneracy occurs, then the couplings of the leading $P = C = (-1)^J$ mesons must dominate over the couplings of their degenerate partners.

We now discuss the inclusion of internal symmetries. If each amplitude in Λ describes scattering in the same s-channel isotopic spin state, then the sign of particular t-channel isotopic spin exchange contributions to Λ will depend on signs of s-t, SU(2) crossing matrix elements. Only the t-channel $I = 0$ amplitudes always contribute with a plus sign.[14] Thus, unless the $I = 0$, $P = C = (-1)^J$ resonances are leading, we can find some Λ which leads to a contradiction with the theorem. It is particularly easy to see why strange mesons cannot be leading. Exchange of strangeness can only contribute to Λ through an off-diagonal contribution which is associated with the indefinite sign of b. The considerations which are necessary for incorporating SU(3) invariance are essentially the same as for SU(2).[14] The conclusion is that the lowest-lying spin-J meson which couples to baryons must have $I = 0$, $P = C = (-1)^J$ or, in case of degeneracy, the coupling of the $I = 0$, $P = C = (-1)^J$ meson must dominate. In an SU(3)-invariant theory, the $I = 0$ meson must also be a unitary singlet. This result should hold for all $J \geq J_0 > \alpha(0)$ the relevant effective trajectory intercept. We expect that the Pomeranchuk should be excluded and therefore we take $J_0 = 1$. In any case, the result should hold for $J \geq 2$.

THE $K\bar{K}$ CHANNEL

Under the assumption that KK resonances do not exist or do not participate in duality, the amplitudes for $K\bar{K} \to K\bar{K}$ can be treated in a fashion similar to the treatment of $B_1\bar{B}_2 \to B_1\bar{B}_2$. This yields information about the coupling of $I=0$ and $I=1$, $P=C=(-1)^J$ mesons to $K\bar{K}$. One finds that the least massive spin-J meson which couples to $K\bar{K}$ must have $I=0$ or, in case of degeneracy, the coupling of the $I=0$ meson must dominate.

MASS INEQUALITIES, J^P ASSIGNMENTS AND $K\bar{K}$ COUPLINGS

From the preceding analyses, we immediately obtain the mass inequalities $M_\omega \leq M_\rho$, $M_\omega \leq M_{K^*}$, $M_f \leq M_{A2}$, $M_f \leq M_{K^{**}}$ for $J=1$ and $J=2$ mesons. These inequalities are clearly in agreement with experiment[15] to within small fractions of resonance widths. Since we have used the zero-width approximation, the ω and the ρ as well as the f and the A2 should probably be considered as degenerate, in which case we should examine the coupling constant inequalities which insure positivity through the dominance of the $I=0$ coupling. For the couplings to $K\bar{K}$ we have $\gamma^2_{\omega K\bar{K}} \geq \gamma^2_{\rho K\bar{K}}$ and $\gamma^2_{fK\bar{K}} \geq \gamma^2_{A2K\bar{K}}$. The couplings of the f and A2 to $K\bar{K}$ are directly measurable but the data are imprecise. The f does appear to couple to $K\bar{K}$.[16]

The g(1680) is thought to be the spin-3 recurrence of the ρ and spin 3 is experimentally favored.[15,17] Since the g couples[17] to $K\bar{K}$, the theorem requires an $I^G=0^-$, $J^P=3^-$ meson with mass $\leq$ 1680 MeV which also couples to $K\bar{K}$. The only plausible candidate listed in the data tables is ϕ(1675). From the theorem and the apparent ϕ-g degeneracy, we conclude that the partial width of the as yet unobserved $K\bar{K}$ decay mode of the ϕ(1675) must be at least as large as the partial width of the observed,[17] charged $K\bar{K}$ decay mode of the g. If the g(1680) couples to any baryon, then the baryon-antibaryon analysis also supports the $J^P=3^-$ assignment for the ϕ(1675).

The data tables[15] indicate the presence of many possible high mass states with $I>0$ among which may be some Regge recurrences with $J \geq 4$. The results obtained here indicate that the identification of any $J \geq 4$, $I>0$ state would immediately provide a reliable upper bound on the mass of a required $I=0$ meson with the same spin and $P=C=(-1)^J$.

Since the least massive spin-2 meson with $I=0$ and $P=C=+1$ appears[15] to be the f(1260), the experimentally possible J^P assignments[15] of 2^- for the η'(958) and A1(1070) would be in dramatic disagreement with positivity and a dual role for these resonances. Experimental evidence against the 2^- assignment for the η'(958) has been reported at this conference.[18] Similarly, there is no known spin-3 meson with $I=0$, $P=C=-1$ and sufficiently low mass to avoid difficulty with the possible $J^P=3^-$ assignment[15] for the K_N(1420). It should be emphasized that the conclusions drawn here do not depend on an assumption of Regge behavior or the assignment of particles to trajectories as is the case with previous arguments[1] based on duality.

MESON-BARYON COUPLINGS

Application of the theorem to Λ immediately provides bounds on the strength of the leading spin-J t-channel exchange contributions to off-diagonal helicity amplitudes $M^s_{\mu\beta\,\lambda\alpha}$. Thus, one finds inequalities among couplings of arbitrary spin baryons to the leading spin-J mesons. As an example, we consider the simple case of vector-meson coupling to nucleons. It is an easy task to calculate t-channel vector-meson exchange contributions to the $N\bar{N} \rightarrow N\bar{N}$ amplitudes, $F^{S}_{\mu\beta\lambda\alpha}$, in terms of the couplings $g\gamma^{\mu} + (if/2m)\sigma^{\mu\nu} q_{\nu}$ where m is the nucleon mass. With $\mu = \beta = \frac{1}{2}$ and $\lambda = \alpha = -\frac{1}{2}$ in Eq. (4), the implication of the theorem for the ω-exchange contribution is found to be

$$a^2 g^2 + 2b\left(M_{\omega} f/2m\right)^2 + c^2 g^2 \geq 0 \ . \tag{5}$$

Since $|b| \leq |ac|$, we can choose $b = -ac = -a^2$ and obtain $g^2 \geq \left(M_{\omega} f/2m\right)^2$. Other choices for the helicities yield no new inequalities. With ω-ρ degeneracy, the inequality becomes

$$g_{\omega}^2 - g_{\rho}^2 \geq \left(M_{\omega}/2m\right)^2 \left|f_{\omega}^2 - f_{\rho}^2\right| . \tag{6}$$

This condition is consistent with Regge-pole models for high energy scattering involving nucleons, vector dominance of the electromagnetic properties of nucleons and nuclear potential analyses although f_{ω} is not well determined.[19]

NONET SYMMETRY

In order to obtain stronger inequalities for vector meson couplings to spin-$\frac{1}{2}$ baryons, we consider an SU(3) coupling scheme with nonet ω-ϕ mixing and with the ϕ decoupled from the proton.[20] Two of the inequalities which we obtain are

$$\left|\frac{f_{\rho}}{g_{\rho}}\right| \leq \frac{2m}{M_{\phi}} \left|\frac{F_V - D_V}{F_M - D_M}\right| \tag{7}$$

and

$$\left|\frac{f_{\rho}}{g_{\rho}}\right| \leq \frac{2m}{M_{\phi}} \left|\frac{F_V}{F_M}\right| \tag{8}$$

where m and M_{ϕ} are the masses of the proton and the ϕ, f_{ρ} and g_{ρ} are couplings of the ρ to the proton and F_V, D_V, F_M, D_M are the f-type and d-type coupling factors for the γ_{μ} and $\sigma_{\mu\nu}$ terms, respectively ($F_V + D_V = F_M + D_M = 1$). In the vector dominance model $F_V = 1$, $F_M = \frac{1}{4}$ and $f_{\rho}/g_{\rho} = 3.7$. Substituting the proton mass, the ϕ mass and the F's and D's in Eq. (7) gives the remarkable, though possibly accidental, result that $|f_{\rho}/g_{\rho}| \leq 3.7$.

In addition to stronger coupling constant inequalities, we find that nonet symmetry together with the theorem leads to further mass inequalities. It is found that due to certain ω and ρ decouplings, the next lowest mass $J = 1$, nonstrange meson above the ω and ρ should again have $P = C = -1$ and $I = 0$, which are just the quantum numbers of the $\phi(1019)$. Other possible low mass $J = 1$ resonances are the A1(1070), B(1235) and D(1285) which satisfy $M_{\phi} < M_{A1}$,

M_B, M_D as they should on account of their most likely quantum number assignments. The lower bound provided by the $\phi(1019)$ could prove interesting in connection with J^{PC} assignments for possible resonances in the neighborhood of the $\eta'(958)$.[15] Since the 2^+ nonet also exhibits nonet symmetry, the mass of the f'(1514) may similarly provide a lower bound on the masses of $J=2$ nonstrange mesons which lie above the f and the A2. This appears to be in accord with experiment.[15]

MESON-BARYON MASS INEQUALITIES

We can extend our approach to include baryon resonances by treating a combination of u-channel–exotic amplitudes of the form

$$a^2(M\bar{M}\to M\bar{M}) + 2b\,(M\bar{M}\to B\bar{B}) + c^2\,(B\bar{B}\to B\bar{B}) \tag{9}$$

in analogy with Eq. (4). Here M denotes a meson and B denotes a baryon. In order to eliminate the possibility of u-channel poles, we assume that there are no dibaryon resonances and that exotic mesons or baryons, if they exist, do not couple to nonexotic mesons. We find that the spin-$\left(J+\frac{1}{2}\right)$, t-channel baryon resonances in $2b(M\bar{M}\to B\bar{B})$ must lie above the lowest-lying, $I=0$, $P=C=(-1)^J$, spin-J meson resonances. When stated in terms of baryon and leading meson trajectories, this implies that $\alpha_B \leq \alpha_M + \frac{1}{2}$, which forces the leading meson trajectory to rise at least as fast as the leading baryon trajectory! The observed low-spin resonances closely satisfy $\alpha_B \leq \alpha_M - \frac{1}{2}$ which is stronger by one unit than we have been able to prove. It might be possible to prove this stronger bound because $\alpha_B \leq \alpha_M + \frac{1}{2}$ does not exhaust the content of positivity.[12] The closeness with which $\alpha_B \leq \alpha_M - \frac{1}{2}$ is satisfied is indicated by the near degeneracy of $\Delta(1236)$ and f(1260) for $J=\frac{3}{2}$ and $J=2$, N(1688) and $\phi(1675)$ for $J=\frac{5}{2}$ and $J=3$.

CONCLUSION

In conclusion, we observe that the agreement between established experimental data, previous models for couplings, and our results tends to indicate that duality does apply to $B\bar{B}$ systems. This provides some support for the existence of exotic mesons[5] which couple to baryons. Unfortunately, we have only lower bounds on the masses of the exotic mesons.

We have chosen to state some of our results in terms of Regge trajectories. However, the derivation of our results does not depend on the assignment of particles to trajectories or on the assumption of Regge behavior. In this respect, the present approach is considerably more general than other approaches to duality.[1]

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$$M(s,t) = \sum_{k=0}^{K-1} \left(s^k/k!\right) M^{(k)}(0,t) + s^K \sum_{i=1}^{\infty} R_i(t)/s_i^K \left(s - s_i\right),$$

which converges for all finite $s \neq s_i$. The number of subtractions required at $t = 0$ is denoted by J_0. Positivity of coefficients (d) in the polynomial residues $R_i(t)$ implies that the required number of subtractions is a nondecreasing function of real positive t and that more $(J+1)$ subtractions are required at some larger value of t if and only if there is a singularity in t at that point. After the $(J+1)$-th subtraction has been performed, the singularity in question is absorbed into subtraction terms of the form $\left(s^k/k!\right) M^{(k)}(0,t)$ with $k \leq J$. If the singularity in

$$M^{(J)}(0,t) = -J! \sum_{i=1}^{\infty} s_i^{-J-1} R_i(t)$$

is a simple pole, then positivity of the coefficients in $R_i(t)$ implies that the residue of this pole in $M^{(J)}(0,t)$ is positive. Since this pole in t is not present in $M^{(k)}(0,t)$ for $k > J$, it occurs in $M(s,t)$ with a polynomial residue in which the coefficient of the highest power of s is positive.
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