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AXIAL VECTOR ANOMALIES AND THE SCALING PROPERTY OF FIELD THEORY*

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ABSTRACT

A proof of the Adler-Bardeen theorem is given with the aid of the Callan-Symanzik equation.

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The recent realization that the processes $\gamma \to 3\pi$ and $2\gamma \to 3\pi$ will provide basic information[1] about the PCAC triangle anomaly[2] has provoked new interest in this subject. If the anomaly is to provide a test of the relevance of the renormalized perturbation series to hadron physics, it is clearly essential that the value of the anomaly remains the same up to any finite order in perturbation theory.[3] Let us elaborate. Consider a perturbative calculation of the amplitude

$$R_{D\mu\nu}(k,q) = i\int d^4x\, d^4y\, e^{i(kx+qy)} < 0\,|T\partial A(0)\; V^*_\mu(x)\; V^*_\nu(y)|\,0>$$

$$= \epsilon_{\mu\nu\lambda\sigma} k^\lambda q^\sigma\, f(k^2, q^2, kq) \tag{1}$$

in any renormalizable quantum field theory with fermions and a partially conserved axial-current. This theory may be, for example, quantum electrodynamics, or the σ-model, or a quark-gluon model. (In what follows the discussion is given for QED. It is straightforward to modify the discussion to a form appropriate for other theories.) The notation V^*_μ indicates that when calculating $R_{D\mu\nu}$ we omit those diagrams in which the vector current $V_\mu = \bar{\psi}_0\gamma_\mu\psi_0$ hooks eventually onto a photon propagator.[4]

The theorem alluded to above then states the following: to any finite order in perturbation theory $f(0,0,0)$ is given by the basic fermion triangle graph. This is an extraordinary assertion for it tells us that PCAC and gauge invariance imply the existence of a spectacular cancellation among the infinite[5] collection of Feynman diagrams, thus providing a unique opportunity to confront renormalized perturbation theory with data.[6] Moreover, this theorem provides a springboard for several other deductions on the behavior of field theories.[7]

A proof of this important theorem was given by Adler and Bardeen.[8,9] In this paper we propose an alternative proof. Our proof avoids the formal interchanges which appeared in Ref. 8 and which have caused some uneasiness. This paper also serves to illustrate the Callan-Symanzik[10] equation at work. So let us begin by writing down the version of this equation appropriate for current correlation functions:

$$\left[\lambda(\alpha)\, m\frac{\partial}{\partial m} + \beta(\alpha)\frac{\partial}{\partial \alpha}\right] R_{D\mu\nu}(k,q) = -\frac{1}{2} R_{SD\mu\nu}(0,k,q) + R_{D\mu\nu}(k,q) \tag{2}$$

Here $R_{SD\mu\nu}(p,k,q)$

$$\equiv i \int d^4z\, d^4x\, d^4y\, e^{i(pz+kx+qy)} \langle 0|TS(z)\,\partial A(0)\, V^*_\mu(x)\, V^*_\nu(y)|0\rangle \tag{3}$$

and $S = 2im_0\bar{\psi}_0\psi_0$. (All unrenormalized quantities are denoted with the subscript 0.)

We now sketch the derivation of Eq. (2). Let $R^o_{D\mu\nu}$ and $R^o_{SD\mu\nu}$ be the unrenormalized counterparts of $R_{D\mu\nu}$ and $R_{SD\mu\nu}$. Then by definition

$$m_0\frac{\partial}{\partial m_0} R^o_{D\mu\nu}(k,q) = -\frac{1}{2} R^o_{SD\mu\nu}(0,k,q) + R^o_{D\mu\nu}(k,q) \tag{4}$$

The partial differentiation in Eq. (4) is performed with the bare coupling constant and the cutoff held fixed. The term $R^o_{D\mu\nu}$ appears in Eq. (4) since ∂A explicitly depends on m_0. We now recall that the operators ∂A, S, and V^*_μ are in fact cutoff independent, thanks to the PCAC and CVC Ward identities.[4, 11] Thus $R^o_{D\mu\nu}(k,q) = R_{D\mu\nu}(k,q)$ and $R^o_{SD\mu\nu}(p,k,q) = R_{SD\mu\nu}(p,k,q)$. Introducing the definitions $\lambda(\alpha) \equiv m_0/m\,(\partial m/\partial m_0)$ and $\beta(\alpha) \equiv m_0\,(\partial\alpha/\partial m_0)$ we obtain Eq. (2). $\lambda(\alpha)$ and $\beta(\alpha)$ are clearly cutoff independent and depend only on α.

We next expand $f(k^2, q^2, kq)$ in powers of momenta:

$$f(k^2,q^2,kq) = c + d\frac{k^2}{m^2} + e\frac{q^2}{m^2} + f\frac{kq}{m^2} + \text{terms higher order in } k^2, q^2, \text{ and } kq \tag{5}$$

Simple dimension counting shows that c, d, e, f, etc. are functions of α only. Such an expansion certainly exists with some nonzero radius of convergence since our fermion mass m does not vanish. Now $R_{SD\mu\nu}$ satisfies a Ward identity:

$$i(p+k+q)^{\lambda} R_{S\lambda\mu\nu}(p,k,q) = R_{SD\mu\nu}(p,k,q) - 2R_{D\mu\nu}(k,q) \tag{6}$$

where

$$R_{S\lambda\mu\nu}(p,k,q) = i\int d^4z\, d^4x\, d^4y\, e^{i(pz+kx+qy)} \langle 0\,|TS(z)\, A_{\lambda}(0)\, V_{\mu}^{*}(x)\, V_{\nu}^{*}(y)\,|\,0\rangle \tag{7}$$

We now argue that the Ward identity in Eq. (6) is free from anomalies. As a consequence of crossing symmetry and gauge invariance, any anomalous term in Eq. (6) must have the form $a\epsilon_{\mu\nu\lambda\sigma}k^{\lambda}q^{\sigma}$. Since absorptive parts satisfy normal Ward identities[2] a is a polynomial in the momenta. Weinberg's theorem[12] then shows that a=0 to any finite order in renormalized perturbation theory.

One is happy to note that the same expression appears on the right-hand side of Eq. (2) and Eq. (6). This enables the proof to proceed as follows. Crossing symmetry implies that $R_{S\lambda\mu\nu}(0,k,q) = \epsilon_{\mu\nu\lambda\sigma}(k-q)^{\sigma} A(\alpha)$ + terms higher order in momenta as $k, q \to 0$. Gauge invariance, however, forces $A(\alpha)$ to vanish. The Ward identity in Eq. (6) now tells us that in the momentum expansion

$$R_{SD\mu\nu}(0,k,q) - 2R_{D\mu\nu}(k,q) = \epsilon_{\mu\nu\lambda\sigma}k^{\lambda}q^{\sigma}B(\alpha) + \ldots$$

the coefficient of expansion $B(\alpha) = A(\alpha) = 0$.[13] Referring to Eq. (2) we learn that

$$\beta(\alpha)\,\frac{dc(\alpha)}{d\alpha} = -\,\frac{1}{2}\,B(\alpha) = 0 \tag{8}$$

Hence $c(\alpha)$ is independent of α. This completes the proof.[14]

The proof just given holds for theories with only one nonvanishing mass such as QED. It is easily generalized, however, to theories with several nonvanishing masses, the reason being that the number of Callan-Symanzik equations is equal to the number of masses in theory. For example, in a theory with a fermion mass m and a boson mass μ we would have two equations of the type

$$\lambda \frac{\partial c}{\partial z} + \beta \frac{\partial c}{\partial g} = 0 \tag{9a}$$

and

$$\lambda' \frac{\partial c}{\partial z} + \beta' \frac{\partial c}{\partial g} = 0 \quad . \tag{9b}$$

Here $z \equiv \mu^2/m^2$ and c(z, q) is defined by the expansion $R_{D\mu\nu} \to \epsilon_{\mu\nu\lambda\sigma} k^\lambda q^\sigma \; c\left(\frac{\mu^2}{m^2}, g\right)$ as $k, q \to 0$. $\lambda, \lambda', \beta, \beta'$ are functions of μ^2/m^2 as well as g. Clearly, we reach the same conclusion as before, namely that $c\left(\frac{\mu^2}{m^2}, g\right)$ does not depend on g. It then follows that $c\left(\frac{\mu^2}{m^2}, g\right)$ does not depend on μ^2/m^2 either.

In applications[15] so far, the right-hand side of the Callan-Symanzik equation is usually eliminated by appealing to Weinberg's theorem and going into the deep Euclidean region. In this paper, we observe that the right-hand side of the Callan-Symanzik equation for $R_{D\mu\nu}$ (Eq. (2)) appears also in a PCAC Ward identity (Eq. (6)). This information turns out to suffice for our purposes. We note here that this feature is quite general.

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2. For a review see S. L. Adler in Lectures on Elementary Particles and Quantum Field Theory, 1970 (MIT Press, Cambridge, Mass., 1971) and R. Jackiw in Lectures on Current Algebra and Its Applications (Princeton University Press, Princeton, New Jersey, 1972).
3. A nonperturbative approach to the PCAC anomaly has been given by K. G. Wilson, Phys. Rev. 179, 1499 (1969) and worked out by R. Crewther, Phys. Rev. Letters 28, 1421 (1972). See also S. S. Shei, to be published.
4. The standard vector Ward identity $i(p'-p)^{\mu}\Gamma^{o}_{\mu}(p,p') = S_F^{-1}(p) - S_F^{-1}(p')$ does not imply that $Z_2\Gamma^{o}_{\mu}$ is cutoff independent. Rather, we learn from the decomposition $\Gamma^{o}_{\mu} = \Gamma^{*}_{\mu} + \left[g_{\mu\nu}(p-p')^2 - (p-p')_{\mu}(p-p')_{\nu}\right] F^{\nu}(p,p')$ that $Z_2\Gamma^{*}_{\mu}$ is cutoff independent but Z_2F_{μ} may be cutoff dependent. V^{*}_{μ} corresponds to Γ^{*}_{μ}.
5. These statements are all predicated on the belief that any amplitude is completely described by the infinite collection of Feynman diagrams given in the Dyson series. At issue is the question whether or not nonperturbative effects are automatically included.
6. There are two issues. Firstly, is the PCAC anomaly relevant to hadron phenomena? Secondly, if so, can the value of the PCAC anomaly be calculated in renormalized perturbation theory? See for example R. Aviv and A. Zee, Ref. 1.
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13. In the σ-model the vanishing of $B(\alpha)$ implies a low energy theorem relating the processes $2\gamma \rightarrow \sigma\pi$ and $2\gamma \rightarrow \pi$. See Ref. 1.

14. At the Gell-Mann-Low eigenvalue $\beta(\alpha_{GML}) = 0$. However, one may still conclude $\frac{dc(\alpha)}{d\alpha} = 0$ at $\alpha = \alpha_{GML}$ in renormalized perturbation theory since up to any finite order $\beta(\alpha)$ is a polynomial.

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