SLAC-PUB-1080
(TH)
August 1972

On the Convergence of Separable Expansions for the t-Matrix*

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ABSTRACT

We obtain theorems on the convergence of separable approximations for t-matrices which derive from local potentials. We prove that convergence is impossible in the operator norm and the Hilbert Schmidt norm. This result is universal and independent of the particular method used to construct the separable approximation.

(Submitted to the Journal of Mathematical Physics)

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*Work supported in part by the U. S. Atomic Energy Commission and the City University of New York Faculty Research Award Program

I. INTRODUCTION

This paper studies the convergence of separable expansions for the off-shell two-body t-matrix. Numerous authors[1] have constructed different schemes for obtaining specific finite rank approximations for the t-matrix and have studied their convergence in differing model problems. The general aim of all these works is to obtain an accurate expansion of the t-matrix which will be suitable for solving Faddeev's equations for the three-body scattering problem. Here we study the convergence of separable t-matrix expansions in an abstract format and obtain a theorem which states that convergence in the operator norm is impossible. We prove this general result for any t-matrix which is derived from a potential that has some local part.

II. THE NON-COMPACTNESS OF THE t-MATRIX

In this section we shall prove that a t-matrix derived from a local potential is non-compact. It is this non-compact property that makes the convergence of finite-rank approximations difficult. Here the t-matrix we examine is the solution of the Lippmann-Schwinger equation

$$t(z) = v - v\, g_o(z)\, t(z) \tag{1}$$

for a two-body interaction v, a complex energy variable z, and a resolvent $g_o(z) = (h_o - z)^{-1}$ expressed in terms of the free hamiltonian h_o. The nature of the solutions of this equation expressed in momentum space have been studied in detail by Faddeev.[2] The conditions imposed by Faddeev on the local potential in Eq. (1) are that it satisfies a boundedness property, A, in momentum space

$$\text{A:} \quad |v(\vec{p} - \vec{p}\,')| \leq C/(1 + |\vec{p} - \vec{p}\,'|)^{1+\theta}, \qquad \theta > \frac{1}{2}$$

where C is a constant. The potential is also assumed to satisfy a smoothness property, B, defined by the Holder condition,

$$\text{B:} \quad |v(\vec{p} - \vec{p}') - v(\vec{p} + \vec{\Delta p} - \vec{p}')| \leq C|\vec{\Delta p}|^{\mu}/(1 + |\vec{p} - \vec{p}'|)^{1+\theta}$$

for all $|\vec{\Delta p}| < 1$, $\mu > 0$. When Eq. (1) is written as an integral equation in momentum space it takes the form

$$t(\vec{p}, \vec{p}'; z) = v(\vec{p} - \vec{p}') - \int \frac{v(\vec{p} - \vec{p}'')\, t(\vec{p}'', \vec{p}'; z)\, d^3\vec{p}''}{\vec{p}''^2 - z} \tag{3}$$

The results of Faddeev that we need in this work are that when the conditions A and B are satisfied then Eq. (3) has a unique solution for all z not at the bound-state energies of $h = h_o + v$. In this case the solution to Eq. (3) satisfies the estimate

$$|t(\vec{p}, \vec{p}'; z)| \leq C_1/(1 + |\vec{p} - \vec{p}'|)^{1+\theta} \tag{4}$$

In what follows we shall analyze t(z) as a linear operator on the Hilbert space, $\mathscr{H}$, of square integrable functions in the three-dimensional momentum variables, i.e., the norm of $f \in \mathscr{H}$ is

$$||f||_2 = \left(\int |f(\vec{p})|^2 d^3\vec{p} \right)^{1/2} \tag{5}$$

We now want to show that t(z) is non-compact in $\mathscr{H}$. This result is the content of the following two propositions.

Proposition 1. Let condition A be satisfied by the potential v and $\mathscr{I}m\, z \neq 0$ then t(z) is non-compact in $\mathscr{H}$.

Proof:

We first note that A implies $v g_o(z)$ is compact. In fact $v g_o(z)$ is a Hilbert-Schmidt operator. This follows by direct calculation. The Hilbert-Schmidt operator norm is defined as

$$\|g_o(z)v\|^2_{H.S.} = \iint \left| \frac{v(\vec{p}-\vec{p}')}{\vec{p}'^2 - z} \right|^2 d^3\vec{p}\, d^3\vec{p}' \leqq 2\pi^2 \frac{\mathrm{Re}\sqrt{z}}{\mathscr{Im}\ z} \int |v(\vec{p})|^2 d^3\vec{p} \tag{6}$$

The last integral on the right exists if $\theta > 1/2$. So $g_o(z)v$ is compact.

Now let us demonstrate that $t(z)$ is non-compact. First we note that the condition that v is local means that v is a multiplication operator when expressed in coordinate space. Thus it is non-compact. Sincē the Fourier transformation from coordinate space to momentum space is a unitary transformation v is non-compact in $\mathscr{H}$. Now we suppose $t(z)$ is compact. We know $v g_o(z)$ is compact, and so will be the product $v g_o(z)t(z)$. Equation (1) tells us that v is the sum of two compact operators. Thus v must be compact. This is a contradiction. So we have shown that $t(z)$ is non-compact. This establishes proposition 1.

We extend the domain of validity of proposition 1 to include the entire z plane, excepting a small neighborhood around the bound-state poles of $t(z)$. This extension follows at once from the following lemma.

Lemma 1. Let conditions A and B be satisfied then the difference $t(z_1) - t(z_2)$ is compact for all z_1 and z_2 in the upper (or lower) half z-plane which excludes the discrete spectra of $\mathscr{H}$.

Proof:

This result is easily established by direct calculation. We use the well-known identity which contains the full off-shell unitarity in the two-body scattering problem, viz

$$t(z_1) - t(z_2) = (z_2 - z_1)\, t(z_1)\, g_o(z_1)\, g_o(z_2)\, t(z_2) \tag{7}$$

The right hand side of Eq. (7) can be proven Hilbert-Schmidt by using the left to obtain a finite bound on the norm.

$$
\begin{aligned}
\| t(z_1) - t(z_2) \|^2_{H.S.} &= \iint \left| t(\vec{p}, \vec{p}\,'; z_1) - t(\vec{p}, \vec{p}\,'; z_2) \right|^2 d^3\vec{p}\, d^3\vec{p}\,' \\
&= |z_2 - z_1|^2 \iint \left| \int \frac{t(\vec{p}, \vec{p}\,''; z_1)\, t(\vec{p}\,'', \vec{p}\,'; z_2)\, d^3\vec{p}\,''}{(\vec{p}\,''^2 - z_1)(\vec{p}\,''^2 - z_2)} \right|^2 d^3\vec{p}\, d^3\vec{p}\,' \\
&\leq |z_2 - z_1|^2 \iint \left[\int \frac{|t(\vec{p}, \vec{p}\,''; z_1)|\, |t(\vec{p}\,'', \vec{p}\,'; z_2)|\, d^3\vec{p}\,''}{|\vec{p}\,''^2 - z_1|\, |\vec{p}\,''^2 - z_2|} \right]^2 d^3\vec{p}\, d^3\vec{p}\,'
\end{aligned}
\tag{8}
$$

For $\mathscr{I}m\, z \neq 0$ and employing Faddeev's estimate Eq. (4), we can change the order of integration to obtain

$$
\| t(z_1) - t(z_2) \|^2_{H.S.} \leq |z_2 - z_1|^2 \left| \int \frac{d^3\vec{p}\,''}{|\vec{p}\,''^2 - z_1|\, |\vec{p}\,''^2 - z_2|} \right|^2 \left| \int \frac{C\, d^3\vec{p}}{(1 + |\vec{p}|)^{2+2\theta}} \right|^2
$$

with

$$
|z_2 - z_1|^2 \left| \int \frac{d^3\vec{p}\,''}{|\vec{p}\,''^2 - z_1|\, |\vec{p}\,''^2 - z_2|} \right|^2 = 4\pi^4 \left| \sqrt{z_2} - \sqrt{z_1} \right|^2 ,
$$

which is valid for z_2 and z_1 in the same half plane, and

$$
\left| \int \frac{C}{(1 + |\vec{p}|)^{2+2\theta}} d^3\vec{p} \right| = C_1 < \infty \qquad \text{for} \quad \theta > 1/2
$$

We have for all $z = 0$

$$
\| t(z_1) - t(z_2) \|^2_{H.S.} \leq 4\pi^4 C_1^2 \left| \sqrt{z_2} - \sqrt{z_1} \right|^2 \tag{9}
$$

But the bound on the right may be continued on to the real axis. So the lemma is proved.

We can establish the generalization of proposition 1.

Proposition 2. Let v satisfy conditions A and B then t(z) is non-compact for all z not belonging to the discrete spectra of h.

Proof:

This follows trivially from the above lemma and proposition 1. Let z_1 lie along the upper portion of the cut along the positive real axis in the complex z plane, i.e., $z_1 = s + io$ where s is positive. Let $\mathcal{Im}\, z_2 > 0$. Now suppose $t(z_1)$ compact. Then

$$t(z_2) = t(z_1) + (t(z_2) - t(z_1)) \tag{10}$$

implies $t(z_2)$ is compact since it is the sum of two compact operators. This contradicts proposition 1 so $t(z_1)$ must be non-compact.

III. THE CONVERGENCE OF SEPARABLE EXPANSIONS

We now turn to the implications of proposition 2 for the convergence of finite rank approximations to t(z). All separable approximations take the form

$$\begin{aligned} t(\vec{p}, \vec{p}'; z) &\approx t^N(\vec{p}, \vec{p}'; z) \\ t^N(\vec{p}, \vec{p}'; z) &= \sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij}(z)\, f_i(\vec{p}; z)\, g_j(\vec{p}'; z) \end{aligned} \tag{11}$$

where the f_i and g_j are square integrable and $c_{ij}(z)$ are constants and N is the order of the finite rank approximation $t^N(z)$. We summarize our conclusion in two propositions.

Proposition 3. Let $t^N(z)$ be any finite rank approximation described above then the Hilbert-Schmidt norm of the difference $t(z) - t^N(z)$ is infinity.

Proof:

Assume $||t(z) - t^N(z)||_{H.S.} = B < \infty$, for some $t^N(z)$. Thus $t(z) - t^N(z)$ is compact. The operator $t^N(z)$ is finite rank, so $t(z)$ must be compact. This is a contradiction, so we must have

$$||t(z) - t^N(z)||_{H.S.} = \infty \qquad \text{for all } t^N(z) \ . \tag{12}$$

A somewhat less demanding norm for convergence than the Hilbert-Schmidt is the operator norm. For any linear operator A on $\mathscr{H}$, this norm is defined by

$$||A|| = \inf_{f \in \mathscr{H}} \frac{||Af||_2}{||f||_2} \tag{13}$$

Our last proposition states that convergence in the operator norm is impossible.

Proposition 4. There does not exist any sequence of separable approximations $\left\{t^N(z) \, ; \, N = 1, \infty\right\}$ such that

$$\lim_{N \to \infty} ||t(z) - t^N(z)|| = 0 \tag{14}$$

Proof:

Assume that Eq. (14) is true for some sequence $\left\{t^N(z)\right\}$ then $t(z)$ is the limit in the operator norm of a sequence of compact operators and is therefore compact.[4] This contradicts the non-compactness of $t(z)$ so Eq. (14) cannot be true.

The results we obtain above of course do not preclude a weaker type of convergence. For example it would be possible

$$||(t(z) - t^N(z)) f||_2 \to 0 \qquad \text{as } N \to \infty$$

for a fixed f in $\mathscr{H}$. What our results do provide is a universal upper bound on the type of convergence possible for separable expansions, irregardless of the method which is used to construct the expansion.

In the results stated above we have assumed the potential is a purely local one. However the only important aspect of the potential our proofs required is that the potential was non-compact. If we add to any non-compact operator a compact operator the sum remains non-compact. Thus our results extend to potential which are a sum of a local part and a compact part, provided that conditions A and B are satisfied.

Acknowledgments

The author would like to thank Professor R. S. Phillips for helpful discussions of the proofs given above.

REFERENCES

1. For example see I an H. Sloan and T. J. Brady, Variational Approach to the On- and Off-Shell t-matrix, (University of Maryland preprint, 1972). A popular separable expansion is unitary pole expansion discussed in E. Harms Phys. Rev. C1 (1970) 1667. The paper by D. Bolle' contains an extensive list of references on separable expansions. D. Bolle' "Separable Approximations to the Off-Shell Two-Body t- and K-matrix at Positive Energies" (University of Leuven preprint, 1971).
2. L. D. Faddeev, Mathematical Aspects of the Three-Body Problem in Quantum Scattering Theory (Davey, New York, 1965) Sec. 4
3. See reference 2 for the standard proof of this identity.
4. A proof of this fact is contained in F. Riesz and B. Sz-Nagy, Functional Analysis (Ungar, New York, 1955) p. 178.