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ANALYTIC CONTINUATION IN THE DIMENSION OF SPACE-TIME*

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I. Introduction

Very recently[1-5] a method has been proposed for the renormalization of field theory, by making use of a continuation in the space-time dimensions. While this method borrows the technique from analytic renormalization, it is much simpler than the latter and more convenient for gauge theories.

The simplicity of the method and its promising developments perhaps justify these straightforward comments concerning the status of the matter and some open chances.

The basic idea is the well known observation that ultraviolet divergencies in Feynman integrals may be viewed as a consequence of the space-time dimension being 4. Models in two dimensional space-time are free from this desease.[6]

It is convenient to consider the parametric integral representation of a Feynman integral in a scalar field theory[7]

$$F_{(n)}(p_i) = \Gamma\left(\ell - \frac{nk}{2}\right) \exp\left[i\,\frac{\pi}{2}\left(\ell - \frac{nk}{2}\right)\right] \times \int_0^1 d\alpha_i \cdots d\alpha_\ell \, \frac{\delta(1 - \sum \alpha_i)}{\left[C(\alpha_i)\right]^{n/2}} \left[\frac{D(\alpha_i,\, p_i)}{C(\alpha_i)}\right]^{\frac{nk}{2} - \ell} \tag{1.1}$$

when ℓ is the number of lines in the graph, k is the number of loops, n the dimension of space-time, $C(\alpha_i)$ and $D(\alpha_i,\, p_i)$ the familiar parametric functions determined by the topology of the graph.

Because of the gamma function one easily sees that any given integral (1.1) (fixed ℓ and k) becomes divergent when the dimension of space-time is increased. The representation (1.1) suggests a "natural" interpolating function $F_{(d)}(p_i)$ defined for complex values of d, and identical to $F_{(n)}(p_i)$ on integer values of d.

In Section II, the arbitrariness of the choice of the interpolating function is discussed.

Given a function $F_d(p_i)$ it is possible to study its Laurent expansion around $d = 4$. For those integrals such that the formal representation (1.1) is divergent, one now throws away the singular part of the Laurent expansion and keeps only its regular (i.e., finite) part. As discussed in Section II, this procedure leads to an arbitrary polinomial that may be added to the finite part. It has been shown in (5) that this prescription for the Laurent expansion is indeed equivalent to the more familiar Bogoliubov-Parasiuk-Hepp procedure of Taylor expanding in the external momenta. The arbitrariness of the present method only amounts to the choice of the subtraction point; this is the usual freedom of the renormalization group.

Example

Let us consider the "propagator" graph in a ϕ^4 theory (Fig. 1).

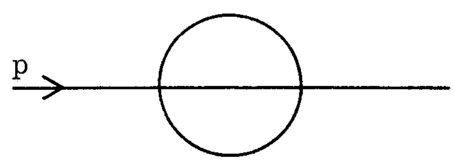

Fig. 1

In an n-dimensional world one finds

$$F_n(p^2) = \int_0^{+\infty} \frac{d\alpha_1 d\alpha_2 d\alpha_3}{\left[C(\alpha_i)\right]^{n/2}} \; e^{iD(\alpha_i,\, p^2)/C(\alpha_i)} \tag{1.2}$$

where

$$C(\alpha_i) = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3$$

$$D(\alpha_i,\, p^2) = \alpha_1\alpha_2\alpha_3 p^2 - (\textstyle\sum \alpha_i)\, C(\alpha_i) m^2 + i\epsilon$$

By doing a scale transformation (8) $\alpha_i \to \rho\alpha_i$, and integrating over the ρ variable, one has

$$F_n(p^2) = \Gamma(3-n) e^{\frac{i\pi}{2}(3-n)} \int_0^1 \frac{d\alpha_i\, \delta(1-\sum_1^3 \alpha_i)}{\left[C(\alpha_i)\right]^{n/2}} \left(\frac{D}{C}\right)^{n-3} \tag{1.3}$$

This may be generalized to an interpolating function $F_d(p_i)$; the finite part of its Laurent expansion for $d \sim 4$ is

$$F_{Rin}(p^2) = -i \int_0^1 \frac{d\alpha_i\, \delta(1-\sum \alpha_i)}{\left[C(\alpha_i)\right]^3} \left[D(\alpha_i, p^2) \log \frac{D(\alpha_i, p^2)}{C(\alpha_i)} + a p^2 + b\right] \tag{1.4}$$

The constants a and b are fixed by the choice of the subtraction point. If one requires that $F_{Rin}(p^2 = 0) = \left.\frac{\partial F(p^2)}{\partial p^2}\right|_0 = 0$ then

$$F_{Rin}(p^2) = i \int_0^1 \frac{d\alpha_i\, \delta(1-\sum_1^3 \alpha_i)}{\left[C(\alpha_i)\right]^3} \left\{\alpha_1\alpha_2\alpha_3 p^2 \left[1 - \log\left(1 - \frac{\alpha_1\alpha_2\alpha_3 p^2}{C(\alpha_i)m^2}\right)\right]\right.$$

$$\left. + C(\alpha_i)m^2 \log\left(1 - \frac{\alpha_1\alpha_2\alpha_3 p^2}{C(\alpha_i)m^2}\right)\right\} \tag{1.5}$$

It is not difficult to check that this expression is identical to the B.P.H. prescription

$$F_{Rin}(p^2) = (1 - m_2)\, F(p^2)$$

where $m_\mu f(x)$ is the Taylor expansion of f(x) around the origin, truncated at order μ.

II. Interpolation and Arbitrariness

An interpolating function plays a role only if one wants a framework for considering field theory models for different numbers of space-time dimensions $n = s + 1$, s being the number of space dimensions. If one only considers renormalization, then any analytic extrapolation around $n = 4$ is adequate and no question of interpolation arises. Here it is assumed that the framework mentioned above is desiderable. Then in perturbation theory one finds integrals like Eq. (1.1), defined for all integers less than some given $\bar{n}$ and one looks for an interpolating function. The relevant asymptotic behavior of a unique interpolating function should be $e^{k|\mathrm{Re}\, d|}$, where $k < \pi$ for $|\mathrm{Re}\, d| \to -\infty$; this is obviously violated along the real axis by $\lim_{n \to -\infty} F_n(p_i)$. Therefore if $\bar{F}_d(p_i)$ is the most obvious interpolating function, we may consider as well

$$'F_d(p_i) = \Gamma\left(\ell - \frac{dk}{2}\right) \left\{ e^{\frac{i\pi}{2}\left(\ell - \frac{dk}{2}\right)} \int_0^1 \frac{d\alpha_i\, \delta(1 - \sum \alpha_i)}{\left[C(\alpha_i)\right]^{d/2}} \left[\frac{D}{C}\right]^{\frac{dk}{2} - \ell} + g(d)\, P(p_i) \right\} \tag{2.1}$$

where $g(d)$ is an entire function of d which vanishes at all integers (say $\sin \pi d$) and $P(p_i)$ a polynomial in the external momenta, with arbitrary coefficients. The following requirements on $P(p_i)$ seem "reasonable":

1) It should be a polynomial in the Lorentz invariant variables, say $p_i \cdot p_j$.

2) It has real coefficients. This may be regarded as part of the definition of an extended unitarity, for d arbitrary.

3) $P(p_i)$ does not grow faster at ∞ than the Feynman integral itself. This fixes the degree of the polynomial to be less than or equal to $\frac{dk}{2} - \ell$, in the quadratic Lorenz invariants.

The interpolating function $F_d(p_i)$ may now be continued to the right half plane Re $d > \bar{n}$ and one may write its Laurent expansion for $d \sim 4$. Because of the requirements 1-3 one sees that the arbitrary polynomial to be added to the finite part of a Feynman integral is indeed equivalent to the choice of the subtraction point in a renormalizable theory.

One might think that the arbitrariness associated with regularization "a la Gelfand" of the integral (1.1) could be different. It is interesting to see that it leads in a simple way to requirements 1 and 3.

In fact one looks at regularization of the integral

$$\int_0^{+\infty} \rho^{\nu-1} e^{i\rho \frac{D(\alpha_i, p^2)}{C(\alpha_i)}} d\rho , \quad \text{where} \quad \text{Im} \quad D > 0$$

In the region Re $\nu < 0$, if $\overline{\text{Re}}\ g(\rho^{\nu-1})$ is a regularization of $\rho^{\nu-1}$ as a generalized function, then any other regularizations may be obtained by adding to it a functional with support in the origin, say $g(\rho) = \sum_{i=0}^{M} c_i \, \delta^{(i)}(\rho)$. All these regularizations are to coincide in the space of test functions where the functional $\rho^{\nu-1}$ exists. This fixes the degree M of the highest derivative of a delta function to be $M = -\nu$ and leads to the definition of a renormalized integral modulo an arbitrary polynomial satisfying the requirements 1 and 3.

III. Q.E.D.

The main advantage of the present method over other regularization, say Pauli-Villars regularization, is in the renormalization of gauge fields. As anticipated in Ref. 2 and 3, this regularization preserves the gauge properties of the theory. The Ward identities are satisfied for every value of the regulator parameter d. This may be shown most clearly in Q.E.D. by direct calculation in the lowest non-trivial order.

In a space-time with n dimensions ($n = s + 1$, s being a positive integer) one finds a set of n Dirac matrices γ_i, that satisfy $\{\gamma_i, \gamma_j\} = 2g_{ij}$. The continuation to complex values of the dimension is done after performing sums and/or traces of the Dirac matrices.

A straightforward set of rules is given in the Appendix. One may then compute the lowest order nontrivial fermion self energy $\Sigma(p)$:

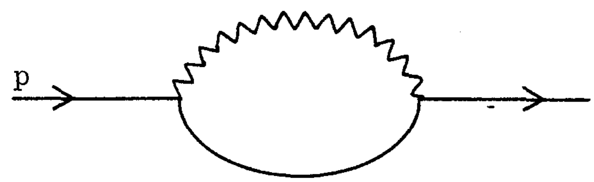

FIG. 2

$$\Sigma(p) = \frac{\alpha\, 4\pi}{(2\pi)^n i} \sum_{\ell=1}^{n} g^{\ell\ell} \int dq \, \frac{\gamma^\ell \left[\gamma\cdot(p-q) + m\right]\gamma^\ell}{\left[(p-q)^2 - m^2\right](q^2 - \mu^2)}$$

$$= - \frac{4\pi\,\alpha\, e^{-\frac{i\pi n}{4}}}{(2\pi)^n} (\pi)^{n/2} e^{\frac{i\pi}{2}\left(2 - \frac{n}{2}\right)} \Gamma\left(2 - \frac{n}{2}\right) \int_0^1 \frac{\left[(2-n)\alpha\gamma\cdot p + nm\right] d\alpha}{\left[\alpha(1-\alpha)p^2 - \alpha\mu^2 - (1-\alpha)m^2\right]^{2-\frac{n}{2}}}$$

The photon has been given a mass μ.

The lowest nontrivial order vertex function (Fig. 3)

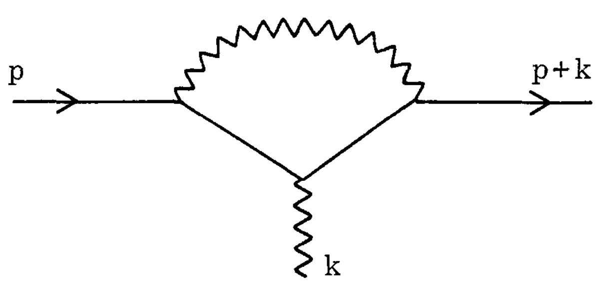

FIG. 3

$$\Gamma^{\mu}(p,\ k) = \frac{i\,\alpha\, 4\pi}{(2\pi)^n} \int dq\ \frac{\sum_{\ell}(i)^2 g^{\ell\ell}\gamma^{\ell}[\gamma\cdot(p-q)+m]\gamma^{\mu}[\gamma\cdot(p-q+k)+m\ \gamma^{\ell}]}{(q^2-\mu^2)\,[(p-q)^2-m^2]\,[(p-q+k)^2-m^2]}$$

$$= \frac{4\pi\alpha}{(2\pi)^n}\ e^{-\frac{i\pi n}{4}}\ (\pi)^{n/2} \left\{ -\ e^{+\frac{i\pi}{2}\left(2-\frac{n}{2}\right)}\, \Gamma\left(2-\frac{n}{2}\right) \frac{(2-n)^2}{2}\ \gamma^{\mu} \int_0^1 \frac{d\alpha_i\, \delta(1-\sum_1^3 \alpha_i)}{B^{2-n/2}} \right.$$

$$\left. +\ i\, e^{\frac{i\pi}{2}\left(3-\frac{n}{2}\right)}\, \Gamma\left(3-\frac{n}{2}\right) \int_0^1 \frac{d\alpha_i\, \delta(1-\sum\alpha_i)}{B^{3-n/2}}\ [Y] \right\}$$

where

$$B = p^2\alpha_1\alpha_2 + k^2\alpha_2\alpha_3 + (p+k)^2\alpha_1\alpha_3 - (\alpha_1\mu^2 + \alpha_2 m^2 + \alpha_3 m^2)$$

and

$$Y = (4-n)\,\gamma\cdot(\alpha_1 p - \alpha_3 k)\gamma^{\mu}\ \ \gamma\cdot\lceil \alpha_1 p + (\alpha_1+\alpha_2)k\rceil - 2\gamma\lfloor\alpha_1 p + (\alpha_1+\alpha_2)k\rfloor\gamma^{\mu}\gamma\cdot(\alpha_1 p - \alpha_3 k)$$

$$+\ m(n-4)\gamma^{\mu}\gamma\cdot k + 2m(n p^{\mu} + 2k^{\mu}) -\ \ \alpha_3 k^{\mu} + (\alpha_2+\alpha_3)p^{\mu}]\,2m\,n$$

$$+\ (2-n)m^2\gamma^{\mu}.$$

at $k = 0$

$$\overline{B}\ =\ B(k=0) = p^2\alpha_1(\alpha_2+\alpha_3) - (\alpha_1\mu^2 + \alpha_2 m^2 + \alpha_3 m^2)$$

$$\overline{Y}\ = Y(k=0) = (2-n)\alpha_i^2\ \gamma\cdot p\gamma^{\mu}\gamma\cdot p + 2m\ n\ \alpha_1 p^{\mu} + (2-n)m^2\gamma^{\mu}$$

By using the identity, proved in Ref. 10

$$\Gamma(3-\tfrac{d}{2})\ \gamma^{\mu} \int_0^1 \frac{(1-\alpha)(m^2-\alpha^2 p^2)d\alpha}{\overline{B}^{\,3-\frac{d}{2}}}$$

$$= \Gamma(2-\tfrac{d}{2})\ \frac{\gamma^{\mu}}{2} \int_0^1 \frac{(d-2)(1-\alpha)-2\alpha}{\overline{B}^{\,2-\frac{d}{2}}}\ d\alpha$$

one may verify that the Ward identity

$$\Gamma^{\mu}(p, 0) = - \frac{\partial \Sigma(p)}{\partial p_{\mu}}$$

holds for every value of the regularizing parameter d.

IV. Open Problems

A number of problems seem to be particularly fit for investigation by the present method. One is certainly the occurrence of Ward anomalies. To study the fermion loop anomaly one first has to define γ_5 in an n-dimensional space. The first proposal (3) that $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ implies that it anticommutes with any γ_i, $i = 1, \cdots 4$ but it commutes with any other Dirac matrices. As was immediately realized, this leads to ambiguities. It may be useful to recall[11] that in an n-dimensional space, n an even integer, one has a set of n Dirac matrices that anticommute and a further matrix, defined as the product of the n Dirac matrices, that yet anticommutes with all of the n matrices. In an odd dimensional space, this last "product" Dirac matrix is to be added to the set of the first n, in order to have a set of Dirac matrices equal to the space dimension. By doing that however it becomes impossible to construct a further Dirac matrix that anticommutes with all of this basic set. This impossibility is indeed related to the different meaning of the parity operator in spaces with even or odd dimensions and to the difficulty[12] to have a CPT invariant Dirac theory in a space time with odd dimension. One is then led to consider only space times of n dimensions, with n even. The obvious generalization for γ_5 is then the Dirac matrix product of all n matrices. Investigations are being done to determine the role of the arbitrariness of the interpolating function in the Ward anomalies.

An interesting problem is the investigation of nonpolynomial Lagrangians. As is well known[13] the imaginary part of the momentum space representation of the superpropagator in a theory with exponential interaction is fixed by unitarity. The real part is undertermined because of the possible addition of an entire function. Such an entire function may be eliminated by requiring[14] that the real part of the momentum space representation $S(p^2)$ of a superpropagator vanish in the limit $p^2 \rightarrow \infty$.

In the framework of this method this requirement amounts to the choice of $p^2 = \infty$ as a subtraction point.[15] If this choice of a special subtraction point were imposed as a boundary condition in renormalization theory, then the so called "nonrenormalizable" theory could be handled as well as the familiar renormalizable theories.[16]

Acknowledgment

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Appendix

In an n-dimensional space-time one may determine a set of n Dirac matrices γ_i that satisfy the basic relation of the Clifford algebra: $\{\gamma_i, \gamma_j\} = 2g_{i_d} I$. From that basic relation one immediately derives:

$$\sum g^{\ell\ell}\gamma^\ell\gamma^k\gamma^\ell = (2 - n)\gamma^k$$

$$\sum g^{\ell\ell}\gamma^\ell\gamma^a\gamma^b\gamma^\ell = (n - 4)\gamma^a\gamma^b + 4\, g^{ab}$$

$$\sum g^{\ell\ell}\gamma^\ell\gamma^a\gamma^b\gamma^c\gamma^\ell = (4 - n)\gamma^a\gamma^b\gamma^c - 2\gamma^c\gamma^b\gamma^a$$

and so on.

From the integral

$$I = \int d^n p\, e^{i(ap^2 + bp\cdot k)} = i\, e^{-i\frac{\pi n}{4}}\, e^{-i\frac{b^2k^2}{4a}} \left(\frac{\pi}{a}\right)^{n/2}$$

by repeated differentiation one has

$$I_\mu = \int d^n p\, p_\mu\, e^{i(ap^2 + bp\cdot k)} = \frac{-ib}{2a}\, k_\mu \left(\frac{\pi}{a}\right)^{n/2} e^{-i\frac{\pi n}{4}}\, e^{-i\frac{b^2k^2}{4a}}$$

$$I_{\mu\nu} = \int d^n p\, p_\mu p_\nu\, e^{i(ap^2 + bp\cdot k)} = \left(\frac{1}{2a}\right)\left[\frac{ib^2}{2a}\, k_\mu k_\nu - g_{\mu\nu}\right]\left(\frac{\pi}{a}\right)^{n/2} e^{-i\frac{\pi n}{4}}\, e^{-i\frac{b^2k^2}{4a}}$$

References

1. A number of authors (2 - 5) have recently and independently proposed a renormalization scheme by using an analytical continuation in the dimension of space-time. One should note that very similar techniques had been used years ago by C. G. Bollini and J. J. Giambiagi, N. Nakanishi and E. Speer.
2. C. G. Bollini and J. J. Giambiagi, to be published in Nuovo Cimento.
3. G. t'Hooft and M. Veltman, to be published in Nuclear Physics.
4. J. F. Ashmore, Lettere Nuovo Cimento 4, 289 (1972).
5. G. M. Cicuta and E. Montaldi, Lettere Nuovo Cimento 4, 329 (1972).
6. Infrared divergences get worse if the number of space time dimensions is decreased, but they are ignored here since no massless particles are considered.
7. A convenient reference is N. Nakanishi, Graphy Theory and Feynman Integrals, (Gordon and Breach, 1971), Chapter 2.
8. R. Eden, et al., The Analytic S-Matrix (Cambridge Univ. Press, 1966) p. 141, or Ref. 7.
9. If one is not interested in much framework, but only in renormalization, there is no question of an interpolating function, and an arbitrary continuation of Eq. 1.1 in the neighborhood of n = 4 will do the job.
10. P. Butera, G. M. Cicuta, E. Montaldi, to be published.
11. Brauer and H. Weyl, Am. Jour. of Math. 57, 425 (1935).
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13. M. K. Volkov, Comm. Math. Phys. 7, 289 (1968).
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15. If one wants to renormalize the exponential superpropagator term by term, by taking its finite part at any order in the small coupling constant, special care is necessary with the test function space, as shown by G. Lararides and A. A. Patani, IC/72/4 Trieste preprint, Feb. 1972, to be published.

16. Very similar proposals and techniques were made by W. Güttinger, Fort. der Physik, 14, 483 (1966).