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# SUM RULES AND BOUNDS ON SCATTERING AMPLITUDES*

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## ABSTRACT

Using sum rules obtained from crossing and analyticity, and unitarity bounds on scattering amplitudes, we show how new relations between low energy and high energy scattering can be derived. These relations can provide tests of a wide range of theoretical ideas. As examples, we discuss several inequalities obtained for $\pi$-$\pi$ and $\pi$-N scattering. For $\pi$-$\pi$ scattering, a number of relations involving the asymptotic behavior of total cross sections are presented, including bounds limiting the size of violations of the Pomeronchuk Theorem. Using Finite Energy Sum Rules for $\pi$-N scattering, we derive new types of bounds and show how they can be used to probe such things as the nature of the Pomeron trajectory and the assumption of s-channel helicity conservation. Finally, we introduce inequality constraints between partial wave amplitudes of different isospin, and indicate how they can be used to explore the nature of exchange degeneracy, absence of exotics and duality.

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# I. INTRODUCTION

Certain general principles, namely unitarity, crossing symmetry, and some form of analyticity, severely restrict the allowed behavior of scattering amplitudes. During the past several years, many interesting inequalities have been derived which follow solely from these principles, or from these principles combined with a few pieces of experimental information, or a few additional theoretically plausible assumptions. In this paper, we discuss several bounds at fixed energies, enforcing unitarity through the use of Lagrange inequality multipliers. We then show two ways in which crossing and analyticity can be introduced into the problems by combining our results either with a Froissart-Gribov expression for crossed channel scattering, or with Finite Energy Sum Rules. Both these approaches yield relations between low and high energy scattering. In Section II, we describe the fixed energy bounds problems with which we shall deal, and we introduce a constraint which can be used to explore the nature of duality, absence of exotics, and exchange degeneracy. Using $\pi$-$\pi$ scattering as an example, Section III develops the formalism for using the Froissart-Gibov formula with the results of Section II, to provide bounds on various quantities. In Section IV, the results of Section II are combined with Finite Energy Sum Rules. We illustrate, using spinless particles, the kinds of relations this marriage produces, and we go on to describe one further interesting problem. Section V develops the necessary formalism for applying the techniques of Section IV to the experimentally accessible case of $\pi$-N scattering. We then present some numerical examples to indicate when FESR bounds can be more restrictive than bounds derived at fixed energies. In Section VI, we conclude with a brief summary of what we have accomplished.

## II. FIXED ENERGY BOUNDS

To illustrate the type of fixed energy bound we need for later applications, consider the expression

$$A(\sigma;z) = \sum_{\ell} (2\ell + 1) \left(a_\ell^1 + (-1)^\sigma a_\ell^2\right) P_\ell(z) . \tag{1}$$

We wish to bound this expression given the conditions

$$0 \leq a_\ell^I \leq 1$$

and

$$\sum\nolimits_T^I = \sum (2\ell + 1)\, a_\ell^I \qquad I = 1, 2.$$

In physical applications, these constraints correspond to imposing unitarity and fixing the total cross sections for different isospin values. Using the formalism of Lagrange multipliers generalized to include inequality constraints,[1] our auxiliary function reads

$$\mathscr{L} = \pm A(\sigma; z) + \sum_{I=1,2} \alpha^I \left[\sum\nolimits_T^I - \sum_\ell (2\ell+1)\, a_\ell^I\right] + \sum_\ell (2\ell+1)\lambda_\ell^I a_\ell^I (1 - a_\ell^I) \tag{2}$$

where the ± sign in front depends on whether we seek an upper (+) or a lower (-) bound. Notice that there is no coupling between $I=1$ and $I=2$ amplitudes.

We need to consider four cases corresponding to $\sigma = 0, 1$ and $z \gtrless 1$. Suppose, first, that $\sigma = 0$. If $z < 1$, $P_\ell(z)$ is a damped oscillating function of $\ell$, and it is clear that if for a given value of I, $\sum(2\ell+1)\, a_\ell^I$ converges, so will $\sum(2\ell+1)\, a_\ell^I P_\ell(z)$. It is easy to show that in this case we can get both an upper and a lower bound on A. The solution for the upper bound is[2]

$$A(0; z) \leq \sum_{I=1,2} \sum_{\ell \in B^I} (2\ell+1) \ ; \qquad z < 1 \tag{3}$$

where $\ell \in B^I$ if $P_\ell(z) > \beta^I$, and $\beta^I$ is determined by

$$\Sigma_T^I = \sum_{\ell \in B^I} (2\ell + 1).$$

For the lower bound we find

$$A(0;z) \geq \sum_{I=1,2} \sum_{\ell \in C^I} (2\ell + 1) \; ; \qquad z < 1 \qquad (4)$$

where $\ell \in C^I$ if $P_\ell(z) < \gamma^I$, and $\gamma^I$ is determined by

$$\Sigma_T^I = \sum_{\ell \in C^I} (2\ell + 1).$$

While the upper and lower bounds appear to be mathematically similar, their behavior as a function of z is much different. The upper bound falls off slowly away from its fixed value at $z = 1$, while the lower bound very rapidly becomes negative as z decreases from 1. Consider next $z > 1$. In this case, $P_\ell(z)$ is a monotonically increasing function of $\ell$, and convergence of $\sum (2\ell + 1) a_\ell^I$ does not guarantee that $\sum (2\ell + 1) a_\ell^I P_\ell(z)$ will exist. Therefore, with the constraints at hand, we can get only a lower and not an upper bound. The lower bound will be given by expression (4) if we substitute a value of $z > 1$.

Suppose now that $\sigma = 1$. Since there is no coupling between $I=1$ and $I=2$, an (upper, lower) bound on A will be given by the (upper, lower) bound for the $I=1$ term, minus the (lower, upper) bound for the $I=2$ term. If $z < 1$, the lower bound for $I = (1, 2)$ behaves badly near $z = 1$, and consequently our (lower, upper) bound on A will be of little use (at least at a fixed energy). Still worse is the fact that if $z > 1$, we can get no finite upper or lower bound on A at all. To rectify this situation, we introduce a further constraint. We require that $a_\ell^2 \leq a_\ell^1$ for all $\ell$.

We shall use constraints of this form in later sections and it will be clear then how they can be used to explore the nature of duality.

When we include this additional constraint, the most convenient way to write the auxiliary function is:

$$\mathscr{L} = \pm A(\sigma;z) + \alpha^1 \left[\sum{}_T^1 - \sum_\ell (2\ell+1)\, a_\ell^1\right] + \alpha^2 \left[\sum{}_T^2 - \sum_\ell (2\ell+1)\, a_\ell^2\right]$$

$$+ \sum_\ell (2\ell+1) \left\{ \lambda_\ell (1 - a_\ell^1) + \phi_\ell (a_\ell^1 - a_\ell^2) + \xi_\ell a_\ell^2 \right\}, \qquad (5)$$

where $\lambda_\ell$, $\phi_\ell$, and $\xi_\ell$ are inequality multipliers. It is clear that the values of $\sum_T^1$ must satisfy $\sum_T^1 \geq \sum_T^2$. If $\sigma = 0$, this additional constraint does not change the solutions to the variational problems. However, when $\sigma = 1$, the situation is much different. For $z < 1$, we have a much improved upper bound to A, and an improved (though still badly behaved near $z = 1$) lower bound. To illustrate how our additional constraint provides better bounds, we present in Figure 1 the partial wave amplitudes which maximize A. For simplicity, we have assumed that $\sum_T^1$ is small enough to require contributions only from the first positive region of $P_\ell(z)$. $a_\ell^1 = 1$ for $0 \leq \ell \leq L_1$ as is shown by the solid line, and $a_\ell^2 = 1$ for $L_2 \leq \ell \leq L_1$, as is shown by the dashed line. All other $a_\ell^I$ are zero. As $\sum_T^2$ increases, $L_2$ moves closer to zero, the point $L_1$ being fixed by the size of $\sum_T^1$. If we had not required $a_\ell^2 \leq a_\ell^1$, the $a_\ell^2$'s would have been nonzero in a region centered about the minimum of $P_\ell(z)$ as a function of $\ell$, and would have resulted in a much higher upper bound on A.

The solutions of the other problems behave in a similar way. For $z < 1$ the lower bound to A becomes finite and well behaved, while the upper bound is still infinite. These results are summarized in Table I.

TABLE I

| | $z < 1$ | | $z > 1$ | |
|---|---|---|---|---|
| | No additional constraints | $a_\ell^2 \leq a_\ell^1$ | No additional constraints | $a_\ell^2 \leq a_\ell^1$ |
| $\sigma = 0$, upper bound | good | good | $\infty$ | $\infty$ |
| $\sigma = 0$, lower bound | bad | bad | good | good |
| $\sigma = 1$, upper bound | bad | good | $\infty$ | $\infty$ |
| $\sigma = 1$, lower bound | bad | bad (but better) | $-\infty$ | good |

Bounds which are finite, but badly behaved near $z=1$, are denoted as "bad." Bounds which are well behaved near $z=1$ are denoted as "good."

## III. APPLICATIONS TO THE FROISSART-GRIBOV FORMULA

Consider the Froissart-Gribov formula for $\pi$-$\pi$ scattering:[3]

$$\left(\frac{y+1}{y}\right)^{\frac{1}{2}} y^{-\ell} f_\ell^I(y) = \int_0^\infty K_\ell(y,x) \sum_{I'} \alpha_{II'} \sum_n (2n+1)\, a_n^{I'}(x)\, P_n\left(1+\frac{2y+2}{x}\right) dx, \tag{6}$$

where x and y are the squares of the center-of-mass momentum in the t and the s channels, respectively,

$$K_\ell(y,x) = \frac{2}{\pi y^{\ell+1}} \left(\frac{x+1}{x}\right)^{\frac{1}{2}} Q_\ell\left(1 + \frac{2x+2}{y}\right) ,$$

$$f_\ell^I(y) = e^{i\delta_\ell^I(y)} \sin\delta_\ell^I(y) \quad ,$$

and

$$a_\ell^I(x) = \operatorname{Im} f_\ell^I(x) \quad .$$

The crossing matrix for $\pi$-$\pi$ scattering is

$$\alpha_{II'} = \begin{pmatrix} \frac{2}{3} & 2 & \frac{10}{3} \\ \frac{2}{3} & 1 & \frac{-5}{3} \\ \frac{2}{3} & -1 & \frac{1}{3} \end{pmatrix}$$

and we work in units where $m_\pi^2 = 1$.

Taking the limit $y \longrightarrow 0$ and keeping the two leading terms on the right, we have:

$$\lim_{y\to 0} \left(\frac{y+1}{y}\right)^{\frac{1}{2}} f_\ell^I(y) = \int K_\ell(0,x) \sum_{I'} \alpha_{II'} \sum_n (2n+1)\, a_n^{I'}(x) \left\{ P_n\left(1+\frac{2}{x}\right) + \right.$$

$$\left. + \frac{y}{x+1}\left[(x+1)^{\frac{1}{2}} P_n^1\left(1+\frac{2}{x}\right) - \frac{(\ell+1)}{2} P_n\left(1+\frac{2}{x}\right)\right]\right\} \tag{7}$$

with

$$K_\ell(0,x) = \frac{\ell!}{\pi(2\ell+1)!!}\left(\frac{2}{x}\right)^{\frac{1}{2}} (2x+2)^{-\left(\ell+\frac{1}{2}\right)}$$

Now, for small values of $y$, we may write the effective range formula for the phase shifts as

$$\frac{y^{\ell+1/2}}{\sqrt{y+1}} \cot\delta_\ell^I(y) = \frac{-2}{T_\ell^I} + M_\ell^I\, y \quad ,$$

where $T_\ell^I$ is the scattering length and $M_\ell^I$ is the effective range. With this parameterization, the left-hand side of Eq. (7) may be written for small y as

$$\frac{\sqrt{y+1}}{y^{\ell+1/2}}\, f_\ell^I(y) = \frac{-1}{2}\, T_\ell^I + \frac{1}{4}\left(T_\ell^I - M_\ell^I\right) y \equiv \mathscr{F}_\ell^I + \mathscr{E}_\ell^I\, y \tag{8}$$

We may now identify the coefficients of like powers of y in Eq. (7) and (8).

We apply the results of Section II in the following way: Suppose $\mathscr{F}_\ell^I$ or $\mathscr{E}_\ell^I$ is known for some value of $\ell$ and I, and that the $a_n^{I'}(x)$ are known for $x < c$. For a fixed value of x above c, we notice in Eq. (7) that both in the term proportional to 1 and the term proportional to y, the coefficients of $a_n^{I'}$ are monotonically increasing or decreasing functions of $\ell'$, depending on the sign of $\alpha_{II'}$. We may therefore solve a bounds problem for all values of $x > c$ as in Section II (with $z > 1$) and insert the solution into our expressions for $\mathscr{F}_\ell^I$ or $\mathscr{E}_\ell^I$. The result will schematically look like:

$$\mathscr{B} - \int_0^c dx\, F\left(x,\, a_n^{I'}(x)\right) \geq \int_c^\infty dx\, G\left(x,\, \sum\nolimits_T^{I'}(x)\right),$$

where $\mathscr{B}$ is $\mathscr{F}_\ell^I$ or $\mathscr{E}_\ell^I$.

As a concrete example, we shall now discuss the bounds obtained from $\mathscr{F}_1^1$. The problem is to minimize

$$\sum_{n,\,\text{even}} (2n+1)\frac{2}{3}\, a_n^0 P_n\left(1+\frac{2}{x}\right) + (2n+3)\, a_{n+1}^1 P_{n+1}\left(1+\frac{2}{x}\right) - \frac{5}{3}(2n+1)\, a_n^2 P_n\left(1+\frac{2}{x}\right). \tag{9}$$

Because $\alpha_{12}$ is negative, we need a constraint limiting the size of $a_n^2$ with respect to $a_n^0$ and $a_{n+1}^1$. The additional constraints we impose are

$$a_n^2 \leq a_n^0 \qquad \text{and} \qquad a_n^2 \leq \frac{2n+3}{2n+1}\, a_{n+1}^1 \quad .$$

These constraints are quite reasonable. Since the I = 2 channel is exotic, and since we only apply these constraints at high energies, they may be taken as simple extensions of exchange degeneracy and duality, backed by the observation that Regge contributions to total cross sections always appear to be positive.[4] The factor $\frac{2n+3}{2n+1}$ in the second inequality is included to allow $\sum_T^2$ to be as large as $\sum_T^1$. Of course, $a_n^2$ must also always be less than 1. One might have preferred a relation of the form $a_n^2 \leq a_{n-1}^1$ between the I=2 and I=1 amplitudes, but this condition is not sufficient to guarantee a finite bound.

To illustrate the usefulness of this bound, let us first assume that for $x > c$,

$$\sum_T^{I'}(x) = \frac{x\sigma_T^{I'}(x)}{8\pi}$$

is independent of I'. The partial wave amplitudes which minimize expression (9) are

$$a_n^o = a_n^2 = \frac{2n+3}{2n+1}\, a_{n+1}^1 = 1, \qquad \text{for } n \leq N$$

and zero otherwise. N is determined by fitting $\sum_T(x)$. Using the approximation

$$P_n\left(1+\frac{2}{x}\right) - P_{n-1}\left(1+\frac{2}{x}\right) \simeq \frac{2n}{x} \tag{10}$$

valid for large x, we find:

$$\mathscr{F}_1^1 - \frac{1}{6\pi}\int_0^c \frac{dx}{(x+1)^{3/2}\, x^{1/2}} \sum_{I'} \alpha_{1I'} \sum_n (2n+1)\, a_n^{I'}\, P_n\left(1+\frac{2}{x}\right) \tag{11}$$

$$\geq \frac{1}{3\pi}\int_c^\infty \frac{dx}{(x+1)^{3/2}\, x^{3/2}} \left\{\frac{\Sigma_T}{6} + \left(\frac{\Sigma_T}{3} - \frac{1}{8}\right)\left(1+\sqrt{8\,\Sigma_T+1}\right)\right\} ,$$

where the leading terms on the right-hand side of the inequality have cancelled. Assuming a functional form for $\sigma_T(x)$ for $x > c$ (for instance, $\sigma_T(x) = \text{const.}$, or $\sigma_T(x) = C + \frac{D}{\sqrt{x}}$, as suggested by Regge theory), we can easily evaluate the right-hand side of expression (11), and get an upper bound on the total cross section. As long as $\frac{x\sigma_T(x)}{3\pi} \gtrsim 1$ for $x > c$, an easily fulfilled condition, the right-hand side of expression (11) is positive definite. If evaluation of the left-hand side yields a negative number, we would have to conclude either that our dual constraints are not satisfied, or that the Froissart-Gribov expression for the P-wave scattering length does not converge. We can further test the first possibility by examining the expression for $\mathscr{E}_1^1$. The derivation of a bound using this term of the expansion in y follows exactly the above discussion, and the analog of expression (11) is

$$\mathscr{E}_1^1 - \frac{1}{6\pi}\int_0^c \frac{dx}{(x+1)^{5/2}x^{1/2}} \sum_{I'} \alpha_{1I'} \sum_n (2n+1)\, a_n^{I'} \left[(x+1)^{1/2} P_n^1\left(1+\frac{2}{x}\right) - P_n\left(1+\frac{2}{x}\right)\right] \geq \frac{1}{6\pi}\int_c^\infty \frac{dx}{(x+1)^{5/2}x^{1/2}} \left\{ P'_{L+1}\left(1+\frac{2}{x}\right) + \left(\frac{L}{2}+1\right)\left[\frac{L^3}{x} + L^2\left(\frac{4}{x}+\frac{4}{3}\right) + L\left(\frac{10}{3}+\frac{3}{x}\right) + 1\right]\right\}, \qquad (12)$$

where L is the even integer determined by

$$\sum\nolimits_T = \left(\frac{L}{2} + 1\right)(L+1), \quad \text{and}$$

where we have again assumed that the $\sum_T^{1'}(x)$ are independent of I' for $x > c$. The Froissart-Gribov expansion for $\mathscr{E}_1^1$ is expected to converge, so if the left-hand side of inequality (12) is negative, we have a strong indication that our duality constraints are violated in nature.

Because the expression for $\mathscr{E}_1^1$ converges more rapidly than the expression for $\mathscr{F}_1^1$, a high-energy upper bound on $\sigma_T$ derived from $\mathscr{E}_1^1$ will be more sensitive to the low-energy data than a bound derived from $\mathscr{F}_1^1$. Furthermore, because of the negative value of $\alpha_{12}$, the leading terms in the large x region of the integral cancel, and both these bounds are more sensitive to the low-energy data than the bound derived using the D-wave I=0 scattering length. However, the form of the crossing matrix for the P-wave scattering length does allow us to derive less convergent upper bounds than we can obtain from the I=0 D-wave scattering length if we do not assume that the $\sum_T^{I'}(x)$ are all the same for $x > c$. It is of particular interest to derive bounds on $\Delta\sigma_T(x) = \sigma_T(\pi^+\pi^-) - \sigma_T(\pi^+\pi^+)$, which should approach zero as $x \to \infty$, by the Pomeronchuk theorem. For comparison, we list below the results of two problems. Equation (13) gives an upper bound for $\Delta\sigma_T(x) = \sigma_T(\pi^+\pi^-) - \sigma_T(\pi^+\pi^+)$ in terms of the D-wave I=0 scattering length. Expression (14) gives an upper bound for $\Delta\sigma_T(x)$ in terms of the P-wave scattering length.

As before, we have used the duality constraints in the derivation of the P-wave bound. We have not used approximation (10) in either expression. We have for the D-wave:[2]

$$\mathscr{F}_2^0 - \int_0^c K_2(0,x) \sum_{I'} \alpha_{0I'} \sum_n (2n+1)\, a_n^{I'}\, P_n\left(1+\frac{2}{x}\right) \geq$$

$$\geq \int_c^\infty K_2(0,x)\left[\frac{2}{3}P'_{L_2+1}\left(1+\frac{2}{x}\right) + 2\,P'_{L_1+2}\left(1+\frac{2}{x}\right)\right] \tag{13}$$

where

$$\Delta \sum_T \equiv \frac{x}{8\pi}\,\Delta\,\sigma_T = \frac{1}{4}(L_1+2)(L_1+3) + \frac{1}{6}(L_2+1)(L_2+2)$$

and the even integers $L_1$ and $L_2$ must satisfy

$$2P_{L_1+1}\left(1+\frac{2}{x}\right) = P_{L_2}\left(1+\frac{2}{x}\right) \quad .$$

For the P-wave scattering length, we find:

$$\mathscr{F}_1^1 - \int_0^c K_1(0,x) \sum_{I'} \alpha_{1I'} \sum_n (2n+1)\, a_n^{I'} P_n\left(1+\frac{2}{x}\right) \geq$$

$$\geq \int_c^{\infty} K_1(0,x)\left[\frac{2}{3} P'_{L+1} + P'_{L+2}\right] , \qquad (14)$$

where the even integer, L, is determined by

$$\Delta \sum\nolimits_T = \frac{1}{12}(L+2)(5L+11).$$

Since the right-hand side is much less convergent in expression (14) than in expression (13), the bound derived on $\Delta\sigma_T(x)$ from (14) will be much less sensitive to the low energy data than the bond derived from (13).

In general, if we assume a form for $\Delta\sigma_T(x)$, the integrals can be done using the techniques outlined in Reference 3, and these bounds will limit the parameters used to describe $\Delta\sigma_T(x)$. Parameterizations of recent Serpukov data[5] on $\sigma_T(\pi^- p) - \sigma_T(\pi^+ p)$ indicate that bounds relating this quantity to the $\pi$-N charge exchange cross sections may be violated at high energies. The Pomeronchuk theorem bounds presented above can be applied to the case of $\pi$-N scattering, and will provide independent restrictions on $\sigma_T(\pi^- p) - \sigma_T(\pi^+ p)$, which do not depend on the high energy charge exchange cross section. Since the upper bound on the $\pi$-$\pi$ cross section derived in Reference 3 was relatively small, we expect that bounds on $\Delta\sigma_T(x)$ for $\pi$-$\pi$ or $\pi$-N scattering derived by the above technique will also be quite restrictive. However, if we assume that the Pomeronchuk theorem is violated, and $\Delta\sigma_T(x) \not\to 0$ as $x \to \infty$, we can use expression (13) but not (14) to derive a bound, since in this case, the Froissart-Gribov expression for the

P-wave scattering length does not converge. To use P-wave information, we would need to use the Froissart-Gribov expression for $\mathscr{E}_1^1$.[6]

In the above discussion, we have concentrated on bounds for the total cross sections, without including any other experimental constraints. One popular variation of this problem is to include the elastic cross section as a constraint in the original bounds problems. With realistic ratios of $\sigma_{e\ell}/\sigma_T$ (say, $\sigma_{e\ell}/\sigma_T \lesssim 1/2$), the upper bounds on $\sigma_T$ are generally decreased by a factor of about 1/2. This is because with the inclusion of the elastic cross section, the variational problem is no longer a strictly linear one, and a substantial number of the partial wave amplitudes are forced to lie in the interior of the unitarity circle. The discussions of these problems proceed in a manner analogous to the ones above.

## IV. APPLICATIONS TO FINITE ENERGY SUM RULES-SPINLESS CASE

We now turn our attention to ways in which the information contained in Finite Energy Sum Rules can be used to derive new bounds on scattering amplitudes. We shall first illustrate the method by discussing some problems using spinless particles. Neglecting isospin, the Finite Energy Sum Rule for the scattering of spinless particles may be written

$$\int_0^{\nu_o} \nu^n \mathrm{Im}\, F(\nu,t)\, d\nu + \int_{\nu_o}^{N} \nu^n \mathrm{Im}\, F\, d\nu = \sum_i \frac{\beta_i(t) N^{\alpha_i(t)+n+1}}{(\alpha_i+n+1)} , \qquad (15)$$

where $\nu^n \mathrm{Im}\, F(\nu,t)$ is antisymmetric in $\nu = \frac{s-u}{2}$, the mass of the external particles is 1, and $\nu_o$ is the threshold value of $\nu$ in the s-channel. For $\nu > \nu_o$, we may write $\mathrm{Im}\, F(\nu,t) = \sum_\ell (2\ell+1)\, a_\ell(\nu)\, P_\ell\left(1+\frac{4t}{2\nu-4-t}\right)$. If we know

$\sum_T(\nu) = \sum(2\ell+1)\, a_\ell(\nu)$ for $\nu > \nu_0$, we may derive bounds on Im $F(\nu, t)$ as we did in Section II when we considered $z < 1$. For example, deriving an upper bound on Im F, fixing $\sum_T$, and unitarity, and using Eq. (15) above, we have

$$\sum_i \frac{\beta_i(t) N^{\alpha_i(t)+n+1}}{(\alpha_i + n + 1)} - I(\nu_0) \leq \int_{\nu_0}^{N} \nu^n \sum_{\ell \in B} (2\ell+1) P_\ell\left(1 + \frac{4t}{2\nu - 4 - t}\right) d\nu\,, \quad (16)$$

where $I(\nu_0) = \int_0^{\nu_0} \nu^n \,\mathrm{Im}\, F\, d\nu$ and B is the set of integers such that $P_\ell\left(1 + \frac{4t}{2\nu - 4 - t}\right) > \alpha$, $\alpha$ being determined by the condition that $\sum_{\ell \in B} (2\ell+1) = \sum_T$. We can use expression (16) in a number of ways. Knowing $I(\nu_0)$ and the Regge parameterization of F, we have a lower bound on a certain average over the total cross section. Knowing both the total cross sections as a function of $\nu$ and the high energy behavior of F, we have a bound on $I(\nu_0)$. This application may be useful in cases such as baryon-antibaryon scattering, where many intermediate states are accessible below threshold. One of the most useful ways of reading the inequality, however, is as an upper bound on Regge behavior given $I(\nu_0)$ and the total cross section as a function of $\nu$. Notice, in particular, that this approach circumvents one of the most troublesome problems in the use of FESR's: We require no detailed information about low- and intermediate-energy partial-wave amplitudes in order to limit the behavior of the Regge parameterization at non-zero values of t. Of course, whatever additional detailed information we provide (such as partial-wave amplitudes in some low-energy region) will improve the result, but this input is not necessary to get usable, interesting information.[7]

In realistic cases, such as $\pi$-N scattering, the amplitudes to be used in the FESR's are linear combinations of s-channel isospin amplitudes. It will therefore be of interest to examine the nature of the FESR bounds derived for the linear combinations discussed in Section II with $z < 1$. This we shall do in Section V. Since we can also derive bounds on the Regge parameters at a fixed, high value of $\nu$, we may ask when use of the FESR can be expected to yield a better result than the bound derived at a fixed energy. In general, that question cannot be definitely answered without specifically carrying out the calculations. However, the following observations are useful:

(1) The FESR's are written as integrals over $\nu$ with t fixed, so smaller values of $\nu$ correspond to large values of $\theta$, the center-of-mass scattering angle. In the case of the bounds derived fixing only $\sum_T$ (and perhaps imposing the additional constraints between partial wave amplitudes of different isospin), the bound does not fall fast enough as $\theta$ gets large to remain very restrictive. On the other hand, if $\sigma_T(x)$ does not vary rapidly with energy $\sum_T(x) \sim x$, and gets smaller at lower energies, improving the FESR bound. The detailed interplay between these two effects will determine the size of the FESR bound. (2) The FESR gives us a bound on a certain combination of Regge parameters which is not the same as the Regge expression for the imaginary part of the amplitude. Therefore, when dealing with a single FESR bound, we can derive different bounds on the high-energy amplitude depending on, say, what we choose for $\alpha_i(t)$. This is a degree of freedom we do not have when deriving a fixed energy bound. We shall see an example in Section V of how these bounds can vary from the fixed energy result.

After the observations of the preceding paragraph, we would like to discuss a type of problem which is likely to yield quite good high-energy bounds, and which will almost certainly benefit from the application of FESR's. Neglecting isospin, we consider the problem of maximizing

$$F = \sum (2\ell + 1)\, a_\ell P_\ell(z) \, , \qquad z < 1$$

Fixing

$$\sum\nolimits_T = \sum (2\ell + 1)\, a_\ell$$

$$\left(\frac{d}{dz} F\right)_{z=1} = \sum (2\ell+1) \frac{\ell(\ell+1)}{2}\, a_\ell \equiv F'$$

and using unitarity,

$$0 \le a_\ell \le 1 \; .$$

The solution is easily found using standard techniques, and we have:

$$\sum_\ell (2\ell+1)\, a_\ell\, P_\ell(z) \le \sum_{\ell \in B} (2\ell + 1) P_\ell(z)$$

where

$$\ell \in B \;\; \text{if} \;\; P_\ell(z) > \alpha + \beta\ell(\ell+1) \; ,$$

and $\alpha$ and $\beta$ are determined by requiring

$$\sum\nolimits_T = \sum_{\ell \in B} (2\ell + 1)$$

and

$$F' = \sum_{\ell \in B} (2\ell + 1) \frac{\ell(\ell+1)}{2} \, .$$

Inserting this into the FESR gives an expression identical to (16), only now the definition of B is different.

One reason for the limited usefulness of FESR's in regard to the bounds problems of Section II is that many total cross sections increase at lower energies, and this further decreases the effectiveness of the low-energy region of the Sum Rule in providing a good bound. In the present problem, however, an increase in $\sum_T$ decreases the upper bound on F as long as F' does not increase too much. A common experimental situation is that total cross sections increase and slopes of the diffractive peak, defined as $F'/\sum_T$ decrease as the energy decreases. If constant total cross sections and slopes of forward peaks result in roughly equivalent fixed energy bounds on F(x, t) at different energies, then any increase in $\sigma_T(x)$ and decrease in F'(x) at small values of x will improve the value of the FESR bound.

## V. $\pi$-N SCATTERING

The approach of the previous section will be most useful when it is applied to experimentally accessible reactions. As an example, we develop in this section the formalism necessary to deal with $\pi$-N scattering, and discuss some of the interesting problems that can be pursued. We then present numerical examples of some FESR bounds and compare them with both fixed-energy bounds and with values for the amplitudes which they bound.

It is convenient to define s-channel partial-wave amplitudes of definite parity as the generalized coordinates of the variational problem,

$$f^I_{\ell\pm} = e^{i\delta^I_{\ell\pm}} \sin\delta^I_{\ell\pm}$$

where $I = \frac{1}{2}$ or $\frac{3}{2}$ is the s-channel isospin. Unitarity may be written in terms of these amplitudes as

$$a^I_{\ell\pm} \equiv \text{Im}\, f^I_{\ell\pm} \geq \left| f^I_{\ell\pm} \right|^2 \quad .$$

It is convenient to write the invariant amplitudes, A' and B, as

$$A'^I = \alpha_+ F^I_+ + \alpha_- F^I_-$$

$$B^I = \beta_+ F^I_+ + \beta_- F^I_-$$

where

$$F^I_+ = \sum_\ell (\ell+1) \left( f^I_{\ell+} + f^I_{(\ell+1)-} \right) \frac{P'_{\ell+1} + P'_\ell}{z+1}$$

$$F^I_- = \sum_\ell (\ell+1) \left( f^I_{\ell+} - f^I_{(\ell+1)-} \right) \frac{P'_{\ell+1} - P'_\ell}{z-1}$$

and

$$\alpha_+ = \frac{4\pi}{k^3} \left[ m(E - \sqrt{s}) + \frac{\nu E}{2(1 - t/4m^2)} \right]$$

$$\alpha_- = \frac{4\pi}{k^3} \left[ E\sqrt{s} - m^2 - \frac{m\nu}{2(1 - t/4m^2)} \right]$$

$$\beta_+ = \frac{4\pi E}{k^3}$$

$$\beta_- = \frac{-4\pi m}{k^3} \quad .$$

The square of the momentum in the center-of-mass is $k^2$. The nucleon mass is m, and E is the center-of-mass nucleon energy. The s-channel helicity non-flip and flip amplitudes are

$$F^I_{++} = F^I_{--} = F^I_+ \cos\left(\frac{\theta}{2}\right)$$

and

$$F^I_{+-} = -F^I_{-+} = F^I_- \sin\left(\frac{\theta}{2}\right)$$

with $\theta$ the center-of-mass scattering angle.

The amplitudes with definite symmetry properties under $\nu \longrightarrow -\nu$ are amplitudes with definite t-channel isospin. They are given by

$$A'^{(+)} = \left(\frac{2}{3}\right)^{1/2} A'^{1/2} + \left(\frac{8}{3}\right)^{1/2} A'^{3/2} \; ; \quad I_t = 0$$

and

$$A'^{(-)} = \frac{2}{3} A'^{1/2} - \frac{2}{3} A'^{3/2} \; ; \qquad I_t = 1$$

with similar relations for the B amplitudes. $B^{(+)}$ and $A'^{(-)}$ are antisymmetric under $\nu \longrightarrow -\nu$, while $B^{(-)}$ and $A'^{(+)}$ are symmetric. We can, therefore, use even moment sum rules for $B^{(+)}$ and $A'^{(-)}$ and odd moment sum rules for $B^{(-)}$ and $A'^{(+)}$. Total cross sections will be used as constraints in the variational problems. They are given by

$$\sum\nolimits_T^I = \frac{k^2 \sigma_T^I}{4\pi} = \sum_\ell (\ell + 1)\left(a^I_{\ell+} + a^I_{(\ell+1)-}\right) \quad .$$

Suppose we now do the variational problems in which we fix the $I = \frac{1}{2}$ and $I = \frac{3}{2}$ total cross sections at some energy, impose unitarity, and seek upper bounds on the imaginary parts of the four amplitudes $A'^{(\pm)}$ and $B^{(\pm)}$. The $A'^{(-)}$ and $B^{(\pm)}$ bounds are badly behaved near $\theta = 0$ in the sense of Section II, while the $A'^{(+)}$ bound is well behaved. There are two reasons for the bad behavior of three of the bounds, both of which can be understood on the basis of the discussion in Section II. The $A'^{(-)}$ and $B^{(-)}$ bounds are badly behaved because of the minus sign in the $\left(I_t = 1,\ I_s = \frac{3}{2}\right)$ crossing matrix element. The $B^{(\pm)}$ bounds are badly behaved near $\theta = 0$ because the coefficient $\beta_-$, which connects the B amplitudes to the s-channel helicity-flip amplitude, does not go to zero as $\theta \to 0$. This means that the coefficients of $f_{\ell +}$ in the B amplitudes near $\theta = 0$ have the type of behavior, as a function of $\ell$, which results in a bad bound. This problem does not arise in the A' amplitudes because $\alpha_-$ goes to zero sufficiently rapidly as $\theta \to 0$.

One can surmount each of these problems and in doing so, can provide intriguing tests of various ideas. The problem of the minus sign in the crossing matrix can be handled by noticing that $\pi^- p$ total cross sections are larger than $\pi^+ p$ total cross sections. Making the same sort of assumptions as we made in Sections II and III, we can require

$$a_{\ell\pm}^{3/2} \leq a_{\ell\pm}^{1/2}$$

for all $\ell$ at large energies. While this constraint seems reasonable in light of the behavior of total cross sections, it has nothing directly to do with exchange degeneracy in the context of duality. It would, therefore, be quite interesting to test these kinds of assumptions in both the present problem and in the cases such as $\pi$-$\pi$ scattering (see Section III), where the concepts of duality and absence of exotics is more directly involved, and see if they are violated in either case.

With this additional constraint, we can get good bounds on $A'^{(-)}$ in addition to $A'^{(+)}$, but we are still faced with the bad behavior of the $B^{(\pm)}$ bounds. We can alleviate this situation by limiting the size of the s-channel helicity-flip amplitudes, which are proportional to $F_-^I$. In particular, we can improve the bounds on all our amplitudes, and change the $B^{(\pm)}$ bounds from bad to good (assuming we also require $a_{\ell\pm}^{3/2} \leq a_{\ell\pm}^{1/2}$ for $B^{(-)}$) by constraining appropriate combinations of s-channel helicity-flip partial wave amplitudes to be zero. This constraint will be particularly interesting in case of the $I_t = 0$ amplitudes, since at high energies these amplitudes should be Pomeron-dominated, and there is some feeling that the Pomeron conserves s-channel helicity.[8] The remarks of the last two paragraphs are summarized in Table II, where we show the character of the upper bounds obtained under various assumptions.

TABLE II

| | No additional assumptions | $a^{3/2}_{\ell\pm} \leq a^{1/2}_{\ell\pm}$ | SCHC | SCHC and $a^{3/2}_{\ell\pm} \leq a^{1/2}_{\ell\pm}$ |
|---|---|---|---|---|
| $\mathrm{Im}\, A'^{(+)}$ | good | good | good | good |
| $\mathrm{Im}\, A'^{(-)}$ | bad | good | bad | good |
| $\mathrm{Im}\, B^{(+)}$ | bad | bad | good | good |
| $\mathrm{Im}\, B^{(-)}$ | bad | bad | bad | good |

Character of fixed energy upper bounds on various amplitudes. In all cases, $\sum_T^{1/2}$ and $\sum_T^{3/2}$ are given and unitarity is imposed.

To illustrate the utility of our technique, we present in Fig. 2 and 3 the results of some upper bounds on $\mathrm{Im}\, A'^{(+)}$ derived under various assumptions. In Fig. 2, we have plotted upper bounds on $A'^{(+)}$ at $s = 30\ \mathrm{GeV}^2$ as a function of t. Curve B is the fixed-energy upper bound on $\mathrm{Im}\, A'^{(+)}$ derived assuming $\sum_T^I$ at $s=30$ ($I=\frac{1}{2}$ and $\frac{3}{2}$), and unitarity. Curves A and C are bounds derived from FESR's, assuming $\sum_T^I$ at all energies above threshold to $s=30$, and unitarity.[9] Since the upper limit in the FESR integral is so large, the contributions from the integral below threshold are negligibly small. To extract a bound on $A'^{(+)}$

from the FESR result, we have assumed a 2-pole (P and P') high-energy parameterization for $A'^{(+)}$. A is the FESR upper bound assuming $\alpha_P(t) = 1$ and $\alpha_{P'}(t) = \frac{1}{2} + t$, and C is the FESR upper bound assuming $\alpha_P = 1 + \frac{1}{2}t$ and $\alpha_{P'} = \frac{1}{2} + t$. In addition, both A and C assume values for the ratio of the P and P' residues gleaned from Ref. 10. For comparison, we have also plotted in curves D and E the predictions for the $A'^{(+)}$ amplitude at $s = 30\ \mathrm{GeV}^2$, using the residue functions derived by Harari and Zarmi[10] with the two choices of $\alpha_P(t)$ described above. We can now clearly see the kinds of circumstances under which FESR bounds can give better results than fixed-energy bounds: Curve C falls below curve B and is therefore more restrictive.

In Fig. 3, we have presented the results of similar calculations, only now we have imposed the additional constraint of s-channel helicity conservation. Curves D and E are identical to curves D and E of Fig. 2, while the bounds represented by curves A, B, and C are the same as those of Fig. 2, with the addition of the SCHC constraint.[11] While the general trend of these bounds is the same as those of Fig. 1, they are all 15 - 20% lower at $t = -1.0\ \mathrm{GeV}^2$ than the corresponding bounds of Fig. 1. This is especially interesting, since $t = -1.0$ at $s = 30$ still represents a quite small scattering angle, and one might have thought that SCHC would not be a severe constraint so close to the forward direction.

Of course, since fixing total cross sections only fixes the scale of hadronic reactions, the particular bounds we have discussed so far do not follow the data very closely. However, adding only one more constraint can give bounds that fall quite rapidly with t. To illustrate this, we have plotted in Fig. 3 (curve F) an estimate of the fixed-energy upper bound to $\mathrm{Im}\,A'^{(+)}$ assuming, in addition to $\sum_T^I$, unitarity, and SCHC values for the elastic $\pi^{\pm}p$ cross sections and the charge exchange cross section $\pi^- p \to \pi^0 n$.[9] We see that this bound falls much more rapidly than curve B, and is quite close to curves D and E. (It may, in fact, fall below curve D.)

If we include these additional constraints in the calculation of FESR bounds, they will follow the trend of curve F, and will closely restrict the allowed behavior of $\mathrm{Im}\,A'^{(+)}$, limiting the acceptable parameterizations of this amplitude, and providing severe tests of a number of theoretical ideas.

## VI. CONCLUSION

We have shown in this paper ways in which unitarity bounds can be incorporated into sum rules obtained from analyticity and crossing to yield relations between low and high energy scattering. Not only may these relations be used to test the general principles of unitarity, crossing symmetry, and analyticity, but with the inclusion of further constraints, less highly cherished theoretical ideas and procedures of phenomonological analysis may be explored. As examples, we have shown ways in which the Froissart-Gribov expressions can be used to limit violations of the Pomeronchuk theorem and also give us insight into the nature of duality and absence of exotic resonances. Using Finite Energy Sum Rules, we have illustrated how we can learn still more about semi-local duality, and we have furthermore shown how FESR bounds can be used to test such things as SCHC and the nature of the Pomeron trajectory.

We have considered these problems to indicate how sum rules may be used to derive improved bounds and test interesting physical ideas. Use of these techniques can lead to many powerful results that will severely limit the allowed behavior of scattering amplitudes.

## ACKNOWLEDGMENTS

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## FOOTNOTES

1. M. B. Einhorn and R. Blankenbecler, Ann. Phys. 67, 480 (1971).

2. Most of the problems considered in this and in the following sections involve only linear constraints. In these problems, there are only one or two nonzero partial wave amplitudes which lie in the interior of the unitarity circle. (This is not always true of linear problems, but it is true of the ones discussed here.) We have neglected these partial wave amplitudes in presenting our solutions. In general, this is justified because of the large number of partial wave amplitudes contributing to the solution. However, if the value of a constraint used in the problem gets very small, this procedure may not be allowable. This situation can be handled in a straightforward way, and is most likely to occur in the problems leading to expressions (13) and (14). Even here, however, if the predictions of duality and Regge theory are reliable, $\Delta^{-}_{T} \sim \sqrt{s}$, and neglecting the partial wave amplitudes which do not saturate unitarity will be justified.

3. Some of the techniques used in this section were first discussed by R. Blankenbecler and R. Savit, Phys. Rev. D, to be published in June.

4. In Section V we shall discuss mathematically similar constraints in connection with $\pi$-N scattering. The physical motivation in that case, however, does not directly involve duality, and comparisons of the two types of problems may be useful for understanding the nature of duality (see Section V for further discussion).

5. S. P. Denisov et al., Phys. Letters 36B, 415 (1971); S. P. Denisov et al., Phys. Letters 36B, 528 (1971); S. M. Roy, Saclay Preprint D.Ph-T/72.20 (March 1972).

6. In general bounds similar to the ones discussed above can be derived for other linear combinations of isospin cross sections, and should substantially improve previous results. For a general discussion of these results, see the fine review article by S. M. Roy, ibid.

7. From a phenomonological point of view, such a bound may be especially useful in cases such as Compton scattering off nucleons, where total cross sections are fairly well known, but a reliable detailed partial wave analysis does not exist. (I thank Y. Avni for this comment.)

8. See for example, F. J. Gilman, J. Pumplin, A. Schwimmer and L. Stodolsky, Phys. Letters, 31B, 387 (1970), and H. Harari and Y. Zarmi, Phys. Letters, 32B, 291 (1970). It has also been proposed that the Pomeron conserves s-channel spin. Some insight into the correct option (if either is correct) can be gained by replacing the SCHC constraint with the s-channel spin conservation constraint, and comparing the bounds.

9. E. Flaminico, et al., CERN/HERA 70-5 and CERN/HERA 70-7 (1970).

10. H. Harari and Y. Zarmi, Phys. Rev. 187, 2230 (1969). The curves D and E in Figures 1 and 2 represent their results after a rescaling of about 15%, so that their amplitudes at t=0 correspond to the correct values of $\pi^{\pm}p$ total cross sections.

11. In the derivation of bounds A and C, we have for simplicity assumed SCHC even at low energies. However, since N is so large, and since we have used $\nu \, \mathrm{Im}\, A'^{(+)}$ as the integrand of the FESR, this does not significantly affect our result.

## FIGURE CAPTIONS

1. Partial wave amplitude which maximize $A(1, z)$ with $z < 1$ and $a_\ell^2 \leq a_\ell^1$. (See discussion of Eq. (5) for full explanation.)
2. Fixed energy and FESR upper bounds on $\mathrm{Im}\, A'^{(t)}$ at $s = 30\ \mathrm{GeV}^2$ without assumption of SCHC. (See Section V for full explanation.)
3. Fixed energy and FESR upper bounds on $\mathrm{Im}\, A'^{(t)}$ at $s = 30\ \mathrm{GeV}^2$ assuming SCHC. (See Section V for full explanation.)

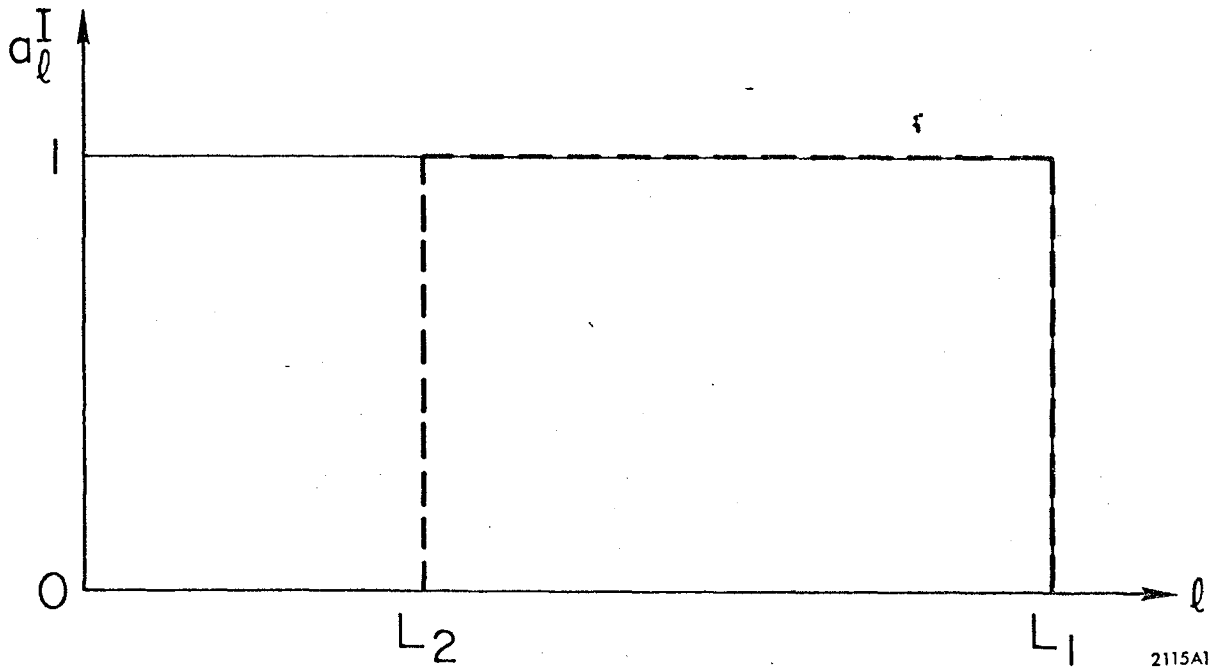
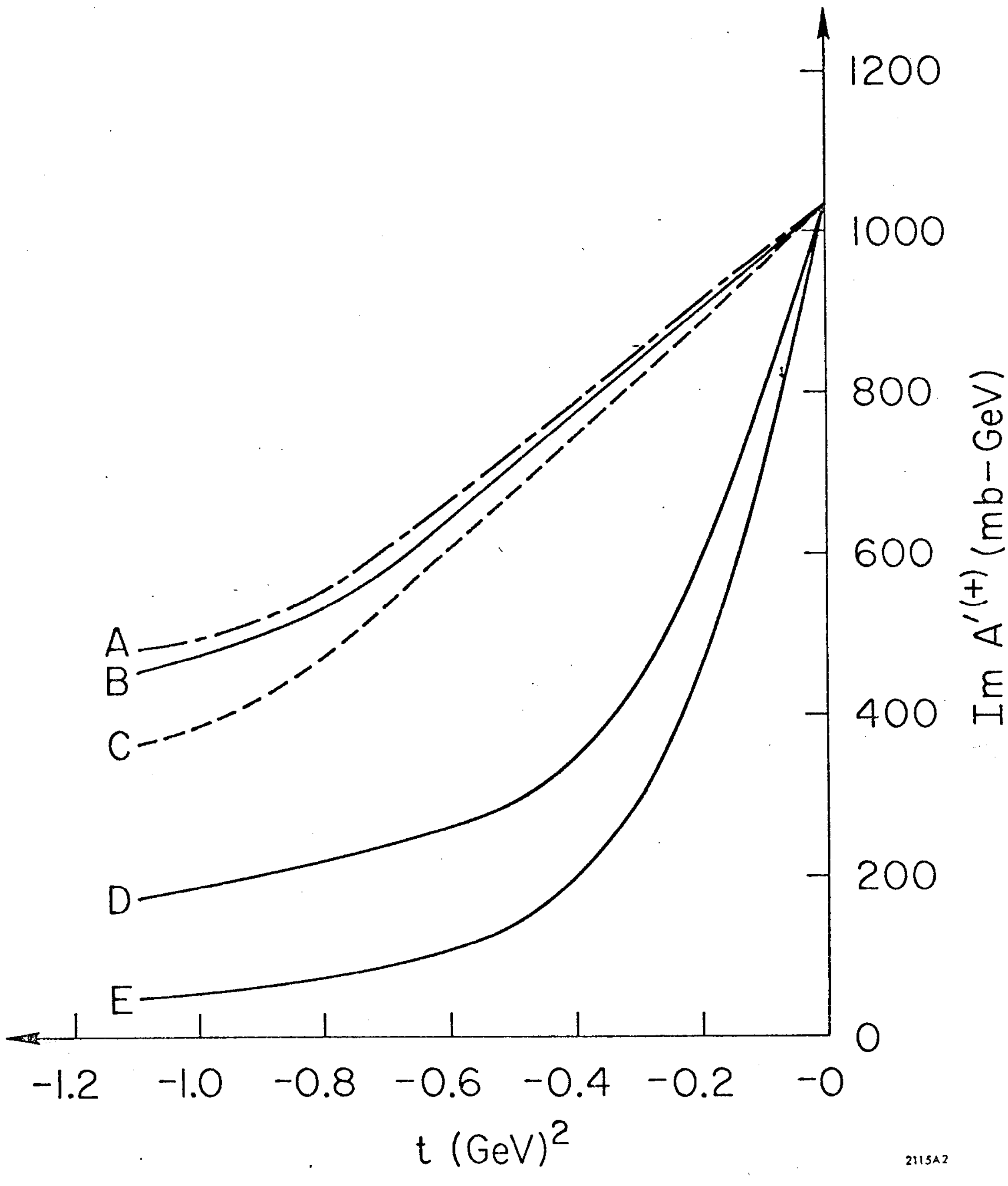
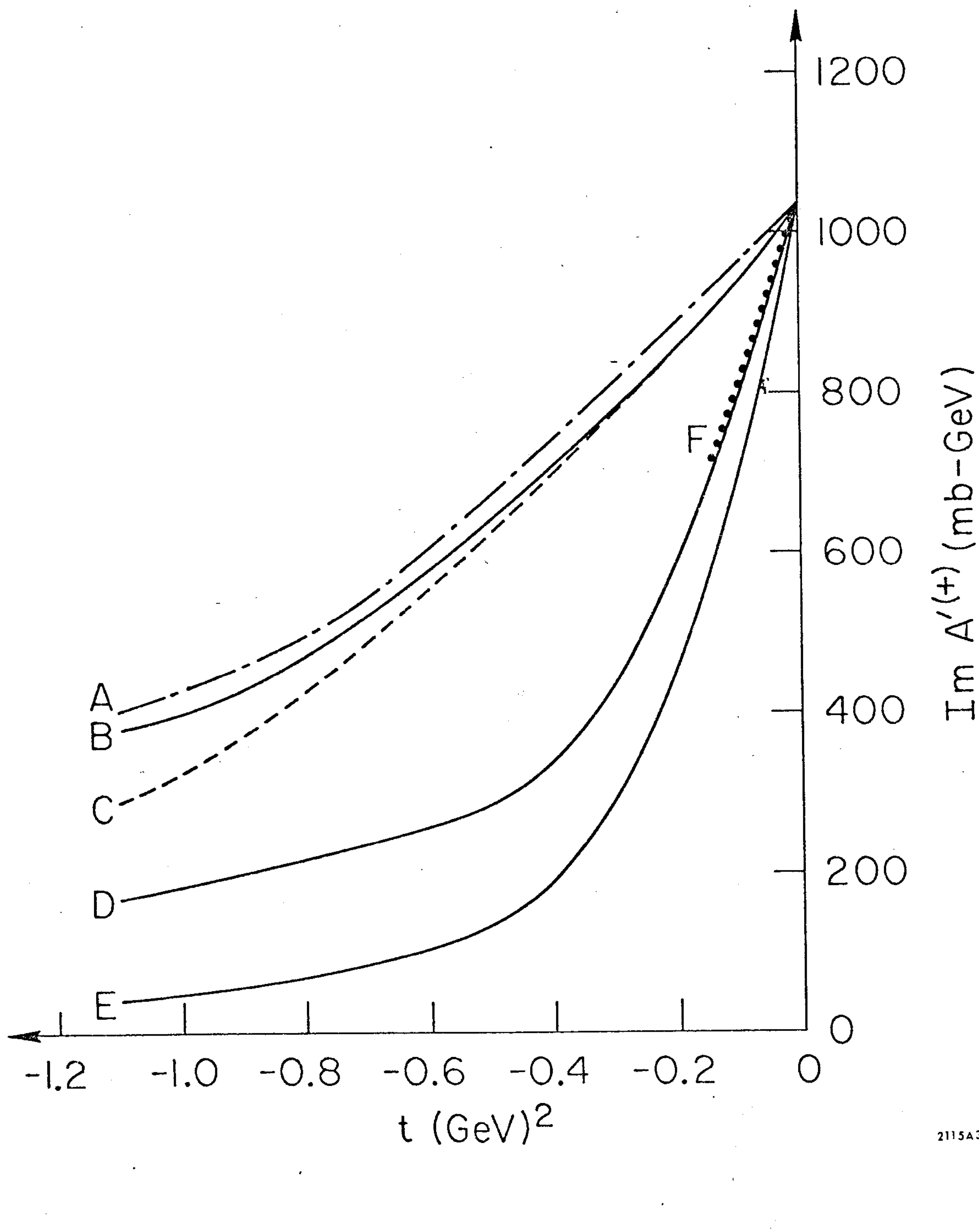



Fig. 1



1200
1000
800
600
400
200
0
Im A'(+) (mb-GeV)
A
B
C
D
E
-1.2
-1.0
-0.8
-0.6
-0.4
-0.2
-0
t (GeV)2
2115A2


Fig. 2



1200
1000
800
600
400
200
0
Im A′(+) (mb-GeV)
A
B
C
D
E
F
-1.2
-1.0
-0.8
-0.6
-0.4
-0.2
0
t (GeV)2
2115A3


Fig. 3