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ALGEBRAIC COMPUTATION TECHNIQUES IN QUANTUM ELECTRODYNAMICS*

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- INTRODUCTION -

The basic equations of quantum electrodynamics are extraordinarily elegant and simple. As far as we know, a complete description of the motion and mutual interactions of electrons, muons, and photons is provided by Maxwell's equations and the Dirac theory of the leptons. In a more complete theory we also include their electrodynamic couplings with the hadrons. In fact, quantum electrodynamics is the basic microscopic theory of the hydrogen atom, atomic physics, chemistry, classical electrodynamics, etc. In all of its critical tests - both in the high energy, short distance domain and low energy precision measurements - the complete success of the theory makes one believe that, in fact, mathematical physics does say something about the physical world [1].

Despite the simplicity of the underlying formulae of quantum electrodynamics, calculations of higher order perturbation effects are sufficiently complex, that organization via computer technique is essential. Still, subtle mathematical analysis is required to

1) carry out the covariant renormalization procedure to remove the ultraviolet divergencies

2) carefully handle infrared divergencies (as regulated by a photon mass parameter λ) which occur in individual Feynman diagram contributions

3) handle the algebraic manipulations in as compact a fashion as possible to avoid incredibly large numbers of terms, and

4) Carry out the analytic or, if necessary, numerical integration.

These are the main questions which I wish to discuss here. In the next sections I will outline the basic approach to renormalization theory used in several recent calculations including a discussion of the intermediate renormalization method which has been found useful for isolating the infrared dependence. A general systematic algebraic approach to the automatic reduction of multiloop Feynman diagrams to parametric form is then discussed. This approach is readily utilized, for example, by the algebraic simplification program REDUCE written by A.C. Hearn [2] . Some brief remarks about numerical integrations are also made. Application of these techniques to fourth and sixth order calculations of the lepton vertex are described, and a brief comparison with experiment is given in Section III. In Section IV, some new computation techniques which are applicable to quantum electrodynamics are presented.

II - ANALYSIS AND EVALUATION OF FEYNMAN DIAGRAMS

There are three central problems in the computation of higher order diagrams in the perturbation theory of quantum electrodynamics [3] :

(i) the application of the covariant renormalization procedure,

(ii) the reduction of multiloop integrals to Feynman parametric form, and

(iii) the calculation of the parametric integrals.

In this section, I wish to review and discuss some of the techniques in these areas which have been found useful in various calculations of the lepton vertex. This includes calculations of (a) the slope $F_1'(0)$ of the Dirac form factor of the electron - for the fourth order electrodynamic contribution to the Lamb shift [4] , (b) the light-by-light scattering contribution to the anomalous moment of the electron and muon [5] , and (c) the vacuum polarization contribution to the anomalous moments [6] . The relevant diagrams are shown in figure 1 .

A) The renormalization procedure of quantum electrodynamics as derived by Feynman, Schwinger, Tomonaga, and Dyson [3] leads unambiguously to finite radiative corrections.

To a given order in perturbation theory, the bare charge e_o and mass m_o parameters are chosen to guarantee that the e-e scattering amplitude yields the Coulomb amplitude e^2/q^2 in the forward limit, and insure that the pole in the electron's propagator is located at $s = p^2 = m^2$, where e and m are the measured charge and mass.

Alternatively, this last condition involving m may be replaced by the physical requirement that the Compton amplitude must approach the Thomson value e^2/m in the threshold $(m \to 0)$ limit.

In practice, the charge and mass renormalization are accomplished by performing subtractions at $q^2 = 0$ and $p^2 = m^2$ in the photon and lepton propagators, respectively. For the photon, the renormalized propagator takes the Kallen-Lehmann form [8]

$$i\,D_{\mu\nu} = \frac{g_{\mu\nu}}{q^2 + i\varepsilon} - \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) \frac{1}{\pi} \int_0^\infty \frac{dt}{t} \frac{\mathrm{Im}\,\pi(t)}{t - q^2 + i\varepsilon}$$

$$\to \frac{g_{\mu\nu}}{q^2 + i\varepsilon} \qquad (q \to 0)$$

and for the electron, proper and improper self-energy bubbles can be summed into the form for the propagator

$$-\,iS_F(p) = \frac{1}{\not p - m_o - \Sigma(p) + i\varepsilon} \underset{\not p \to m}{\to} \frac{1}{1-B}\,\frac{1}{\not p - m}$$

with

$$\Sigma(p) = m - m_0 + (\not p - m)B + (\not p - m)^2\, \Sigma_f(p) \quad .$$

The Ward Identity $Z_1 = Z_2 \equiv \frac{1}{1-B}$ guarantees that the wave function renormalization contributions from B exactly cancel against vertex renormalization contributions Z_1 obtained at $q^2 = 0$ from the on-shell vertex graphs and subgraphs.

Thus because of the Ward Identity, the wave-function renormalization contributions B may be implicitly neglected if vertex subgraph contributions are explicitly subtracted on the mass-shell :

$$\Lambda_\mu^f = \Lambda_\mu(p^2, p'^2, q^2) - \Lambda_\mu(m^2, m^2, 0)$$

In the ladder and corner graph contributions to the fourth order vertex (see figure 1a), the subtraction necessary to remove the internal logarithmic divergence of the vertex subgraph is easily accomplished by subtracting (even at the stage of parametric integrals) the parallel form obtained with the vertex subgraph taken with its legs constrained to the mass-shell. The alternate, more standard procedure, which utilizes an integral representation in p^2, p'^2, and q^2 for the renormalized vertex subgraph is in fact much more complicated -especially for algebraic computer - oriented calculations.

The above procedure also allows an immediate application of the very useful <u>intermediate renormalization</u> technique [10] . In the usual method, as described above, the ultraviolet divergencies are cancelled at the expense of introducing in individual diagrams infrared logarithmic divergencies in the photon mass λ . (In the case of the anomalous moment (to all orders) and the slope of the Dirac form factor $F_1'(0)$ (at fourth or higher order) the infrared divergences cancel in complete results [11]). The vertex subgraph renormalizations, however, can be grouped as

$$[\Lambda_\mu(p'^2, p^2, q^2) - \Lambda_\mu(0,0,0)]$$
$$+ \; [\Lambda_\mu(0, 0, 0) - \Lambda_\mu(m^2, m^2, 0)]$$

The first finite contribution in fact has no infrared divergence, and the second, which can be computed simply from the next lower order calculation, isolates the infrared dependence in a simple analytic fashion. In fact, there is considerable flexibility in the choice of the subtraction amplitude which allows the cancellation of the ultraviolet divergence and is infrared-free, and a choice other than $\Lambda_\mu(0,0,0)$ may be made for convenience of analytic calculation. Moreover, this extended method [6] has been applied to the self-energy renormalizations as well, allowing a straightforward isolation of the infrared pieces [12] .

There is also one other special case of renormalization in quantum electrodynamics which occurs when photon-photon subgraph contributions need to be considered. The polarization tensor of fourth rank representing photon-photon scattering is (using the notation of ref. 5(b)) .

$$\Pi_{\varkappa\rho\sigma\mu}(-p_1, p_2, p_3, -\Delta)$$

$$= \frac{-ie^4}{(2\pi)^4} \int d^4p_6 \mathrm{Tr}[\gamma_\varkappa(\not p_6 - m_e)^{-1}\ \gamma_\rho(\not p_7 - m_e)^{-1}$$

$$\times\ \gamma_\sigma(\not p_8 - m_e)^{-1}\ \gamma_\mu(\not p_9 - m_e)^{-1}$$

$$+ \text{ five other terms - regularization constant term}].$$

Although individual terms of $\Pi_{\varkappa\rho\sigma\mu}$ are logarithmically divergent for large p_6 , the sum is convergent and well-defined and gauge-invariant if properly renormalized. In fact by differentiating the condition of gauge-invariance :

$$\Delta^\nu\ \Pi_{\varkappa\rho\sigma\nu}(-p_1, p_2, p_3, -\Delta) = 0$$

with respect to Δ^μ (regarding, e.q. as Δ, p_1, and p_3 as the independent variables), we obtain

$$\Pi_{\varkappa\rho\sigma\mu}(-p_1, p_2, p_3, -\Delta)$$

$$= -\Delta^\nu \frac{\partial}{\partial\Delta_\mu}\ \Pi_{\varkappa\rho\sigma\nu}(-p_1, p_2, p_3, -\Delta)$$

in which the cancellation of the ultraviolet divergencies is manifestly evident from the beginning. This form is eminently suitable for the calculation of the photon-photon-subgraph contribution to the anomalous moment [5], since the required linear dependence on the external momentum is explicit from the start and Δ may be set to zero elsewhere. We note that the differentiation effectively adds one further electron propagator to the calculation. As we shall see, this means this is the only sixth order calculation which ultimately involves six powers of loop momenta in the numerator structure.

B) Several very elegant techniques have been developed for reducing multiloop integrations of Feynman graphs to parametric form :

$$I = \int \prod_{r=1}^{R} d^4\ell_r \, \frac{F(p_1 \ldots p_n)}{\prod_{j=1}^{n} (p_j^2 - m_j^2)} = \int_0^1 \prod_{j=1}^{n} dz_j \; \delta(1 - \sum_{k=1}^{n} z_k) \; \frac{N(z)}{D(z)}$$

there we have considered a graph with R loops and n propagators ; the p_j of course satisfy four momentum conservation at each vertex and the external mass-shell conditions. Among these methods are the graphical techniques of Chisholm [13] , Nakanishi [14] and Kinoshita [15] , and the more standard algebraic techniques of Landau outlined in Bjorken and Drell's book [7] .

The latter method is very straightforward and is quite easy to implement on REDUCE, Hearn's LISP-based automatic computation program [2] , although in all the applications discussed here we have used both techniques to provide an independent check.

We first note that in any order the contributions to the $F_1(q^2)$ and $F_2(q^2)$ form factors of the lepton may be automatically computed from the vertex matrix element ($J^{\mu} = \gamma^{\mu}$ gives the Born normalization)

$$M^{\nu} = -ie\,\bar{u}(p+q/2)\,J^{\nu}\,u(p-q/2)$$

using [16]

$$F_j(q^2) = \frac{1}{4}\,\mathrm{Tr}[(\not{p}+\not{q}/2 + m)\,J^{\mu}(\not{p}-\not{q}/2 + m)\,\Lambda_{\mu}^{(j)}]$$

where $$\Lambda_{\mu}^{(1)} = (3m\,p_{\mu} - p^2\,\gamma_{\mu})\,/\,4\,p^4$$

$$\Lambda_{\mu}^{(2)} = [\,m^2p^2\gamma_{\mu} - m(m^2 + \tfrac{1}{2}\,q^2)\,p_{\mu}]/[q^2(p^2)^2]$$

with $p^2 = m^2 - \frac{1}{4}\,q^2$, $p.q = 0$.

Thus, for example, in the sixth order calculations of the anomalous moment $a = F_2(0)$, [17] , we are required to evaluate a scalar expression

$$I = \int d^4\ell_1 d^4\ell_2 d^4\ell_3 \;\frac{F(p_1 \ldots p_8)}{\prod_{j=1}^{8} (p_j^2 - m_j^2)}$$

where for a given loop momenta labeling we may write

$$p_j = k_j + \ell_j = k_j + \sum_{r=1}^{3} n_{jr}\ell_r$$

where n_{jr} is the projection ($\pm 1,0$) of p_j along ℓ_r . The k_j can be any choice of fixed momenta (independent of ℓr) such that four momentum is conserved at all the vertices. We may then combine denominators

$$I = 7!\int dz_1 \ldots dz_8\;\delta(1-\Sigma z_k)\int \frac{d^4\ell_1 d^4\ell_2 d^4\ell_3\;F(p)}{[\sum_{j=1}^{8} z_j(p_j^2 - m_j^2)]^8}$$

If we now choose the k_j such that

$$\sum_{j=1}^{8} z_j\,k_j\,n_{jr} = 0 \qquad , \quad r = 1,2,3$$

Then the denominator in (5.3) has no $k.\ell$ cross terms :

$$\sum_{j=1}^{8} z_j(p_j^2 - m_j^2) = -D + \sum_{rr'=1}^{3} U_{rr'}\, \ell_r . \ell_{r'}$$

where

$$D = \sum_{j=1}^{8} z_j(m_j^2 - k_j^2)$$

and

$$U_{rr'} = \sum_{j=1}^{8} z_j\, n_{jr}\, n_{jr'}$$

It is easy to see that the k_j are completely and uniquely determined in terms of the external momenta

$$K_j^{\mu} = \frac{1}{U(z)} [A_j(z)\, p^{\mu} + B_j(z)\, q^{\mu}]$$

by 5 conditions of 4-momentum conservation and the three conditions for diagonalization. Here the (cubic) polynomial $U(z) = \det(Urr')$ plays a fundamental role in the evaluation of the diagram. [As a convenient check the denominator term D (positive definite for $q^2 < 0$) can also be written in the form

$$D = \frac{1}{U} [\sum_{j=1}^{8} z_j\, m_j^2\, U + W_p\, m^2 - W_q\, q^2]$$

where W_p and W_q are simple forms [13], [15] which can be read simply from the graph structure]. At this point odd powers of loop momenta may be dropped in the numerator [18] and the integrand $F(p_1 \ldots p_8)$ becomes an even polynomial in ℓ_r .

We are now ready to integrate over the loop momenta. The basic integration over loop momenta is

$$7! \frac{\int d^4\ell_1 \int d^4\ell_2 \int d^4\ell_3}{[-D+ \Sigma\, U_{rr'}\ell_r - \ell_{r'}]^8} = i^3\, \pi^6\, \frac{1}{U^2\, D^2}$$

Note that integrands containing an extra denominator factor $p_k^2 - m_k^2$ (e.g. , from the $\partial/\partial\Delta_v$ differentiation in the light-by-light scattering calculation), may be obtained by parametric differentiation with respect to m_4^2 . Similarly, integrands containing numerator factors of $\ell_j \cdot \ell_k$ can be integrated using succesive parametric differentiation with respect to the $U_{rr'}$. Note that forms like $\ell_r \cdot p\ \ell_s \cdot p$, with explicit dependence on the direction of the external four-vectors can always be removed using tensor methods. The anomalous moment is associated with a linear dependence on q^2 in the numerator trace reduction.

The replacements for numerator polynomials $\ell_j \cdot \ell_k$ can be put into a very simple form using the following technique [4] ; the form $\ell_j \cdot \ell_k$ can be written as a linear combination

$$\ell_j \cdot \ell_k = \sum_{s,s'=1}^{3} \eta_{js}\, \eta_{ks'}\, \ell_s \cdot \ell_{s'}$$

where $s = 1,2,3$ are the indices of the three independent loop momenta. $\ell_s \cdot \ell_{s'}$ is equivalent to

$$\frac{\partial}{\partial \bar{U}_{ss'}} \left(\frac{1}{U^2}\right) = -2\, \frac{1}{U^3}\, \frac{\partial U}{\partial \bar{U}_{ss'}}$$

where to avoid double counting for $s \neq s'$ (U is symmetric) we define

$$\begin{aligned} \bar{U}_{ss'} &= U_{ss'} \quad \text{for} \quad s = s' \\ &= 2\, U_{ss'} \quad \text{for} \quad s \neq s' \,. \end{aligned}$$

Note that

$B_{ss'} = \dfrac{\partial U}{\partial \bar{U}_{ss'}}$ is the signed cofactor of $U_{ss'}$ in U . Consequently $B_{ss'}/U$ is the inverse of the matrix $U_{ss'}$. The polynomials $B_{ss'}$, A_r , B_r , and U determine the complete parametric structure of the graph.

In addition to quadratic terms, numerators with up to six powers of loop momenta ℓ_r appear in the computation of the photon-photon-scattering anomalous moment contribution. An important identity for reducing the required higher order derivatives is

$$U\frac{\partial^2 U}{\partial \bar{U}_{ab}\partial \bar{U}_{cd}} = B_{ab}\, B_{cd} - \frac{1}{2}\, B_{ac}\, B_{bd} - \frac{1}{2}\, B_{ad}\, B_{bc}$$

which holds for symmetric matrices $U_{rr'} = U_{r'r}$.

To prove this it is simplest to consider the more general case where the elements of $U_{rr'}$ are independent, and show for $n \times n$ square matrices $(n \geq 2)$

$$U\frac{\partial^2 U}{\partial U_{ab}\, \partial U_{cd}} = B_{ab}B_{cd} - B_{ad}B_{bc}$$

$$B_{ab} \equiv \frac{\partial U}{\partial U_{ab}}$$

We start from the statement that $B_{ss'}/U$ is the inverse

$$\sum_{j=1}^{3} U_{ij}\, B_{cj} = \delta_{ic}\, U$$

Differentiating w.r.t. U_{ab} gives

$$\delta_{ai}\, B_{cb} + \sum_j U_{ij}\, \frac{\partial^2 U}{\partial U_{ab}\partial U_{cj}} = \delta_{ic}\, B_{ab}$$

Multiplication by B_{id} and summation over i then gives

$$B_{ad}B_{cb} + U\, \frac{\partial^2 U}{\partial U_{ab}\, \partial U_{cd}} = B_{cd}\, B_{ab}$$

as required. The prove for the symmetric case is the same except

$$\frac{\partial U_{ij}}{\partial \bar{U}_{ab}} = \frac{1}{2}\,\delta_{ia}\,\delta_{jb} + \frac{1}{2}\,\delta_{ib}\,\delta_{ja}$$

As a consequence of this identity we carry out successive differentiations of U in a uniform manner in terms of B_{ab}. Thus a numerator term linear or in higher powers of $\ell_a \cdot \ell_b$ is replaced in the integrand according to [4]

$$\ell_a \cdot \ell_b \rightarrow \frac{B_{ab}}{U}\,D$$

$$\ell_a \cdot \ell_b \cdot \ell_c \cdot \ell_d \rightarrow \frac{4D^2}{U^2}\,[4B_{ab}B_{cd} + B_{ad}B_{bc} + B_{ac}B_{bd}]$$

$$\ell_a \cdot \ell_b \cdot \ell_c \cdot \ell_d \cdot \ell_e \cdot \ell_f \rightarrow \frac{D^3}{U^3}\,[8\,B_{ab}B_{cd}B_{ef} + 2\,B_{ab}B_{cf}B_{ed} + \ldots + \frac{1}{2}\,B_{ae}B_{bc}B_{df}]$$

(15 distinct terms)

aside from a simple numeratorfactor which depends on the number of denominators.

Thus, in this manner, all Feynman graphs are reduced systematically to parametric integrals. The renormalized photon and lepton propagator insertions of reducible graphs are no more complicated since they can be written as a spectral sum of free propagators. Alternately, subtraction counter terms can be constructed in parallel and subtracted in momentum or parametric space [5],[6].

In the vertex calculations discussed here [4],[6], all of the above substitutions and traces could be accomplished automatically by straightforward REDUCE [2] substitutions. The programs (1) project the desired form factor by a trace calculation, (2) replace the internal 4-vectors p_j in terms of the expansion :

Let PI = AI x P + BI x Q + LI x S ; enforcing the mass shell condition and retaining only even powers in the scale variable S. (3) Angular averaging is introduced to eliminate explicit dependence on the directions of P and

Q ; e.g. :

```
MATCH     S **2 * P.X * Q.Y = 0
```

(X and Y are free variables)

(4) the substitutions for powers of loop momenta are then made ; e.g.

```
MATCH S ** 4 * X.Y * U.V = C*(4*X.Y * U.V + X.U*Y.V + X.V * Y.U) ;
```

(5) and finally the replacements for the pairs X.Y are made (e.q. $\ell_1 . \ell_2 \rightarrow B_{12}$),. Additionally, the integrands may be simplified even further by Kirchoff laws ; (see App.A, ref.[5] .

After the final stages of algebraic simplification REDUCE produces the integrand in FORTRAN form suitable for numerical integration. Although, we did not do this, the program could have also been used to automatically generate the function form of $A_j(z)$, $B_j(z)$, $B_{jk}(z)$ and U from the 3x3 inhomogeneous system derived from the diagonalization conditions.

C) The integrals over the Feynman parameters (up to 5 dimensions for the slope of the fourth order vertex and 7 for the anomalous moment in sixth order) have been performed numerically using the multi-dimensional FORTRAN program originally developed by G. Sheppey [19] and modified by A.Dufner [5]. In this method one calculates the usual Riemann sum, taking the central value of the integrand from an average of two random points within each hypercube. The difference of the function values is used to compute a variance and error for the integration, on successive iterations the computer readjusts the grid to minimize the variance.

Although the Sheppey program is well-suited to the integrals involved in this calculation, it is still very advantageous to cut down the number integration variables by mapping onto appropriate variables according to the structure of D and U , or in the case of log m_μ^2/m_e^2 or log λ^2 (or then residual terms) make appropriate changes of variables to eliminate or restrictthe near singular behaviour to as few variables as possible. Dramatic charges of variables also allow obvious checks on the convergence and validity of the integrations.

III - BRIEF DISCUSSION OF RESULTS

Recent reviews of the precision tests of quantum electrodynamics have been presented by Drell and myself [1] and de Rafael, Lautrup, and Peterman [1] , and I will only briefly discuss tests of the higher order corrections to the lepton vertex relevant to this talk.

The fundamentaland historic test ofelectrodynamics is the Lamb Shift, the $2S_{1/2} - 2P_{1/2}$ separation in hydrogen and hydrogen-like atoms. The bulk of the Lamb Shift is computed from the 2nd order self-energy of the electron as modified to all orders by the Coulomb field of the proton [20],[21]. In fourth order, (i.e. : 2 photon corrections to the lepton line) it is sufficient [22] to consider only first order effects in the Coulomb field ; and thus knowledge of the electron vertex form factors in fourth order is sufficient to determine the short-distance modifications of the electron-proton interaction. Since the effect of the electron anomalous moment is well-known, the important question is the value for [4]

$$\Delta E_n(n,j,\ell) = \delta_{\ell 0} \; \frac{4(Z\alpha)^4 mc^2}{n^3} \; m^2 \frac{\partial F_1}{\partial q^2}\bigg|_{q^2=0}$$

due to the order α^2 effective r m s radius of the Dirac form factor :

$$\langle r^2 \rangle = 6 \frac{\partial F_1}{\partial q^2}\bigg|_{q^2=0}$$

Higher order terms in q^2 produce corrections of order $\pi Z\alpha$ smaller.

In 1970, Appelquist and I [4] reported a calculation of the $q^2 = 0$ slope of the Dirac form factor in fourth order and found the result

$$m^2 \frac{dF_1^{(4)}}{dq^2}\Bigg|_{q^2=0} = [0.48 \pm 0.07] \frac{\alpha^2}{\pi^2}$$

using the analytic and numerical techniques discussed in Section II. Our result differed from the previous analytic calculation [23] due to an overall sign discrepancy and individual differences in the non-infrared remainder of the cross and corner graphs. More recently there have been additional numerical and analytic calculations which have confirmed, graph by graph, the new results and removed the numerical integration uncertainty. The final result is [see *figure* (1)]

$$m^2F_1'(0) = \left(\frac{\alpha}{\pi}\right)^2\left[-\frac{4819}{5184} - \frac{49}{72}\,\xi(2) + 3\log 2 * \xi(2) - \frac{3}{4}\,\xi(3)\right]$$

$$= \left(\frac{\alpha}{\pi}\right)^2 [0.470]$$

as obtained analytically by R. Barbieri, J. Mignaco and E. Remiddi [24] [graphs (a) - (e)] and A. Peterman [25] [graph (a)]. Graphs (b) and (d) were also calculated and checked numerically by de Rafael, Lautrup and Peterman [26]. This result leads to an increase [see Table I] of

$$0.34\ \text{MHz}\quad Z^4(2/n)^3\ \delta_{\ell 0}$$

from the previous compilations of the Lamb Shift given by Erickson and Yennie [20], and is in fact in good agreement with **recent** experimental results.

Table II is the result of a recent summary of the Lamb Shift given by Erickson [21]. It also includes an improved estimate of the higher order binding corrections to the second order self-energy, and an improved value for the deuteron charge radius for the deuteron time structure. Experimental references are given in ref. [4].

Table I . Revised tabulation of the theoretical contributions to the Lamb interval $\mathcal{L} = \Delta E(2S_{\frac{1}{2}} - 2P_{\frac{1}{2}})$ in H . References to the various entries may be found in Erickson and Yennie (Ref. [20]) and Taylor et al. (Ref. [29]) . The major revision from the compilation of Ref. [20] is the new result for the order $\alpha^2(Z\alpha)^4m$ contribution to the energy shift from the slope of the Dirac form factor in fourth order as given in refs. [4, 24, 25, 26] . The result of Ref. [23] is 0.102 MHz . Note that fourth-order contributions also arise from the anomalous magnetic moment and vacuum polarization corrections. An improved estimate of errors will be given in ref. [1b] .

DESCRIPTION	ORDER	MAGNITUDE (MHz)
2nd Order-Self-Energy	$\alpha(Z\alpha)^4m\{\log Z\alpha,1\}$	1079.32 ± 0.02
2nd Order - Vac.Pol.	$\alpha(Z\alpha)^4m$	- 27.13
2nd Order - Remainder	$\alpha(Z\alpha)^5m$	7.14
	$\alpha(Z\alpha)^6m\{\log^2 Z\alpha,\log Z\alpha,1\}$	- 0.38
4th Order - Self-Energy	$\alpha^2(Z\alpha)^4m$ $F_1'(0)$	0.44
	$\alpha^2(Z\alpha)^4m$ $F_2(0)$	- 0.10
	$\alpha^2(Z\alpha)^5m$	± 0.02
4th Order - Vac.Pol.	$\alpha^2(Z\alpha)^4m$	- 0.24
Reduced Mass Corrections	$\alpha(Z\alpha)^4\frac{m}{M}m\{\log Z\alpha,1\}$	- 1.64
Recoil	$(Z\alpha)^5\frac{m}{M}m\{\log Z\alpha,1\}$	0.36 ± 0.01
Proton Size	$(Z\alpha)^4(mR_N)^2m$	0.13

$$\mathcal{L} = \Delta E(2S_{\frac{1}{2}} - 2P_{\frac{1}{2}}) = 1057.90 \pm 0.05 \text{ (L.E.)}$$

$$\alpha^{-1} = 137.03608(26) \qquad \Delta E(2P_{\frac{3}{2}} - 2S_{\frac{1}{2}}) = 9911.13 \pm 0.11 \text{ (L.E.)}$$

$$\Delta E(2P_{\frac{3}{2}} - 2P_{\frac{1}{2}}) = 10969.03 \pm 0.04 \text{ (L.E.)}$$

<u>Table II</u> . Precision tests of Lamb Shift calculations*

Interval	Theory ($\pm 1\sigma$)	experiment ($\cdot\ \sigma$)	$\frac{\text{theory - exp}}{\sigma}$
H $2S_{\frac{1}{2}} - 2P_{\frac{1}{2}}$	1057.899 ± 0.017	1057.90 ± 0.06 1057.77 ± 0.06	- 0.0 + 2.1
$2P_{3/2} - 2S_{\frac{1}{2}}$	9911.136 ± 0.033	9911.17 ± 0.04 9911.25 ± 0.06 9911.38 ± 0.03	- 0.7 - 1.7 + 5.5
D $2S_{\frac{1}{2}} - 2P_{\frac{1}{2}}$	1059.259 ± 0.028	1059.28 ± 0.06 1059.00 ± 0.06	- 0.3 + 3.9
He^{+} $2S_{\frac{1}{2}} - 2P_{\frac{1}{2}}$	14,044.56 ± 0.64	14,045.4 ± 1.2 14,040.2 ± 1.8	- 0.6 + 2.3
$3S_{\frac{1}{2}} - 3P_{\frac{1}{2}}$	4184.36 ± 0.19	4183.17 ± 0.54	+ 2.1
$3P_{3/2} - 3S_{\frac{1}{2}}$	47,843.46 ± 0.24	47,844.05 ± 0.48	- 1.1

Values are in MMHz . Experiments are listed in Ref. [4] .

* Compiled by G.W. Erickson (to be published).

An even more severe test of quantum electrodynamics is provided by the anomalous magnetic moments of the electron and muon. There are basically three separate components to the sixth order calculation of the electron

(1) The insertion of second and fourth order vacuum polarization loops into the second and fourth order vertex. Fourth order vacuum polarization gives [27,6] [$.0554 \frac{\alpha^3}{\pi^3}$] and second order vacuum polarization [6, 28] ($.154 \pm .009 \frac{\alpha^3}{\pi^3}$

(2) The light-by-light scattering contribution : [5]

$$0.36 \pm .04 \frac{\alpha^3}{\pi^3}$$

obtained via the analysis and 7-dimensional numerical integration, discussed in Section II, and

(3) The non-loop contributions from 28 distinct graphs recently computed by M. Levine and J. Wright [12] : $1.2 \pm .2 \frac{\alpha^3}{\pi^3}$.

The theoretical result through order α^3 (using the Schwinger and Peterman - Sommerfield result [1] is

$$\left(\frac{g-2}{2}\right) = a_e = \frac{1}{2}\frac{\alpha}{\pi} - 0.328479 \frac{\alpha^2}{\pi^2} + 1.46 \frac{\alpha^3}{\pi^3}$$

which can be compared with the very precise Wesley-Rich [30] measurement (using [29] $\alpha - 1 = 137.03608\ (26)$)

$$a_e^{exp} = .0011596577\ (35)$$
$$= \frac{1}{2}\frac{\alpha}{\pi} - 0.328479 \frac{\alpha^2}{\pi^2} + (1.68 \pm .33) \frac{\alpha^3}{\pi^3} ,$$

that is, agreement in the seventh significant figure. In a sense, we can say that the QED prediction for the gyromagnetic ratio $g = 2(1 + a)$ is confirmed in the tenth significant figure, considering the fact that there is no general reason why the unperturbed theory should have the Dirac form beginning with $g = 2$.

The largest sixth order contribution to the difference of electron and anomalous moments comes from the electron-loop photon-photon scattering contribution [5]

$$\Delta\, a_{\mu}(\text{photon-photon}) = (18.4 \pm 1.1)\, \frac{\alpha^3}{\pi^3}$$

which is very large due to a logarithmic dependence $\left[(6.4 \pm 0.1) \log \frac{m_4}{m_e} + \text{const}\right] \frac{\alpha^3}{\pi^3}$ in the electron mass m_e for $m_\mu^2/m_e^2 \gg 1$. The coefficient 6.4 was obtained by doing two integrations analytically and the remaining 5 dimensional integral to high accuracy numerically. The contributions of second and fourth order electron loops in the muon vertex has recently been completed independently by Lautrup, de Rafael, and Peterman [31], and Kinoshita and myself [6].

The total theory result is [1,6]

$$a_\mu^{\text{theory}} - a_e^{\text{theory}} = [\underbrace{616\ (1)}_{\text{QED}} + \underbrace{6.5\ (5)}_{\text{hadronic v.p.}}] \times 10^{-8}$$

The experimental [37] result is

$$a_\mu^{\text{exp}} - a_e^{\text{exp}} = 652\ (32) \times 10^{-8}$$

Although there is good agreement, improved numerical accuracy for the experimental muon anomalous moment is clearly needed.

IV - Computational Techniques in High Energy Physics.

Another area of quantum electrodynamics which can greatly benefit from systematic algebraic computer techniques is the area of higher order high energy processes such as trident and multi-pair production calculations, the now important colliding beam processes of the type $ee \to eeX$ via 2γ annihilation [33] , and hard photon radiative corrections. There is one general comment which is applicable here - one should not start by computing the traces ! The following are examples when the standard approach is awkward and usually impractical :

(1) For the trident process

$$e + p \to e + p + e^+ + e^-$$

there are 8 Bethe-Heitler matrix elements in fourth order, including the exchange diagrams. The individual amplitudes are of course gauge-dependent, and in fact suffer severe cancellations at usual laboratory conditions in the total result. The trace can involve a string of 12 γ-matrices ; moreover the sum of the horrendous results for the traces then involves numerical cancellations often not practical in a Fortran calculation even in double precision because of round-off accuracy. The simplest approach, utilized by Ting and myself [34] , Bjorken and Chen [35] , and Henry and Ehn [36] is to directly evaluate each matrix element for each helicity possibility, sum the 8 amplitudes, square and sum over spins . This can be done in a compact, convenient numerical form.

It also should be noted that the corresponding differential cross-section for spin 0 leptons is much more easier to compute, but has the same basic magnitude, structure and cancellations as the spin $\frac{1}{2}$ case.

We have found it extremely useful to have both calculations for comparison. The spin 0 result which is more compact can, moreover, be used to set-up an initial optimum grid for a first stage numerical integration.

In other higher order calculations, the work of Cheng and Wu [38], and others, and more recently, the ∞-momentum frame calculations of Bjorken, Soper, and Kogut [39] has made it clear that the high energy behaviour of the Feynman amplitude obeys a simple eikonal form ; e.g. for e-e scattering in ladder and crossed ladder approximation

$$T \rightarrow \int d^2 b \left[e^{i\chi(b)} - 1 \right]$$

where χ is the phase obtained by integrating an effective potential from the one photon exchange. Further, Brezin, Itzikson, and Zinn-Justin [10] have shown that the longest range Coulomb part of the 1γ exchange potential can be obtained from an eikonal answer at all energies, (and thus in fact provides a useful starting point for the positronium spectrum). These calculations further emphasize the importance of adding sets of amplitudes before computing cross-sections. Further, the eikonal results demonstrate the importance of including all orders of particle emission and exchange in the scattering amplitude in contrast to "ladder" approximation which usually has misleading analytic behavior. It now seems very desirable to develop a perturbation theory which starts with the eikonal answer, and incorporates shorter range interactions in a perturbative form. Possible applications include the positronium spectrum, and approximations to the Coulomb-Dirac propagator for the Lamb Shift calculations.

Another very exciting area for the study of perturbation theory results is the "∞-momentum frame" technique. In this method one returns to "old-fashioned" time-ordered perturbation theory, choosing, however, a reference frame in which particle energies and longitudinal momenta become large. Thus in the analysis of forward virtual Compton scattering or the elastic form factor, one takes, the target 4-momenta as

$$P_\mu = (E_p, \vec{0}, P) \rightarrow \left(P + \frac{m^2}{2P}, 0, P\right)$$

and the photon momentum as

$$q_e = \left(\frac{m\nu}{2P} , \vec{q}_{\perp} , - \frac{m\nu}{2P} \right)$$

$$q.P \rightarrow m$$

$$q^2 = - \vec{q}^{\,2}_{\perp}$$

where we choose the parameter P such that $P^2 \gg M^2$, $m\nu$.
As P becomes large, the time-ordered diagram with particles moving with longitudinal momentum XP , $X < 0$ (opposite to $\vec{p}$) vanish [41] in order P^{-2} or have a simple limiting (seagull-like) form (the latter case occurs in fermion theories) [42] . Unlike ordinary time-ordered theory, each surviving diagram is only a function of covariant quantities. Unlike dispersion theory the intermediate phase space is trivial with no square roots. In this manner one obtains Feynman-like results but with loop integrations only involving 3 integrations. [Thus, in the sixth order electron moment there are 9 integrations, of which at least two are simple] .

The calculation of the anomalous moment in second order is extremely simple in this framework. In higher orders, various subtleties involving renormalization and the $P \rightarrow \infty$ limit need to be handled carefully. We also mention here that the distribution functions in the fractional longitudinal momentum X , and transverse momenta $\vec{k}$ are of great physical utility in the discussion of the impulse approximation in field theory.
For example, the high energy limit of forward Compton scattering in QED contains a constant term of the form [43]

$$\lim_{\nu \rightarrow \infty} T_1(\nu, 0^\circ) = T(\nu=0) * \int_0^1 \frac{f(x)}{x} \, dx$$

where $f(x)$ is the probability of measuring via the charge operator, an intermediate charged particle with fractional longitudinal momentum X as viewed from the infinite momentum frame. The function $Xf(X)$ is also connected with the large q^2, ν limit of the inelastic form factors of the target electron [42]. Hopefully, the ∞-momentum frame and eikonal techniques will lead not only to efficient calculation, but also will finally lead to interpretative insight into our calculations.

- FOOTNOTES AND REFERENCES -

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$$\tilde{\Gamma}_\mu(p,p') = e^B \Gamma_\mu^{NIR}$$

$$= \gamma_\mu + B\gamma_\mu + \Lambda_\mu^{(2)} + \frac{1}{2} B^2 \gamma_\mu$$

$$+ B \Lambda_\mu^{(2)} + \Lambda_\mu^{(4)} + \ldots$$

where B contains all the infrared dependence and $\Lambda_\mu^{(2)}$, $\Lambda_\mu^{(4)}$ are finite for $\lambda \to 0$. Since B and the F_1 part of $\Lambda_\mu^{(4)}$ vanish like q^2 for $q^2 \to 0$ we see that the fourth order Dirac form factor to order q^2 and $F_2(0)$ are convergent in the infrared.

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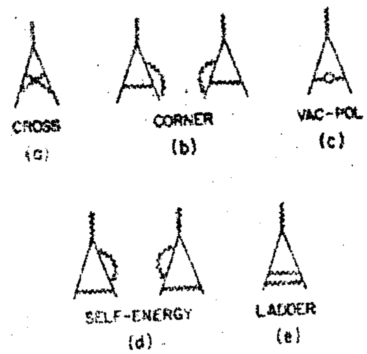

FIG. 1. Feynman diagrams which contribute to the fourth-order electrodynamic corrections to the Lamb shift.

Fig. 1 - Feynman diagrams containing subdiagrams of photon-photon scattering type. The heavy, thin, and dotted lines represent the muon, electron, and photon, respectively. There are three more diagrams obtained by reversing the direction of the electron loop.

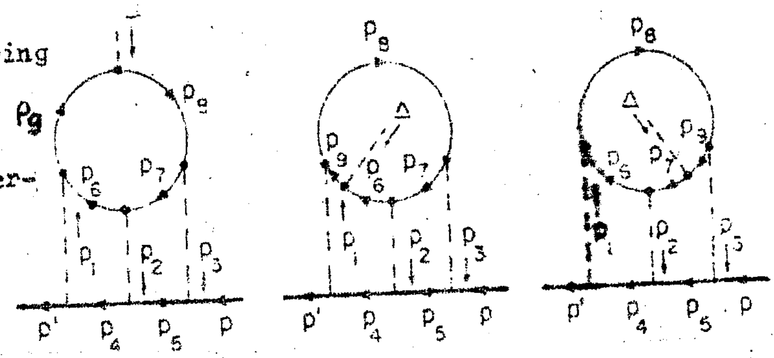

Fig. 1 - Four types of sixth-order Feynman diagrams which contribute to the anomalous magnetic moment of the muon and electron.