SEMI-CLASSICAL RADIATION THEORY

AND

HIGH ENERGY MULTIPRODUCTION[*]

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One of the pleasant things about the recently revitalized field of very high energy reactions is that it brings us back to some very simple ideas first raised in the infancy of the subject, when cosmic rays revealed the existence of high multiplicity events. One such idea — perhaps the first to come to mind — is that there might be some connection or analogy between high energy multiproduction by hadrons and the more familiar features of electrodynamic radiation processes. This is the point we will pursue in these lectures. First we will describe briefly how the characteristics of the secondary spectrum might be understood. Then we will consider an analogy between "radiative tails" or "straggling" in electrodynamics and leading particle behavior in hadron reactions. Along the way we will show how to derive in a simple way the radiative correction formulas for "radiative tails" and "straggling". Our major point will be that these phenomena have their origin in the statistical fluctuations of the radiation field and thus can easily be described on that basis. These ideas have been taken up in recent years by Feynman[1] but they go at least as

far back as Heisenberg[2], who considered treating cosmic ray multi-production a la Bloch-Nordsieck[3], noting that this would give a logarithmic multiplicity growth. The basic idea is that in very high energy processes we have something like the soft-photon infrared limit in electrodynamics. One assumes that setting the incident hadron energy very high allows us to treat the radiated particles in a zero mass, low energy limit and that this limit behaves like that in classical electrodynamics.

Before beginning, however, it might be well to stress that we do not attempt to explain the two basic facts of high energy reactions:

(1) That the transverse momentum is limited. This observation is so often quoted that we will not try to justify it here. A remark perhaps worth making, however, is that if there were some tendency for the transverse momentum to spread out slowly (say as in a random walk with the multiplicity[4]) then since π's are very light, they carry off very little of the momentum of any moving system, and the effect might be hard to see. But heavy particles will carry off more of the transverse momentum, so K's and $\overline{P}$'s might show any such hidden spreading more dramatically. The limitation on the transverse momentum simplifies kinematics considerably and is what allows us to treat things in an essentially one-dimensional way, with $P_{longitudinal} \approx E$.

(2) That leading particles exist. Figure 1 shows the proton emerging in the reaction $P + P \rightarrow P + X$ (X means "anything") at 19 GeV[5] and we see that the proton takes away a substantial fraction of the energy almost always; it is "leading". Note also that this proton spectrum is rather flat, something we will try to explain later. It is roughly flat in the cross section $d\sigma/dP_L$, not in terms of the presently popular "invariant" cross
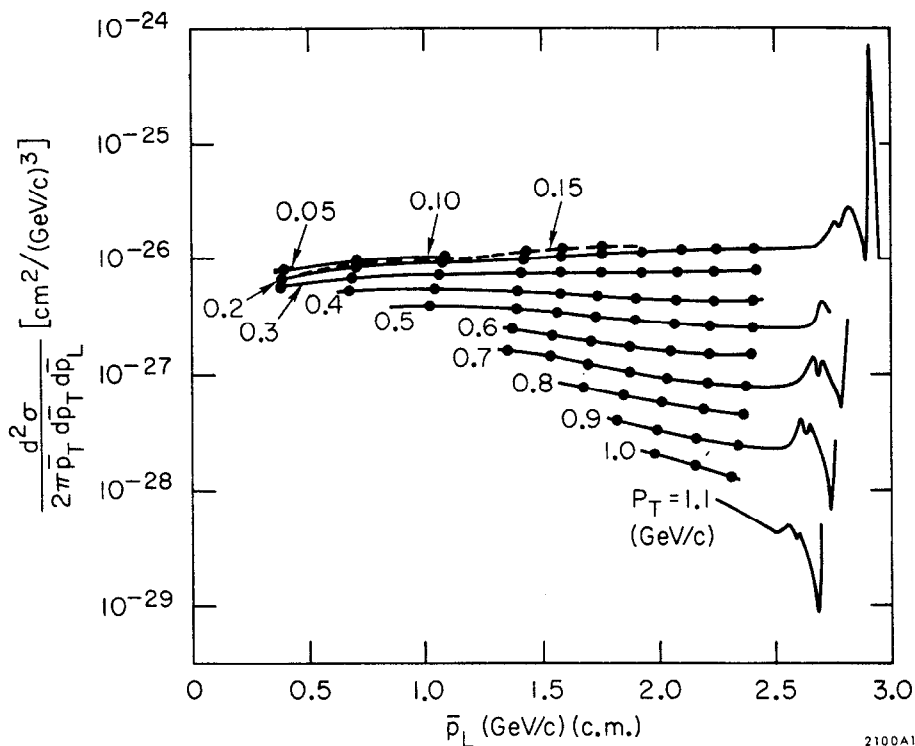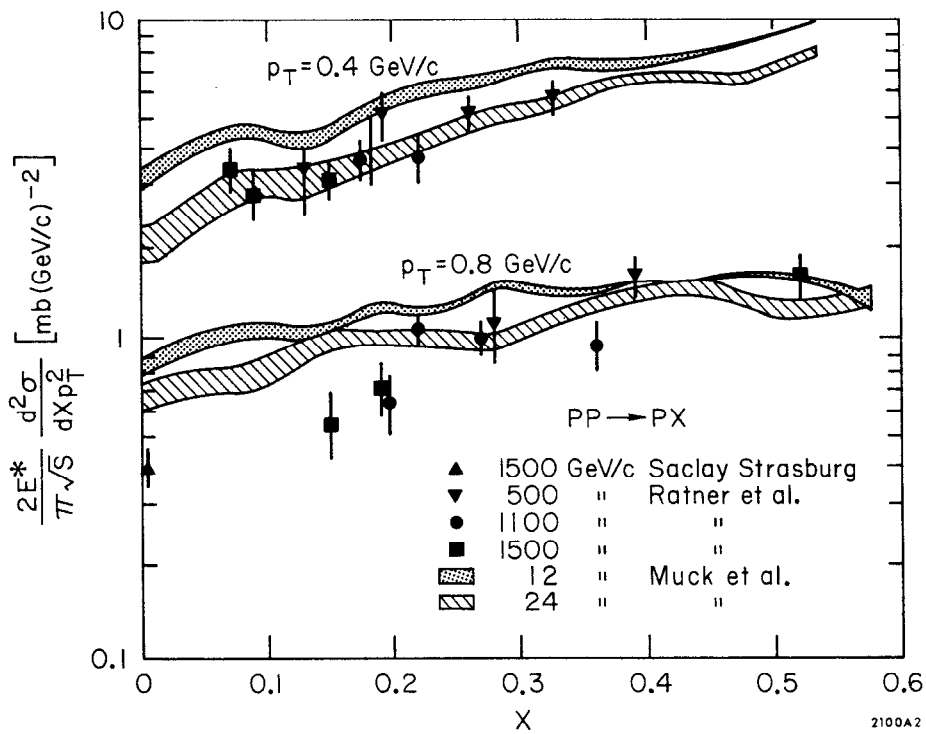
Figure 1

Figure 2

4

section $E(d\sigma/dP_L)$. At the ISR there is also data on the emerging proton at small x (x = Feynman scaling variable = P/P_{max}) shown in Figure 2.[6] The ISR data appears to have the same shape as the bands indicating the low energy data. It also appears to "scale", i.e. the invariant cross section is the same at equivalent values of x. It is unfortunate, since we will be mainly interested in the leading proton, that the ISR data does not cover large x. At very small x we can expect, at high energy, some "protonization" ($\bar{p}p$ production) so that there is no reason to expect the low and high energy data to agree.

As for pion beams, we have seen data at this school for $\pi^- + P \rightarrow \pi^- + X$ which looks like:
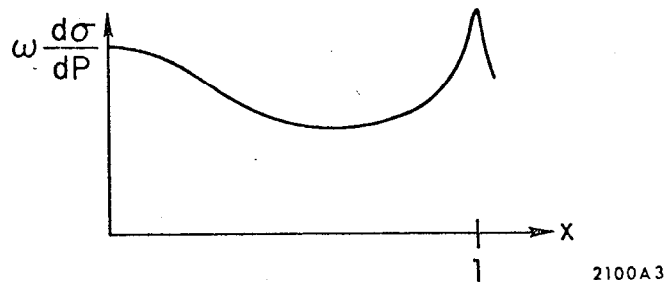

Figure 3

showing again that there is some kind of leading π^-, although clearly the "leading" and "produced" π^-'s get mixed up at small x. We have, as yet, no leading particle data for K beams, but since multiproduction of strangeness is small, it should look like the proton spectra.

I. The Infrared Analogy

The fundamental feature of infrared radiation is that the radiated energy is constant as a function of frequency

$$\frac{dI(\omega)}{d\omega} = c \tag{1}$$

The intensity dI is the energy content of the radiation field in the frequency interval $d\omega$. A scattered charged particle always radiates this spectrum at low frequencies, regardless of the details of the scattering process.[7] Why is this? One way[8] of explaining it is to look at the field of a charged particle as it goes by
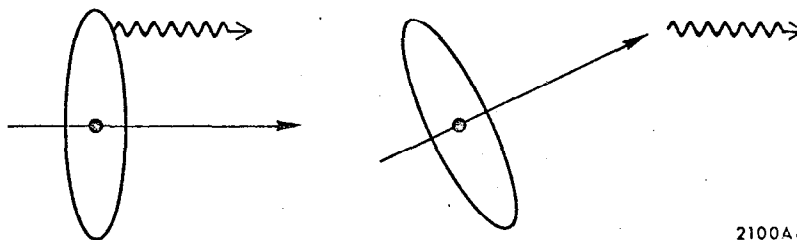
Figure 4

very fast. If the particle suddenly changes direction, the electromagnetic field cannot adjust instantaneously and some of it just keeps going, appearing as photons. The energy in the field, the intensity, is a function of the $\mathcal{E}$ and $\mathcal{H}$ fields: $I \sim \mathcal{E}^2 + \mathcal{H}^2$. Since for very fast moving charges $\mathcal{E}$ and $\mathcal{H}$ are delta function-like pulses in time their Fourier transforms $\mathcal{E}(\omega)$ and $\mathcal{H}(\omega)$ must be approximately constant, and so the intensity is also, hence Eq. (1).

We now assume that something of the same sort takes place in the high energy scattering of hadrons, and that in particular Eq. (1) still holds. I is now the radiated energy in the hadron field. So far we have used no Quantum Mechanics. Now introducing a minimal amount via Planck's relation; namely, $dI = \omega dn$, ($\hbar = 1$) gives us the particle spectrum

$$\omega \frac{dn}{d\omega} = c \ . \tag{2}$$

This, when coupled with the limited P_T distribution, gives a one-dimensional uniform distribution in rapidity Y, or what has been called the "Feynman gas", because $\omega \frac{dn}{d\omega} = \frac{dn}{d\omega/\omega} \cong \frac{dn}{dY}$. It is currently popular to

talk about the invariant cross section $\omega \dfrac{d\sigma}{d^3 P}$, or after integrating out the transverse variables, $\omega \dfrac{d\sigma}{dP_\ell}$. At high energy, though with $\omega \approx P$, this is the same as in Eq. (2). The statement that the invariant cross section is a constant can then be physically interpreted to mean that the energy radiated per unit momentum interval is a constant, c.

We can use Eq. (2) to get the multiplicity by integrating:

$$n = \int_{\omega_0}^{\omega_m} \frac{dn}{d\omega} = c \ln \frac{\omega_m}{\omega_0} . \tag{3}$$

The lower limit, ω_0, presumably stays fixed around the meson mass. The upper limit, on the other hand, must increase as the energy increases and it is reasonable to assume, as in bremsstrahlung, that the "infrared" behavior holds up to some value controlled by the maximum energy available — the "x-ray endpoint". This situation is displayed in Figure 5.
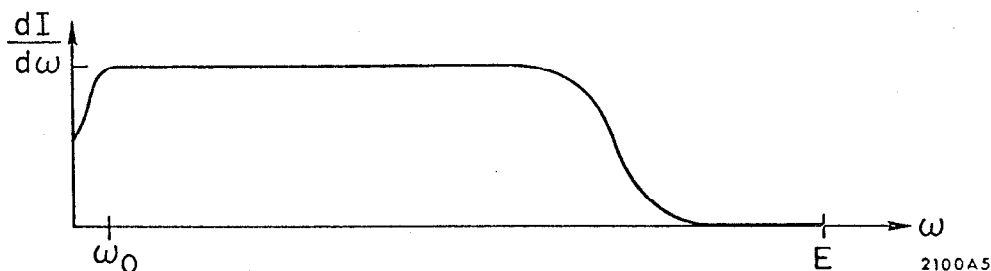

Figure 5

Thus we take $\omega_m \sim E$, giving

$$n = (\text{const}) + c \ln E \tag{4}$$

This is the reasoning used by Heisenberg in 1942 to obtain a logarithmic multiplicity growth. We see that the coefficient c has the significance of the classical $dI/d\omega$. Both the logarithmic grown of n and the spectrum Eq. (2) appear to have been observed experimentally.[6, 11]

Relativistic Transformation Properties

Thus far we have not specified in what frame we work and so to what the energy in Eq. (4) refers. Presumably if we are to treat all the particles as effectively massless we should be in a frame where all the particles are relativistic, such as the center-of-mass. Neither have we taken into account the other incident particle in the reaction; the multiplicity referred to so far comes from only one of the incident particles. In the center-of-mass, for example, Eq. (4) refers to the multiplicity and energy appropriate to one of the incident particles. Thus for the total multiplicity $n_1 + n_2 = c \ln E_1 + c \ln E_2 = c \ln E_1 E_2$. But at high energy $E_1 E_2 = s/4$, giving for the total $n = c \ln s$ (+ const). Remark now if we consider another Lorentz frame similar to the center-of-mass but where E_1 and E_2 are different we will still get the same answer, since $E_1 E_2$ is invariant, although the total multiplicity will be divided up between n_1 and n_2 in a different way. This elegant feature is a reflection of the uniform distribution in rapidity for n given by Eq. (2). This uniform distribution means that if we shift Lorentz frames, giving a translation in the rapidity Y, the distribution looks the same in the new frame but with respect to a shifted zero in Y. Some particles that had positive Y in the center-of-mass move to negative Y, and so look like they come from the other incident particle, but the total remains unchanged. Note that these shifts come about as a result of the shifts in the endpoint energy associated with the incident particles — the constant c itself is an invariant.

II. Radiative Tails and the Leading Particle

Fluctuations

The classical radiation spectrum sketched in Figure 5 is of course an average, summed over many events, and the n in Eq. (4) is the average total multiplicity. Now we must add another quantum mechanical element in considering flucturations. If we look at one particular event there will of course be deviations from the average, as in Figure 6
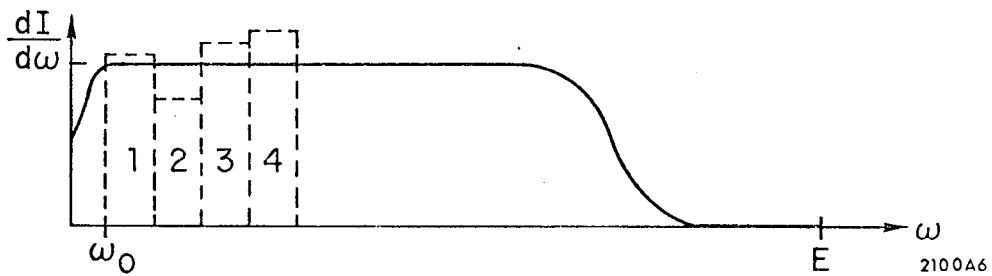

Figure 6

We will assume, as in the case of bremsstrahlung, that these fluctuations are Poisson, that is that the probability of finding n particles in one of the "bins" indicated in Figure 6 is, where $\bar{n}$ is the average number in the bin

$$P(n) = \frac{(\bar{n})^n}{n!} e^{-\bar{n}} . \tag{5}$$

This assumption is equivalent to an assumption of statistical independence in the emission process. That is, the probability of emitting a particle of a given energy and type is always the same, regardless of what else has happened. Obviously, this cannot be exactly true since the probability of emitting anything must go to zero as the energy is used up, and with charged particles we cannot go on emitting so many particles of the same kind that we eventually violate charge conservation. Nevertheless at high energy with many emissions taking place it seems plausible that these constraints are only weakly felt in the bulk of the emissions and that

statistical independence is a reasonable first working hypothesis.[9]

Radiative Tails

These flucturations show up in electrodynamics as the well-known phenomenon (or for experimentalists, the notorious problem) of "radiative tails". The scattering of a charged particle is always accompanied by some electromagnetic radiation. In fact in the real limit of zero mass photons there are always an infinite number of photons emitted[3], due to the $\omega = 0$ singularity of $dn/d\omega$ in Eq. (2). Although the energy thus carried off remains finite, it will fluctuate from event to event. Therefore a particle, like an electron, that radiates easily will have, as in Figure 7, a "tail" extending down from the energy, E_0, it would have for the given process and kinematics if no photons existed.
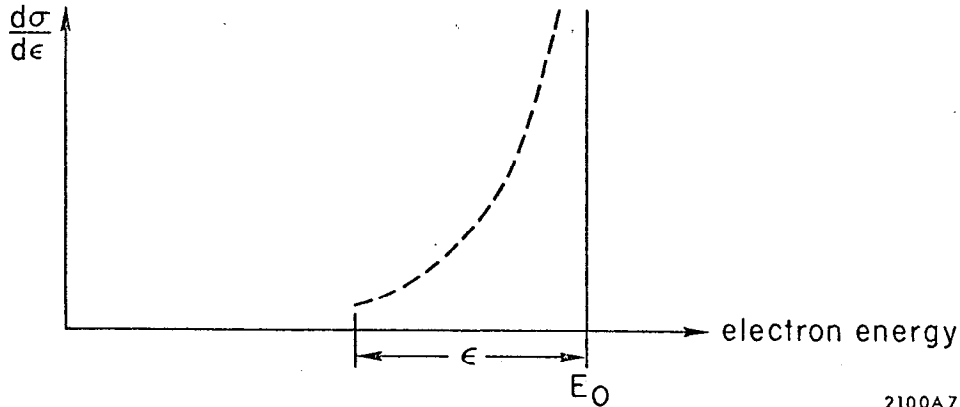
Figure 7

The energy loss, as measured from the nominal endpoint E_0, we call ϵ. A formula can be derived for this "tail" by the methods of quantum electrodynamics[7]. If σ^o is the cross section that would exist without considering these soft photon effects, α the fine structure constant, and A various numerical factors it is

$$\frac{d\sigma}{d\epsilon} = \frac{\alpha A}{\epsilon} \sigma^o (\epsilon/E)^{\alpha A} . \tag{6}$$

The $1/\epsilon$ factor in this formula is easy to understand, it just reflects the $d\omega/\omega$ spectrum of the radiated photons; the low ω "bins" in Figure 6 tend to dominate because of their much higher average number. The $(\epsilon/E)^{\alpha A}$ factor, which make the $\epsilon = 0$ singularity integrable, arises from more subtle multiple photon effects.

We will presently see how this formula has its origin in the energy fluctuations of the radiation field.

Straggling

The spreading of the spectrum in Figure 7 can become quite extensive when an electron beam passes through a macroscopic thickness of material. As a result of the successive bremsstrahlung emissions, electrons originally of energy E_0 may become spread out in energy:
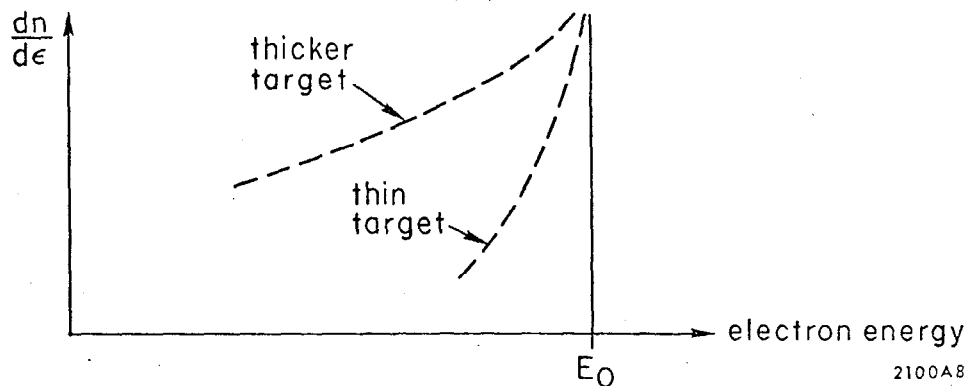
Figure 8

This is called "straggling". In particular we are talking about "bremsstrahlung straggling", to distinguish it from other kinds, such as "Landau straggling", which is the fluctuations connected with energy loss due to ionization, not radiation.

There is something puzzling about Figure 8 for the following reason: when some process, such as an energy-loss mechanism is made up of many random contributions we expect the resultant distribution to have some

average value and then some dispersion, i. e. fluctuations around the average, as in Figure 9.
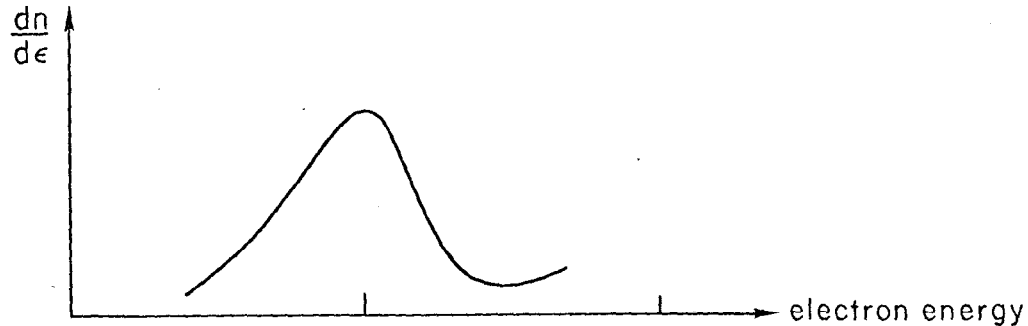
Figure 9

Mathematically, this expectation is enshrined in the "Central Limit Theorem". But Figures 7 and 8 don't look at all like this, although the spectra do result from many photon emissions. On the other hand, the spectrum resulting from "Landau straggling" does resemble Figure 9.

The explanation for the difference in the two cases is interesting because it has some parallel with the high energy problem. In the case of energy loss by ionization, the particle loses a fixed absolute amount of energy in each collision, which comes out to, say, a few MeV per gram.

The total energy loss for an energetic particle is then indeed made up of many small random loses. On the other hand, in the case of bremsstrahlung (for a relativistic electron) the energy loss in a collision is always a fixed fractional or relative amount of the incident electron energy. This can be seen, for example, in the structure of Eq. (6) and has its origin in the fact that when the electron mass plays no role we only have variables like (ϵ/E) at our disposal. This situation (called "Approximation A" in the terminology of Rossi[10]) is analogous to "scaling" in high energy reactions. It means that although we may have many independent emissions, one photon may take off a substantial fraction of the energy — something which essentially cannot happen in the other case and

which leads to non-normal behavior. A high energy analogy to the Landau straggling case would be a total soft pionization model in which all the produced particles, say in the center-of-mass, each had a finite energy, perhaps always less than a few hundred MeV. (This model does not seem to be seriously entertained by any school at present.)

- The formula giving the energy loss distribution of the straggling electrons is, for small thicknesses of the target, just like Eq. (6).

$$\frac{dn}{d\epsilon} = \frac{T}{\epsilon} (\epsilon/E_0)^{-1 + T} \tag{7}$$

The dimensionless parameter T measures the thickness of the target and is called the "radiation length". Its significance (for thin targets) is that it is equal to the average fractional energy loss. If $T = 1/6$, say, then the average energy loss $\bar{\epsilon}/E_0$ as may be calculated from Eq. (7), is also $1/6$. For thick targets the situation is more complicated (see the appendix).

Derivation of the Radiative Tail or Straggling Formulas

It is both amusing in its own right and useful for our generalization to the leading hadron problem that Eqs. (6) and (7) can be derived simply from the information already given about the behavior of the radiation field.

Knowing the behavior of the radiation field is equivalent to knowing the behavior of an outgoing electron or leading particle because the energy appearing in radiation must be the energy lost by the electron or incoming hadron. Therefore the spectrum of the total energy in the radiation is the energy loss spectrum we wish to derive. On the other hand we know, in principle, everything necessary to find the energy in the radiation field.

The number of particles in the field has been taken to follow a Poisson distribution which is only characterized by one parameter, the

average number; but we are also given the average number by Eq. (2).

Looking at Eq. (6) we can say that we know the average number in each "bin" and how the number in the bin fluctuates; we then want to know how the energy in the whole ensemble of bins fluctuates.

This can be calculated as follows: lable all the bins in Eq. (6) $1,2,3,\ldots j\ldots n$. The probability of a given configuration with n_1 in bin 1, n_2 in bin 2... is $P_1(n_1)\,P_2(n_2)\,P_3(n_3)\,\ldots$ where $P(n)$ is given by Eq. (5) with the $\bar{n}$ appropriate to each bin. We are interested in knowing the probability, $P(\epsilon)$, of those configurations which have a total energy ϵ, where the energy in the bins is $n_1\omega_1 + n_2\omega_2 + \ldots$. Therefore

$$P(\epsilon) \propto \sum_{\text{all n's}} \delta(\epsilon - n_1\omega_1 - n_2\omega_2 \ldots)P_1(n_1)P_2(n_2)\ldots P_j(n_j)P_{j+1}(0)\ldots P_m(0)$$

(8)

The bin j is the one where a <u>single</u> meson has energy of ϵ, i.e. $\omega_j = \epsilon$; all bins at higher frequencies must contain no particles. Introducing the Fourier representation of the δ function and Eq. (5) for the $P(n)$ gives

$$P(\epsilon) \propto \sum_{\text{all n's}} \int_{-\infty}^{+\infty} dt\, e^{i\epsilon t}\, \frac{\left(\bar{n}_1 e^{-i\omega_1 t}\right)^{n_1}}{n_1!}\, e^{-\bar{n}_1}\, \frac{\left(\bar{n}_2 e^{-i\omega_2 t}\right)^{n_2}}{n_2!}\, e^{-\bar{n}_2} \ldots$$

(9)

$$\ldots \frac{\left(\bar{n}_j e^{-i\omega_j t}\right)^{n_j}}{n_j!}\, e^{-\bar{n}_j}\, e^{-\bar{n}_{j+1}}\, e^{-\bar{n}_{j+2}} \ldots e^{-\bar{n}_m}\,.$$

Doing the sum on $n_1,\ n_2\ldots$ gives

$$P(\epsilon) \propto \int_{-\infty}^{+\infty} dt\, e^{i\epsilon t}\, e^{(e^{-i\omega_1 t} - 1)\bar{n}_1}\, e^{(e^{-i\omega_2 t} - 1)\bar{n}_2} \ldots e^{(e^{-i\omega_j t} - 1)\bar{n}_j} \ldots$$

$$\ldots e^{-\bar{n}_{j+1} - \bar{n}_{j+1} \ldots \bar{n}_m}\,.$$

(10)

Now if we imagine the bins being very small and numerous the sums in the exponents become integrals so that

$$P(\epsilon) \propto \int_{-\infty}^{+\infty} dt\, e^{i\epsilon t}\, e^{\int_0^{\epsilon} (e^{-i\omega t}-1)\frac{d\bar{n}}{d\omega}d\omega}\, e^{\int_{\epsilon}^{E_m}\frac{d\bar{n}}{d\omega}d\omega} \qquad (11)$$

So far we have assumed nothing except the Poisson — statistically independent — character of the various emissions, and in general Eq. (11) is a complicated expression. It simplifies remarkably, however, if $\frac{d\bar{n}}{d\omega}$ is given by the infrared formula Eq. (2) (taken all the way to $\omega = 0$). For then we have by using variables ω/ϵ and ϵt

$$\int_{-\infty}^{+\infty} dt\, e^{i\epsilon t}\, e^{c\int_0^{\epsilon} (e^{-\omega t}-1)\frac{d\omega}{\omega}} = \frac{F}{\epsilon} \qquad (12)$$

where F is an ϵ independent factor so that

$$P(\epsilon) \propto \frac{1}{\epsilon} e^{-c\ln E_m/\epsilon}$$

$$ \qquad (13)$$

$$P(\epsilon) \propto \frac{1}{\epsilon} (\epsilon/E)^c$$

The maximum energy E_m has been set to the incident energy E, since it only serves as a scale factor in Eq. (13).

Eq. (12) is our basic result. It follows from two assumptions: (1) the Poisson character of the emissions, and (2) the "classical" value for $\bar{n}$, given by $d\bar{n}/d\omega = c/\omega$. If we wish to abandon assumption (2) we can always revert to Eq. (11).

With Eq. (13) we have essentially derived the thin target straggling formula Eq. (7) and the radiative tail formula Eq. (6). Given the form of Eq. (13) which states that $dN/d\epsilon \propto (\epsilon/E)^{c-1}$ the normalization of the straggling formula results from the requirement that the number of

particles in the beam be unchanged, so that $\int_0^E \frac{dN}{d\epsilon} d\epsilon = 1$, while the association $c = T$ follows from the definition of T as the average fractional energy loss. (T in turn can be calculated from the properties of the material[10].) We thus arrive at Eq. (7). Furthermore comparison with Eq. (2) says that the photon spectrum per electron from the target is

$$\frac{dn}{d\omega} = T/\omega \qquad (14)$$

As for the radiative tail in electron scattering, we see that the factor αA in Eq. (6) is given by c or $\alpha A = dI/d\omega$, the classical differential intensity for the process in question. Eq. (6) should, in effect, be read as

$$\frac{d\sigma}{d\epsilon} \sim \frac{1}{\epsilon} \ (\epsilon/E)^{(dI/d\omega)} \qquad (15)$$

This agrees with the treatment according to quantum electrodynamics (as may be verified by comparing the formula for αA on page 230 of reference 7 with the classical radiation formula on page 223.) Note that our treatment avoids any problem of infrared singularities since we have always dealt with the energy fluctuations directly. These are always finite even though the number of photons involved goes to infinity.

The normalization of Eq. (6) may be inferred, as in the straggling case, from the requirement (which we have not derived here) that the cross section, when integrated over the energy loss, must be the non-radiative cross section σ^o.

The Leading Hadron Spectrum[12]

If the secondaries produced in high energy collisions are a kind of "bremsstrahlung" then the leading particle can be thought of as being the "electron". That is, the energy it has lost appears in the radiated particles. Thus its energy loss fluctuates and its spectrum is also given by Eq. (13):

$$\frac{d\sigma}{dP_L} \cong \frac{d\sigma}{d\epsilon} \propto \frac{1}{\epsilon} (\epsilon/E)^c \ . \tag{15}$$

But here we can find c from the experimental multiplicity growth according to Eq. (4), so that this is a relation between the multiplicity growth and the leading particle spectrum.[11]

This experimental value of c appears to be in the range 1-2, or perhaps somewhat larger. If we were to take $c \approx 1$ we see that Eq. (15) gives us the flat proton spectrum in $d\sigma/dP_L$ noted in Figure 1 and 2.

At this point an ambiguity appears, however. In contrast to the real radiation problem where we know that the fundamental statistically independent processes are simply single photon emissions, the hadron problem is not so simple. The effective Poisson emitted objects may be some kind of mixture of π's, ρ's, ω's ..., correlated π pairs and the like. But while the c in Eq. (5) refers to the Poisson emitted objects the experimental c in Eq. (4) refers to the actual (mainly pion) multiplicity. Thus the experimental c of Eq. (4) may have to be reduced before being used in Eq. (15) to predict the leading particle behavior. In the future, detailed examination of the correlations present in multiproduction may tell us exactly what is Poisson emitted and by how much c is to be decreased. For the present we can only say that it seems reasonable to reduce c by

a factor which is "not too big". We stress, however, that it does not work the other way; should the experimental c from Eq. (4) turn out to be too small to fit via Eq. (15) in some process, then our explanation is simply vitiated — we certainly cannot claim to emit less that one pion at a time.

Should our interpretation be borne out, however, we have a simple and amusing interpretation of high energy reactions. When one hadron hits another it is like an electron going through an absorber, and the coefficient c (perhaps adjusted as discussed above) of the multiplicity law tells us how "thick" that absorber is. The flatness of the experimental proton spectrum then means that it sees the other proton as one radiation length[12], or in terms of Eq. (6): hadrons have "α" ~ 1 !

Appendix
"Thick Target Bremsstrahlung"
(The method described here was developed in collaboration with R. Roskies.)

The bremsstrahlung straggling problem has a long and sometimes messy history. The problem has been attacked by the method of the diffusion equation[10] or by Heitler's trick of distorting the photon spectrum to get a tractable approximation.[14] Here we would like to show how to arrive at an exact answer by iterating our simple answer and solving the associated transport problem. Once we go beyond the simple treatment described above, problems arise for two reasons. One has to do with the fact that deviations from the infrared limiting form $d\omega/\omega$ for the photon spectrum occur for photons that have some fraction of the indicent energy. These deviations will then be reflected in the electron's energy loss, particularly when it is large, since it is the "harder" photons that are principally

affected. These effects, while they can be important in practice, may be incorporated in the general method (at least as long as we are in "Approximation A"[10] where the deviations have a "scaling" form as (ω/E)) and do not offer any problems in principle.[15] We will thus not consider this problem any further and will always assume that the radiation from a single interaction is given by Eq. (4). More interesting and perhaps with some relevance to the high energy problem is the second question, connected with "thick targets".

When the target is thick the electrons slow down sufficiently that an initially monoenergetic beam becomes substantially degraded and spread out in energy. In this case we can no longer maintain our basic assumption (1) (following Eq. (13)) concerning the statistical independence of the emissions. This is because our neglect of the energy conservation constraint, which was all right when the beam was not very spread out, has now become serious. An electron which has lost a lot of energy cannot emit a photon near the maximum energy: energy conservation obviously means that the emission of, say, two high energy photons cannot be statistically independent. This is clearly an essential complication and it is interesting to see how we can treat it in a practical case.

Part of the answer we know already, without any further calculation: those electrons which have lost only a small amount of energy still do have a Poisson emission spectrum. Therefore we know that even for a thick target the low energy loss electrons must distribute themselves according to the previous results

$$\frac{dN}{d\epsilon} \propto \frac{1}{\epsilon} (\epsilon/E)^c , \qquad (16)$$

although now we can no longer make a simple identification of the constants.

Since we do know the exact result for a very thin target, however, we can imagine doing the thick target problem by dividing the thick target up into a series of thin slices and iterating the spectrum entering each slice according to the thin target formula.

Let the electron distribution we wish to find be the dn/dE resulting from a monoenergetic (δ function) beam entering the target. We call this function G . It depends on E_0 and the thickness T as well as E. Thus for an arbitrary beam dn_0 incident on the target what comes out is

$$\frac{dn}{dE} = \int G (E, E_0, T) \frac{dn^0}{dE_0} dE_0 \ . \tag{17}$$

Using what we know from Eq. (7) for the G on a thin target, we imagine n the number of slices large so that $T/n \equiv t$ is small so that

$$G (E, E_0, t) = \frac{t}{E_0} \left(\frac{E_0 - E}{E_0}\right)^{-1+t} = \frac{t}{E_0} (1-x)^{-1+t} \ . \tag{18}$$

We introduce the scaled variable $x = E/E_0$, and similarly in each slice there will be an integration variable E_i which we scale to $x_i = E_i/E_0$, while x represents the final energy. Then iteration of Eq. (17) gives

$$G (E, E_0, nt)E_0 =$$

$$\int_x^1 \frac{dx_{n-1}}{x^{n-1}} \left(1 - \frac{x}{x_{n-1}}\right)^{-1+t} \cdots \int_{x_3}^1 \frac{dx_2}{x_2} \left(1 - \frac{x_3}{x_2}\right)^{-1+t} \tag{19}$$

$$\int_{x_2}^1 \frac{dx_1}{x_1} \left(1 - \frac{x_2}{x_1}\right)^{-1+t} (1-x_1)^{-1+t} \ .$$

We have an iteration with a ratio kernel, reflecting the "scaling" character of the problem. It can be brought to a more familiar form involving a difference kernel by using $y_i = -\ln x_i$

$$G(E, E_0, nt)E_0 = G(y, nt)E_0 = t^n \int_0^y dy_{n-1}\left(1 - e^{-(y-y_{n-1})}\right)^{-1+t} \cdots$$

$$\cdots \int_0^{y_3} dy_2\left(1 - e^{-(y_3-y_2)}\right)^{-1+t} \int_0^{y_2} dy_1\left(1 - e^{-(y_2-y_1)}\right)^{-1+t}\left(1 - e^{-y_1}\right)^{-1+t}$$

$$\tag{20}$$

Now we have something precisely in the form suitable for the application of the Laplace transform[16], whose "Faltung theorem" is precisely in the form needed in Eq. (20). Using $\mathscr{L}$ as the symbol for the transform then

$$\mathscr{L}_s\left\{ G(y, nt) \right\} E_0 = (tB(s, t))^n \tag{21}$$

where s is the transform variable and B is the usual Beta function[17] which arises because it is the Laplace transform of the kernel. Now letting $n \to \infty$, keeping $nt = T$ fixed gives

$$(tB(s, t))^n \to e^{-(\gamma + \psi(s))T}, \tag{22}$$

where[17] ψ is the logarithmic derivative of the Γ function and γ the Euler constant $\gamma = 0.57 \ldots$ Inverting Eq. (21) gives us the final answer

$$G(y, T) = \frac{e^{-\gamma T}}{E_0} \mathscr{L}^{-1}\left\{ e^{-\psi(x)T} \right\} \tag{23}$$

with $y = \ln E_0/E$.

If we express $\mathscr{L}^{-1}$ in terms of the usual contour integral, then this agrees with an answer that can be arrived at by the diffusion equation[18]

$$\frac{dN}{dE} = \frac{1}{E_0} e^{-\gamma T} \int_{a-i\infty}^{a+i\infty} e^{sy} e^{-\psi(s)T} ds \qquad (24)$$

It may be verified by direct integration that Eq. (24) satisfies the normali-
zation property

$$\int_0^{E_0} \frac{dN}{dE} dE = \int_0^{E_0} G(E, E_0 T) dE = 1 \qquad . \qquad (25)$$

and the necessary "group" property

$$G(E, E_0, T) = \int_0^{E_0} dE' G(E, E', T-t) G(E', E_0, t) .$$

Footnotes

1. R. P. Feynman, Phys. Rev. Letters 23, 1415 (1969), and in High Energy Collisions, Third International Conference at Stony Brook, New York, edited by J. A. Cole et al., (Gordon and Breach, New York, 1969).

2. W. Heisenberg, Cosmic Radiation (Dover, New York, 1946), p. 124.

3. F. Bloch and A. Nordsieck, Phys. Rev. 52, 54 (1937).

4. L. Stodolsky, SLAC-PUB-864.

5. J. V. Allaby et al., CERN Report No. 70-12 (to be published).

6. Report of J. C. Sens at the Oxford Conference, April 1972.

7. See, for example, D. R. Yennie, in Brandeis Summer Institute 1962 Lectures in Theoretical Physics: Elementary Particle Physics and Field Theory, edited by K. W. Ford (Benjamin, New York, 1963), Vol. I. The case of the infinite range unscreened Coulomb scattering is an exception to the general statement.

8. See R. P. Feynman, reference 1, or E. Fermi, Nuclear Physics (University of Chicago Press, Chicago, Illinois, 1950); p. 53.

9. L. Caneschi and A. Schwimmer, Phys. Rev. D3, 1588 (1971), discuss, for example, the constraints put on a Poisson-like distribution by iso-spin in a multiperipheral model and find that the results are not greatly altered.

10. B. Rossi, High-Energy Particles, Prentice-Hall, New York.

11. If the theory is consistent, the experimental value of c can be found either from the multiplicity law or read off from the value of the "plateau" in $\omega \, dn/d\omega$. It is conceivable, however, that c could vary with energy. Then Eq. (15) would still be good with c taken from $\omega \, dn/d\omega$. Data on the value of c from the total multiplicity is given by

L. W. Jones et al., Phys. Rev. Letters 25, 1679 (1970) giving c ~ 1.
Data from Serpukhov taken with the Mirabelle bubble chamber and
presented at the Oxford Conference tends to suggest a larger value if
the logarithmic multiplicity law is adopted. Data on $\omega \, dn/\omega$ from the
ISR, also presented at Oxford, appear to give a c in this general
range.

12. L. Stodolsky, Phys. Rev. Letters 28, 60 (1972).

13. These speculations raise an interesting point when extended to targets
thicker than 1, i.e. to nuclei. If a proton is one radiation length,
is then carbon two radiation lengths? This question is given some
interest by the old observation in cosmic rays that the inelasticity
increases only slowly with nuclear size (see for example H. A.
Dobrotin at the 1970 Heceg-Novi School). It will be interesting in this
connection to see the energy loss spectrum of protons impinging on
nuclear targets from the high energy NAL proton beam.

14. W. Heitler, "The Quantum Theory of Radiation", Oxford
Section 37.

15. The effects of deviation from the simple infrared limit are discussed
in reference 10. A detailed discussion with many references is
L. W. Mo and Y. S. Tsai, Rev. Mod. Physics 41, 205 (1969).

16. D. V. Widder, "Advanced Calculus" (Prentice-Hall, New York),
Chapter XIII.

17. E. T. Whittacker and G. N. Watson, "Modern Analysis", Cambridge.

18. See reference 10, section 5.8, where the exact formula for A can be
interted in Eq. (16). I would like to thank J. Bjorken for pointing out
the relevance of Rossi's book in this connection.