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OFF-SHELL EXTENSION OF THE PARTIAL WAVE TRANSITION AMPLITUDE*

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ABSTRACT

The Marchenko approach to the inverse problem of scattering theory is transformed into a procedure for the calculation of the (half-) off-shell partial wave transition amplitude from the on-shell amplitude and certain bound state parameters.

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1. Introduction

In many theories for multiparticle systems, like the Faddeev equations for three-particle systems, the input information is not the explicit inter-particle potentials but rather the two particle transition amplitudes $t(\vec{p}', \vec{p}; E + i\epsilon)$. However, two-particle scattering experiments provide direct information only on the on-the-energy-shell part of these amplitudes, corresponding to $|\vec{p}'| = |\vec{p}| = k$, where $k^2/2\mu = E$ is the energy, while in the multiparticle theories the scattering amplitude is in general also needed for off-shell values of the momenta, and for negative energies.

The methods to extract information on the off-shell parts of t from on-shell parts in one way or another exploit the assumption that t corresponds to a unitary S-matrix or, more precisely, that the solutions to the Schrödinger equation corresponding to different energies form a complete set. The most well-known consequence of this so-called unitarity condition on t is of course that the on-shell transition amplitude itself in every partial wave can be parameterized in terms of a <u>real</u> function of energy, the phase-shift $\delta_\ell(k)$,

$$\mathrm{Im}\ t_\ell(k,k; E + i\epsilon) = -\frac{1}{\pi\mu k} \sin\delta_\ell(k) e^{i\delta_\ell(k)} \tag{1.1}$$

Other well-known consequences,[1] also applying to partial wave amplitudes, are that the amplitude can be expressed in closed form in terms of half-off-shell amplitudes,[2,3,4] e.g., for the imaginary part of t_ℓ

$$t_\ell(p', p; E + i\epsilon) = -\pi\mu k\, t_\ell(p', k; E + i\epsilon)\, t_\ell^*(k, p; E + i\epsilon) \tag{1.2}$$

and, moreover, that only the modulus of the half-off-shell amplitude depends on the off-shell momentum, leaving it with the same phase-factor as the corresponding on-shell amplitude,

$$t_\ell(p\ ,k;\ E + i\epsilon) = f_\ell(p\ ,k)\ t_\ell(k,k;\ E + i\epsilon) \tag{1.3}$$

where the half-off-shell factor $f_\ell(p\ ,k)$ is real. Thus, in every partial wave, the completely off-shell amplitude can be parametrized in terms of the two real functions $\delta_\ell(k)$ and $f_\ell(p,k)$.

The remaining general restrictions on t due to the unitarity condition[3] are less transparent, since they are expressed in the form of an integral relation which is quadratic in the half-off-shell amplitude. However, in the important special case when the transition amplitude corresponds to an interaction potential which is diagonal in configuration space (a condition that excludes, for instance, separable interactions) the situation is considerably simplified. From the solutions to the inverse problem of scattering theory,[5] it is known that in this case the quadratic unitarity condition can be reduced to a linear integral equation, through which an in principle unique potential can be deduced using on-shell and bound state properties of the two-particle system. But once the potential is known, the off-shell extension of the transition amplitude is obtainable as the solution to the Lippmann-Schwinger equation. In this way also the off-shell amplitude is uniquely determined by the phase-shift together with, in the case of a partial wave with bound states, the bound state energies and normalizations.

The procedure just outlined can obviously be simplified if the potential itself is of less interest and the main objective is to find the off-shell extension of a given on-shell amplitude. In the present paper, such a simplified procedure is developed, through which the half-off-shell factor $f_\ell(p,k)$ can be obtained from the phase shift and the bound state parameters, and in which the intermediate step of actually calculating the potential has been eliminated. The basic ingredient is the Marchenko integral equation from the theory of the inverse scattering problem,[5,6] but rather than considering its solution in configuration space, as in the conventional discussion of the equation, it is shown that a certain momentum space transform of the solution is closely related to the half-off-shell factor. After this observation, what remains is essentially to develope methods to solve the Marchenko equation that are suitable for the subsequent calculation of the half-off-shell factor. In this paper, the iterative solution will be discussed in some detail, but a more general method based on the Fredholm solution will also be outlined. Finally, it is known that for Bargmann $S_\ell(k)$-matrices the Marchenko equation has a separable kernel and hence a closed form solution. Through a calculation closely related to the Schmidt process it will be indicated how this fact can be used to transform the Marchenko equation into a similar equation for which the above methods of solution are expected to be more rapidly converging.

2. The Half-Off-Shell Factor and the Marchenko Theory

The system under consideration is two nonrelativistic particles interacting via the potential $v_\ell(r)$, and the discussion will be restricted to an uncoupled partial wave in which, for simplicity, there is at most one bound state. Let $f^{\pm}(k;r)$ be the solutions to the Schrödinger equation

$$\left[-\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}+\frac{\ell(\ell+1)}{r^2}+2\mu\, v_\ell(r)\right] U_\ell(k;r) = k^2\, U_\ell(k;r) \tag{2.1}$$

that satisfies the boundary condition at infinity

$$\lim_{r\to\infty} f_\ell^{(\pm)}(k;r)/h_\ell^{(\pm)}(kr) = 1 \tag{2.2}$$

$h_\ell^{(\pm)}(kr) = n_\ell(kr) \pm i\, j_\ell(kr)$ are spherical Bessel functions in the notation of Messiah.[7] The Marchenko treatment of the inverse problem of scattering theory is based on the observation that the solution $f_\ell^{(\pm)}(k;r)$ is related to the free particle solution $h_\ell^{(\pm)}(kr)$ via a real Volterra kernel function $A_\ell(r',r)$,[5]

$$f_\ell^{(\pm)}(k;r') = h_\ell^{(\pm)}(kr') + \frac{1}{r'} \int_{r'}^{\infty} r''dr''\, A_\ell(r',r'')\, h_\ell^{(\pm)}(kr'') \tag{2.3}$$

In order to establish a relation between $A_\ell(r',r'')$ and the off-shell transition amplitude, it is convenient to introduce the outgoing wave scattering solution to the Schrödinger equation (2.1),

$$\psi_\ell^{(+)}(k;r) = \sqrt{\frac{2}{\pi}}\, e^{i\delta_\ell(k)}\, \mathrm{Im}\left[f_\ell^{(+)}(k;r)\, e^{i\delta_\ell(k)}\right] \tag{2.4}$$

where the normalization is such that

$$\int_0^{\infty} r^2 dr\, \psi_\ell^{(+)*}(p';r)\, \psi_\ell^{(+)}(p;r) = \frac{1}{p^2}\, \delta(p'-p) \tag{2.5}$$

The momentum space transform of $\psi_\ell^{(+)}(k;r)$ is related to the transition amplitude through

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} r^2 dr\, j_\ell(pr)\, \psi_\ell^{(+)}(k;r) = \frac{1}{p^2}\, \delta(p-k) - \frac{2\mu}{p^2-k^2-i\epsilon}\, t_\ell(p,k;\, E+i\epsilon) \tag{2.6}$$

Replacing $f_\ell^{(+)}(k;r)$ in (2.4) by the expression (2.3), and using the fact that

$$\int_0^{\infty} r^2 dr\, j_\ell(pr)\, h_\ell^{(+)}(kr) = \frac{p^\ell}{k^{\ell+1}}\, \frac{1}{p^2-k^2-i\epsilon} \tag{2.7}$$

it is straightforward to show that

$$f_\ell(p,k) = \left(\frac{p}{k}\right)^\ell + (p^2 - k^2)\ \frac{k}{\sin\delta_\ell(k)} \int_0^\infty r\,dr \int_0^\infty r'dr'\, j_\ell(pr) A_\ell(r,r') \mathrm{Im}\left[h_\ell^{(+)}(kr')e^{i\delta_\ell(k)}\right] \tag{2.8}$$

where $f_\ell(p,k)$ is the half-off-shell factor introduced in section 1.

For partial waves with $\ell > 0$, the Marchenko theory as introduced above has some unsatisfactory features related to the divergence of $f_\ell^{(+)}(k;r)$ and $h_\ell^{(+)}(kr)$ at the origin. For example, if $f_\ell^{(+)}(k;r)$ from (2.3) is used in (2.4), $\psi_\ell^{(+)}(k;r)$ on the left hand side is proportional to r^ℓ for small r, while the two terms on the right hand side individually diverge as $r^{-\ell-1}$. The same difficulty shows up in (2.8) where the left hand side is finite for $p \to \infty$, while the two terms on the right hand side diverge in this limit. In order to overcome this problem, consider a solution $h_\ell^{R(\pm)}(k;r)$ to the equation (2.1) with the boundary condition (2.2) and a potential

$$v_\ell(r) = -\frac{\ell(\ell+1)}{2\mu r^2}\ \theta(R - r) \tag{2.9}$$

where $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$. The behavior of $v_\ell(r)$ for $r < R$ is rather arbitrary. One only requires that the Schrödinger equation in this region is reducible to an effective s-wave equation for which the irregular solution diverges no worse than r^{-1} at the origin. Another, equally simple choice for the potential would be to take

$$v_\ell(r) = \frac{\ell(\ell+1)}{2\mu}\left(\frac{1}{R^2} - \frac{1}{r^2}\right)\ \theta(R - r) \tag{2.9'}$$

so that the potential is continuous at $r = R$.

Assuming now that $h_\ell^{R(\pm)}(k;r)$ has a representation analogous to (2.3),

$$h_\ell^{R(\pm)}(k;r') = h_\ell^{(\pm)}(kr') + \frac{1}{r'} \int_{r'}^\infty r''dr''\, A_\ell^R(r',r'')\, h_\ell^{(\pm)}(kr''), \tag{2.10}$$

this expression can be combined with (2.3) to give

$$f_\ell^{(\pm)}(k;r') = h_\ell^{R(\pm)}(k;r') + \frac{1}{r'} \int_{r'}^{\infty} r''dr'' \, B_\ell(r',r'') h_\ell^{R(\pm)}(k;r'') \tag{2.11}$$

where B is again a Volterra kernel. In operator notation, $1 + B = (1+A)(1+A^R)^{-1}$, but if A^R is a Volterra operator, so is B. If this representation is introduced in (2.4), the two terms on the right hand side only diverge as r^{-1} at the origin, as in the $\ell = 0$ case, and this divergence is always compensated for by the weight factor r^2 appearing in all integrals. The relation between B and the half-off-shell factor is obtained in the same way as was equation (2.8):

$$f_\ell(p,k) = \left(\frac{p}{k}\right)^\ell + (p^2 - k^2) \frac{k}{\sin\delta_\ell(k)} \int_0^R r^2 dr \, \mathrm{Im}\left[j_\ell(pr) \; h_\ell^{R(+)}(k;r) - h_\ell^{(+)}(kr) \; e^{i\delta_\ell(k)} \right|$$

$$+ \int_0^\infty rdr \int_r^\infty r'dr' \, j_\ell(pr) B_\ell(r,r') \, \mathrm{Im} \; h_\ell^{R(+)}(k;r') \, e^{i\delta_\ell(k)} \Big| \tag{2.12}$$

Here, the integral $\int_0^R r^2 dr(\cdots)$ can be evaluated exactly, and it is easy to verify that it contains a term that exactly cancels the $(p/k)^\ell$ term in (2.12), as expected. By construction, the half-off-shell factor is independent of the parameter R, and the expression (2.12) depends on R only to the extent that to different R-values correspond different kernels $B_\ell(r,r')$.

3. The Marchenko Equation and Its Iterative Solution

Because of the relation (2.12) between the Marchenko Function $B_\ell(r',r)$ and the half-off-shell factor, the problem of computing the half-off-shell continuation of a given on-shell amplitude is equivalent to the determination of $B_\ell(r',r)$, or rather its transform as given in (2.12), from the phase shifts and bound state parameters only. This is exactly what the Marchenko theory amounts to, and our goal is a reformulation of this theory such that the calculation of the half-off-shell factor according to (2.12) becomes simple and straightforward.

As was mentioned in the introduction, one main ingredient in the Marchenko theory is the completeness relation for the regular solutions (2.4) to the Schrödinger equation,

$$\int_0^\infty k^2 dk\, \psi_\ell^{(+)}(k;r')\, \psi_\ell^{(+)*}(k;r) + C\psi_\ell^{B}(r')\,\psi_\ell^{B*}(r) = \frac{1}{r^2}\,\delta(r'-r) \tag{3.1}$$

Here $\psi_\ell^B(r)$ is the bound state wave function corresponding to a binding energy $B > 0$, and C is a normalization constant. $\psi_\ell^B(r)$ is related to $B_\ell(r',r)$ through the formula (2.11) with $k = i\sqrt{2\mu B} \equiv i\kappa$,

$$\psi_\ell^B(r') = h_\ell^{R(+)}(i\kappa;\ r') + \frac{1}{r'}\int_{r'}^\infty r''dr''\, B_\ell(r',r'')h_\ell^{R(+)}(i\kappa;\ r'') \tag{3.2}$$

Let B be the operator corresponding to the kernel $r^{-1}r'^{-1}\, B_\ell(r',r)\ \theta(r-r')$. As an operator relation, the completeness relation (3.1) reads

$$(1+B)(1-F)(1+B^+) = 1 \tag{3.3}$$

where the kernel of the operator F is

$$\frac{1}{r'r}\,F_\ell(r',r) = \frac{1}{\pi}\int_0^\infty k^2 dk\,\mathrm{Re}\left[h_\ell^{R(+)}(k;r')\,h_\ell^{R(+)}(k;r)\,\left(e^{2i\delta_\ell(k)}-1\right)\right]$$

$$- C\,h_\ell^{R(+)}(i\kappa;r')\,h_\ell^{R(+)*}(i\kappa;r) \tag{3.4}$$

After multiplication with $(1+B)^{-1} = 1+\widetilde{B}$, equation (3.3) reads

$$B_\ell^+(r',r)\ \theta(r'-r) = F_\ell(r',r) + \int_r^\infty dr''\ F_\ell(r',r'')B_\ell^+(r'',r) + \widetilde{B}_\ell(r',r)\theta(r-r') \tag{3.5}$$

$$= F_\ell(r',r) + \int_r^\infty dr''F_\ell(r',r'')\,B_\ell^+(r'',r),\quad r'>r \tag{3.6}$$

Equation (3.6) can now for every fixed r be considered as a linear integral equation for $B_\ell^+(r',r)$ in terms of $F_\ell(r',r)$, where $F_\ell(r',r)$ only depends on the phaseshift $\delta_\ell(k)$ and the bound state parameters κ and C in (3.4). In the following, equation (3.6) will be referred to as the (slightly generalized) Marchenko equation.

When considering methods to solve equation (3.6) it should be kept in mind that it is not really $B_\ell^+(r',r)$ but rather the expression (2.12) that is of primary interest. If the half-off-shell function were to be calculated with the help of equation (3.6) as it stands, the first step would be to transform the momentum space data into the configuration space function $F_\ell(r',r)$. The next step would be to solve the equation (3.6) for $B_\ell(r',r)$, and the last step to transform the solution into the momentum space half-off-shell factor $f_\ell(p,k)$. Since in any application of the theory, the on-shell data are known with limited accuracy and in a finite energy range, transformations back and forth between momentum and configuration space should clearly be avoided.[8] The straightforward way to achieve that is to try to convert equation (3.6) into a momentum space Marchenko equation. The result is unfortunately a rather complicated integral equation in two variables with a singular kernel.[9]

The approach to be followed here will be to introduce formal, configuration space solutions to the Marchenko equation (3.6) in the expression (2.12) for the half-off-shell factor, and try to rewrite the result in such a way that all transformations to configuration space are eliminated. For the iterative solution to the Marchenko equation this approach amounts to interchanging order of integration in the expression

$$\int_0^\infty r dr \int_r^\infty r' dr' \; \mathrm{Im}\left[h_\ell^{R(+)}(k;r')e^{i\delta_\ell(k)}\right] B_\ell^+(r',r) j_\ell(pr)$$

$$= \sum_{n=1}^\infty \int_0^\infty r dr \int_r^\infty r' dr' \int_r^\infty dr'' \cdots dr^n \; \mathrm{Im}\left[h_\ell^{R(+)}(k;r')e^{i\delta_\ell(k)}\right] F_\ell(r',r'') \cdots F_\ell(r^n, r) j_\ell(pr) \qquad (3.7)$$

so that all r-integrations are carried out before the k-integrations in the $F_\ell(r',r)$:s. At least in the $\ell = 0$ case this is easily done, as will be shown in detail in the next section. In section 5 the usefulness from numerical point of the resulting expression is demonstrated for the test case of an s-wave spherical well interaction with no bound state. The results in section 7 indicate that this straightforward iterative series for the half-off-shell factor might not converge if the interaction is strong enough to support bound states in the partial wave of interest.

4. The S-Wave Iterative Solution

For the lowest partial wave $\ell = 0$, the change in the order of integrations in (3.7) is particularly simple. Consider the kernel $F_0(r',r)$,

$$F_0(r',r) = \frac{1}{\pi} \int_0^\infty dk \; \mathrm{Re}\left[e^{ik(r'+r)} (S_0(k) - 1)\right] \qquad (4.1)$$

where $S_0(k) = e^{2i\delta_0(k)}$ is the S-matrix in the $\ell = 0$ partial wave, and where it has been assumed that there is no bound state. Since $S_\ell(-k) = S_\ell^*(k)$, the region of integration can be extended to $(-\infty, \infty)$, so that

$$F_0(r',r) = \frac{1}{2\pi} \int_{-\infty}^\infty dk \; e^{ik(r'+r)} (S_0(k) - 1) \, . \qquad (4.2)$$

The imaginary contributions from the integral cancel. All r-integrations in (3.7) can now be carried out with the result for the half-off-shell factor

$$f_0(p,k) = 1 + (p^2 - k^2)\ \frac{1}{2p}\ \frac{1}{\sin\delta_0(k)} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} dq' \cdots dq^n\ \frac{1}{(2\pi)^n}$$

$$\times\ \mathrm{Im}\left\{ e^{i\delta_0(k)}\ \frac{i}{k+q^n+i\epsilon}\ (S_0(q^n)-1)\ \frac{i}{q^n+q^{n-1}+i\epsilon}\ (S_0(q^{n-1})-1) \cdots (S_0(q')-1)\right.$$

$$\times \left[\frac{1}{k+2q'+\cdots+2q^n+p+i\epsilon} - \frac{1}{k+2q'+\cdots+2q^n-p+i\epsilon}\right] \tag{4.3}$$

For the evaluation of the q^ν-integral, $\nu = 1,2,\ldots,n$, the modification of the integrand $(q^{n+1} \equiv k)$

$$S_0(q^\nu) - 1 \rightarrow S_0(q^\nu) - e^{-2i\delta_0(q^{\nu+1})} \tag{4.4}$$

does not change the value of the integral but eliminates the $(q^{\nu+1} + q^\nu + i\epsilon)^{-1}$ singularily, so that

$$f_0(p,k) = 1 + (p^2 - k^2)\ \frac{1}{4}\ \frac{1}{\sin\delta_0(k)} \sum_{n=1}^{\infty} \frac{\Delta_n(k, \frac{k+p}{2}) - \Delta_n(k, \frac{k-p}{2})}{p} \tag{4.5}$$

where

$$\Delta_n(k,q) = -\frac{1}{\pi} \int_{-\infty}^{\infty} dq^n\ \frac{\sin(\delta_0(k) + \delta_0(q^n))}{k+q^n}\ \Delta_{n-1}(q^n, q^n + q) \tag{4.6}$$

$$\Delta_1(k,q) = -\frac{1}{\pi} \int_{-\infty}^{\infty} dq'\ \frac{\sin(\delta_0(k) + \delta_0(q'))}{k+q'}\ \mathrm{Im}\left[e^{i\delta_0(q')}\ \frac{1}{q+q'+i\epsilon}\right] \tag{4.7}$$

These expressions are particularly suited for the numerical calculation of $f_0(p,k)$ from $\delta_0(k)$, as will be discussed in the next section.

5. A Numerical Example: The Spherical Well

As a first test of the usefulness of the off-shell extension formula (4.5-7) from numerical point of view, the s-wave off-shell factor corresponding to a spherical well of range $a = 2$ Fermi and depth $V_0 = 20$ MeV has been calculated. With this choice of parameters, there is no bound state, and the s-wave phase shift is similar to the s-wave singlet n-p phaseshift at low momenta (figure 1).

The functions $\Delta_n(k,q)$ of (4.6-7) have been computed successively on a 24×24 mesh (note that $\Delta_n(k,q) = \Delta_n(-k, -q)$) using 24 point Gaussian quadrature. The values of $\Delta_{n-1}(q^n, q^n + q)$ on the right hand side of (4.6) were obtained by means of linear interpolation in the $\Delta_{n-1}(k,q)$ matrix, and const/q extrapolation outside this matrix. $\Delta_1(k,q)$ was computed in a more careful manner in order to account properly both for the singularity at $q' + q = 0$ and for the rapid variation of the integrand in the neighborhood of $q' = 0$. The singularity was first shifted to the origin, and the δ-function part of it was taken into account explicitly. The $\int_{-\infty}^{0}$ and $\int_{0}^{\infty}$ pieces of the remaining principal part integral was then computed individually using identical 24 point Gaussian quadrature. Figure 2 shows the exact half-off-shell factor $f_0(p,k)$ for $k = 1.0$ Fermi^{-1} together with the result obtained from the formula (4.5-7) and interpolation in the matrix $\sum_{n=1}^{N} \Delta_n(k,q)$ for $N = 5$. The discrepancy at high momenta is mainly due to the fact that only 4 of the 24 meshpoints are in the region $|p| > 2.0$ Fermi^{-1}.

With the numerical calculations organized as outlined above, the computation of Δ_n from Δ_{n-1} is neither harder nor more time consuming than the computation of Δ_2 from Δ_1, and despite the fact that Δ_n is in principle an n-dimensional integral, the computer time required to calculate N terms in the series (4.5) is just proportional to N. This means that the expansion (4.5-7) is useful from a numerical point of view even when the rate of convergence is rather low.

6. The Fredholm Solution

In this section a more general method to find the half-off-shell continuation of the on-shell amplitude will be outlined. It is based on the Fredholm solution to the Marchenko equation (3.6)[10]

$$B_\ell^+(r',r) = Y(r',r;r)/\Delta(r) \tag{6.1}$$

where r is treated as a parameter, and where

$$Y(r',r;r) = \sum_{n=0}^{\infty} Y_n(r',r;r) \tag{6.2}$$

$$\Delta(r) \equiv 1 + \Delta'(r) = \sum_{n=0}^{\infty} \Delta_n(r) \tag{6.3}$$

$$Y_n(r',r'';r) = \int_r^\infty d\rho \quad Y_{n-1}(r',\rho;\ r)\,F_\ell(\rho,\ \ r'') + F_\ell(r',r'')\Delta_n(r),$$

$$r' \geq r,\quad r'' \geq r \tag{6.4}$$

$$\Delta_n(r) = -\frac{1}{n}\int_r^\infty dr'\ Y_{n-1}(r',r';r) \tag{6.5}$$

$$Y_0(r',r'';r) = F_\ell(r',r''),\quad r' \geq r,\ r'' \geq r \tag{6.6}$$

$$\Delta_0(r) = 1 \tag{6.7}$$

The index ℓ has been suppressed on all Y:s and Δ:s. For the calculation of the half-off-shell factor according to equation (2.12), it is suitable to rewrite equation (6.1) as

$$B_\ell^+(r',r) = Y(r',r;r) - B_\ell^+(r',r)\,\Delta'(r) \tag{6.8}$$

and introduce a notation for the transform of $B_\ell^+(r',r)$ that appears in (2.12)

$$B_\ell(k,p) = pk \int_0^\infty rdr \int_r^\infty r'dr' \ \mathrm{Im}\left[h_\ell^{R(+)}(kr')e^{i\delta_\ell(k)}\right] B_\ell^+(r',r)\, j_\ell(pr) \tag{6.9}$$

In terms of $B_\ell(k,p)$, the relation (6.8) is an integral equation

$$B_\ell(k,p) = pk \int_0^\infty rdr \int_r^\infty r'dr' \ \mathrm{Im}\left[h_\ell^{R(+)}(kr')e^{i\delta_\ell(k)}\right] Y(r',r;\, r)\, j_\ell(pr) \quad (6.9)$$

$$- \int_{-\infty}^{\infty} dq\, B_\ell(k,q)\, \Delta'(q,p) \tag{6.10}$$

with the kernel

$$\Delta'(q,p) \equiv qp \ \frac{1}{\pi} \int_0^\infty r^2 dr\, j_\ell(qr) j_\ell(pr) \Delta'(r) \tag{6.11}$$

If in equation (6.10) the r-integrations in the inhomogeneous term and in the kernel are carried out before the k-integrations in the $F_\ell(r',r)$-factors, in the same manner as in the expression (3.7) for the iterative method, all transforms back and forth between momentum and configuration space are eliminated. The final step of solving equation (6.10) once the inhomogeneous term and the kernel have been calculated should be straightforward.

The $\ell = 0$ case is again particularly simple, and the calculations can be carried through in almost the same way as in section 4. Here, however, the procedure is expected to work also if bound states are present.

7. The Schmidt Process

As an introduction to the use of the Schmidt process[10] when solving the Marchenko equation, the model problem of finding the half-off-shell factor corresponding to a Bargmann type s-wave S-matrix will be studied. In this case not only can the half-off-shell factor be written down in closed form but all the integrations in the expansion (3.7) can also be carried out. It is then possible to estimate the rate of convergence in the expansion.

Let $S_0(k)$ be a Bargmann type S-matrix,[11]

$$S_0^B(k) = R(-k)/R(k), \qquad R(k) = (k-i\kappa)/(k+ib) \tag{7.1}$$

If $\kappa > 0$, this S-matrix corresponds to a bound state pole at $k = i\kappa$, an additional pole at $k = ib$, $b > \kappa > 0$ (to represent the dynamical cut), and no other singularities for Im $k \geq 0$. If $\kappa < 0$, there is no bound state, and only the "dynamical cut" pole remains in Im $k \geq 0$.

If there is a bound state, it is further assumed that the normalization constant C in $F_0(r',r)$,

$$F_0(r',r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ik(r'+r)} (S_0^B(k)-1) - \frac{C}{\kappa^2} e^{-\kappa(r'+r)} \tag{7.2}$$

is just $2\kappa^3 \Gamma$, $\Gamma = (b+\kappa)/(b-\kappa)$, so that the contribution from the bound state pole in a contour integration evaluation of $F_0(r',r)$ exactly cancles the explicit bound state term. This choice of C corresponds to a Bargmann potential with a range $\sim b^{-1}$, while any other choice would correspond to a potential with a longer range, $\sim \kappa^{-1}$.[11] In this way the kernel $F_0(r',r)$ of the Marchenko equation has a form that is independent of the sign of κ,

$$F_0(r',r) = -2b\,\Gamma\, e^{-br'}\, e^{-br} \tag{7.3}$$

Moreover, $F_0(r',r)$ is separable in r' and r, so that the Marchenko equation has a closed form solution

$$B_0^+(r',r) = -2b\Gamma \frac{e^{-br'}\, e^{-br}}{1+\Gamma e^{-2br}}, \qquad r' > r \tag{7.4}$$

The half-off-shell factor can also be written in closed form:

$$f_0(p,k) = 1 - (p^2 - k^2)\,\frac{1}{p}\,\frac{1}{\sin\delta_0(k)}\,\mathrm{Im}\left\{e^{i\delta_0(k)}\,\frac{2ib}{k+ib}\int_0^\infty dr\,\frac{\Gamma e^{-2br}}{1+\Gamma e^{-2br}}\,e^{ikr}\sin pr\right\}. \tag{7.5}$$

On the other hand, the iterative expansion for $f_0(p,k)$ is obtained from (2.12) and (3.7)

$$f_0(p,k) = 1 + (p^2 - k^2)\,\frac{1}{p}\,\frac{1}{\sin\delta_0(k)}\,\mathrm{Im}\left\{e^{i\delta_0(k)}\,\frac{ib}{k+ib}\right.$$

$$\left.\times\sum_{n=1}^{\infty}(-1)^n\,\Gamma^n\left(\frac{1}{k+2inb+p} - \frac{1}{k+2inb-p}\right)\right\}. \tag{7.6}$$

It is easy to verify that (7.6) is the result that is obtained if the integral in (7.5) is evaluated with the help of the expansion

$$(1+\Gamma e^{-2br})^{-1} = \sum_{n=0}^{\infty}(-1)^n\,\Gamma^n\,e^{-2nbr} \tag{7.7}$$

The expansion (7.6) for the half-off-shell factor evidently converges when $|\Gamma| < 1$. In other words, when there is no bound state. This result suggests that the absence of bound states is the criteria for convergence of the iterative expansion also in the general case.

Observe that any Bargmann type S-matrix corresponds to a sum of separable terms for $F_\ell(r',r)$, so that the corresponding Marchenko equation has a closed form solution. This suggests that the Schmidt process can be used in a natural way to rearrange the original Marchenko equation into a form for which the iterative and Fredholm solutions are more rapidly convergent, and for which the iterative solution converges when the iterative solution to the original Marchenko equation diverges. As an example, let

$$F_0(r',r) = F_0^B(r',r) + F_0'(r',r) \tag{7.8}$$

where

$$F_0^B(r',r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(r'+r)}\left[S_0^B(k) - 1\right] - \frac{C''}{\kappa^2}\, e^{-\kappa(r'+r)}$$

$$= -2b\Gamma\, e^{-b(r'+r)} \tag{7.9}$$

$$F_0'(r',r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(r'+r)}\left[S_0(k) - S_0^B(k)\right] - \frac{C'}{\kappa^2}\, e^{-\kappa(r'+r)} \tag{7.10}$$

Here, $C'' = 2\kappa^3\Gamma$, and $C' = C - C''$ (if there is no bound state, $C = C' = C'' = 0$ and $\kappa < 0$ is a free parameter). In this way the kernel in the Marchenko equation has been split into a separable term and a non-separable remainder, and through a calculation similar to that familiar from the Schmidt process, the Marchenko equation is transformed into

$$B_0^+(r',r) = -2b\Gamma\,\frac{e^{-br'}e^{-br}}{1+\Gamma e^{-2br}} + \int_r^{\infty} d\rho' d\rho\left[\delta(r'-\rho') - 2b\Gamma\,\frac{e^{-br'}e^{-b\rho'}}{1+\Gamma e^{-2br}}\right] F_0'(\rho',\rho)$$

$$\times\left[\delta(\rho - r) + B_0^+(\rho, r)\right], \qquad r' > r \tag{7.11}$$

The iterative solution to this equation can now be used in equation (3.9) to generate a new series expansion for the half-off-shell factor. As in section 4, the r- and ρ-integrations can be carried out before the k-integrations in the $F_0'(r',r)$-factors, with the only complication that the final r-integration cannot be handled as neatly as before. If the parameters of $S_0^B(k)$ are suitably chosen, the resulting expansion should converge faster than the expansion (4.3), and it can also be expected to converge in some cases when (4.3) diverges.

The Fredholm solution to equation (7.11) can be treated as was the Fredholm solution to the Marchenko equation in section 6.

8. Summary

The problem of computing the off-shell continuation of a given on-shell, partial wave scattering amplitude has been discussed when the underlying interaction is diagonal in configuration space. It has been shown that the methods of the inverse problem of scattering theory can be recast in a form which allows the (in this case essentially unique) half-off-shell continuation of the amplitude to be calculated entirely in momentum space. This avoids the potentially troublesome fourier transforms of the experimental and hence imperfectly known $S_\ell(k)$-matrix. The procedure proposed here has been shown to be satisfactory from a numerical point of view in the simple but not entirely trivial case of an s-wave spherical well interaction, with parameters chosen to simulate the singlet n-p interaction. In more complicated cases, that is in the presence of bound states and/or for higher partial waves, the numerical calculations are still expected to be managable although less straightforward.

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References

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Figure Captions

1. The s-wave phase-shift $\delta_0(k)$ for a spherical well with range 2 fermi and depth 20 MeV. $\hbar = c = 2\mu = 1$.

2. The s-wave half-off-shell factor $f_0(p,k)$ for a spherical well with range 2 fermi and depth 20 MeV, for $k = 1$ Fermi^{-1}. The solid line is the exact value, and the dashed line is obtained from formula (4.5) with $n \leq 5$.

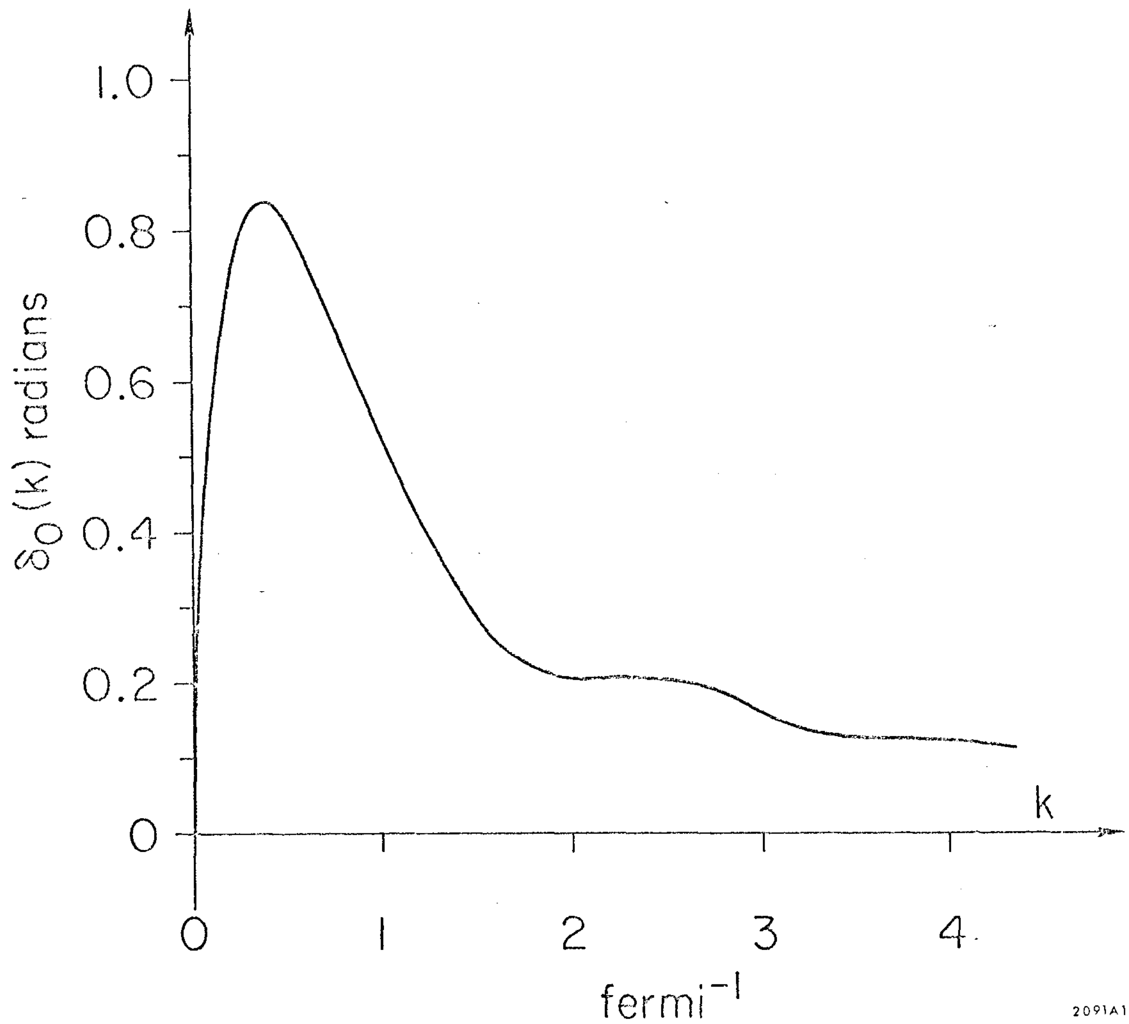

Fig. 1

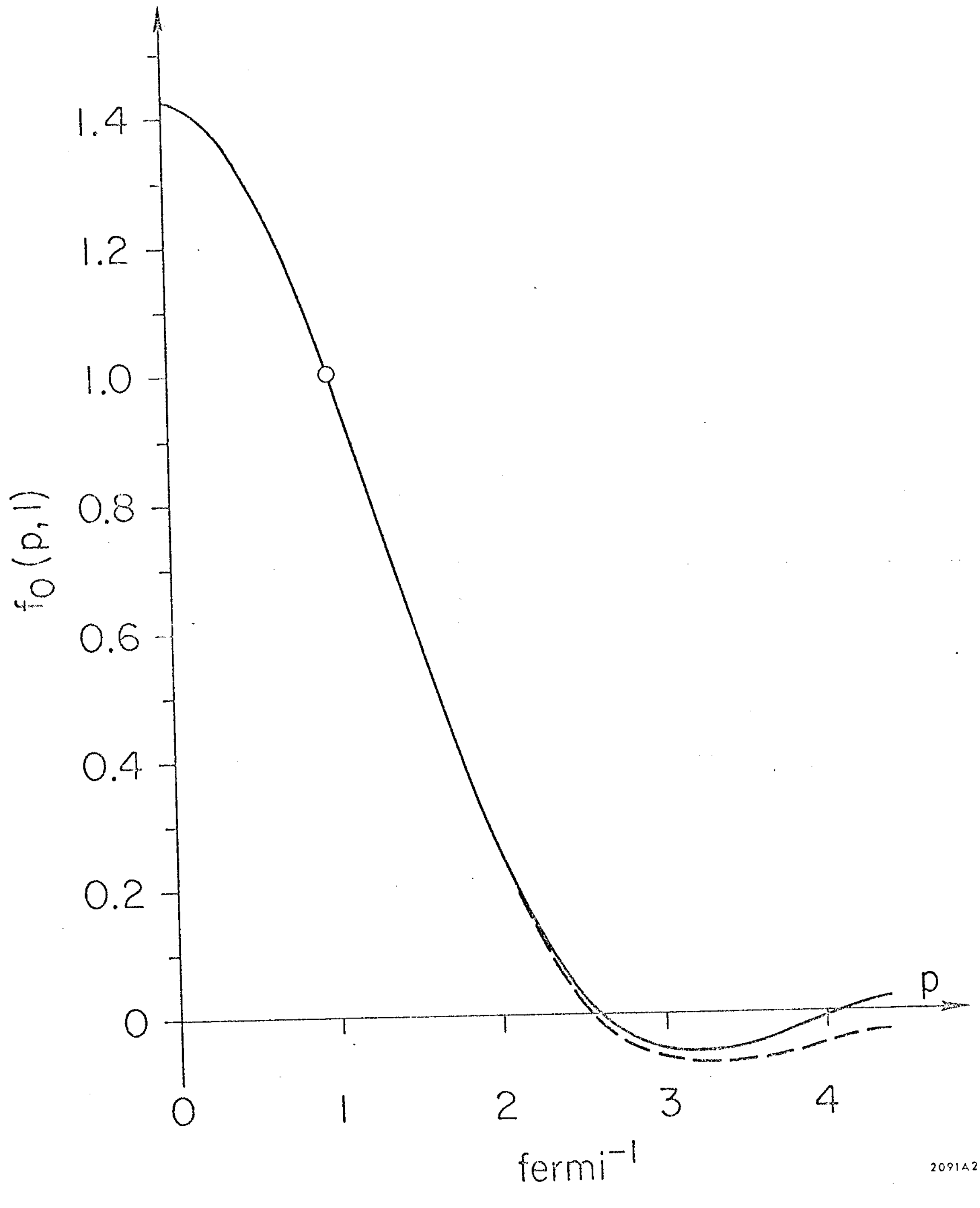

Fig. 2