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CAN QUARKS BE KEPT INSIDE? *

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Abstract

A self-consistent dynamical mechanism is suggested, which appears to be compatible with conventional ideas about relativistic field theory, and which prevents the appearance of the quanta of the field in outgoing or incoming scattering states.

A model calculation is provided which illustrates the mechanism. The model produces a spectrum of conventional hadron states which corresponds to an infinitely rising Regge trajectory. All the states are physical. When SU(3) internal quantum numbers are included, it is argued that the mechanism is stable only if the physical states have zero triality.

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I. Introduction

During the past decade, all the experimental evidence accumulated about hadrons has indicated that they are composite. The theoretical pictures which have been successful in accounting in some way for a range of phenomena have also incorporated a description of the particles which could be called a composite one. We list, very briefly, the outstanding ones which include this feature. We have first the S-matrix bootstrap mechanism.[1] In this context, it became clear that the particle states should be on Regge trajectories, just as bound states in the nonrelativistic Schrödinger equation. Next, the SU(6) ideas[2] were most consistently developed in terms of a nonrelativistic bound quark model.[3] The formulation of current algebra[4] is also most easily seen to be consistent if the currents are formed from quark field operators, with commutation relations which are like those of free fields. If this is so, then the hadron matrix elements should acquire their structure through that in the "states," that is, one should have composite hadrons. The asymptotic behavior of the elastic[5] and inelastic[6] hadron form factors has been qualitatively accounted for in a simple way in a model in which the photon interacts with a structureless virtual current inside the hadron. The asymptotic form is obtained by requiring the appropriate behavior of the bound-state wave function at short distances.

Finally, if the hadrons lie on infinitely rising Regge trajectories,[7,8] there is suggested a composite model of hadrons made from quarks moving in an infinitely deep potential well. The latter feature could also account for the absence of free quarks in the world.[9]

We therefore ask whether it is possible to construct a consistent relativistic model of a deep potential well. In this paper, we should like to suggest one possible mechanism which could accomplish this. Our aim is the development of a composite model of hadrons composed of quarks which cannot escape (be created) in the collisions of physical hadrons. This process will not be forbidden by internal symmetry properties or any kinematic restriction on the quark mass. Indeed, the model mechanism will be consistent with very light quarks such as those favored by quark modelists. The paper will be organized as follows: In Section II, we review well-known facts about infinitely deep wells in the Schrödinger equation. In Section III, we suggest how these features can be generalized to relativistic field theory. In Section IV, we apply the idea to a simple model, and in Section V, we complete the asymptotic spectrum in the model. In Section VI, we discuss the self-consistency of our mechanism. In Section VII, we discuss the SU(3) structure and indicate how the dynamics is stable if the ordinary hadrons have triality zero.

II. Nonrelativistic Wells with Oscillator Walls

In order to motivate the development of the relativistic picture of a deep well, we should like to review trivial and well-known facts about the nonrelativistic Schrödinger equation. The purpose will be solely to help to orient the reader when we discuss the real problem. The transition from the nonrelativistic context to the relativistic is often easiest if one works in momentum space nonrelativistically. Therefore, we write the Schrödinger equation for a particle interacting with a potential V,

$$\left(E - \frac{p^2}{2m}\right)\psi_E(p) = \int \frac{d^3q}{(2\pi)^3} V(p-q)\, \psi_E(q) \quad . \tag{2.1}$$

In the case of finite range forces, the potential V(k) is regular at zero momentum transfer, i.e., V(0) is finite. Consequently, no care is required in the integration of the right-hand side at $p = q$. However, since when $E > 0$, the right-hand side of (2.1), in general, has no desire to vanish when $p^2 = 2mE$, a pole is produced in $\psi_E(p)$ at $p^2 = 2mE$. The residue at the pole, that is, the right-hand side of (2.1), evaluated at $p^2 = 2mE$, is the scattering amplitude. This is the asymptotic amplitude of the coordinate space wave function at large distances from the potential. Since we must integrate over the pole to solve (2.1), boundary conditions are required to specify the integration path and to characterize the degeneracy of the eigenstate. These determine whether we have an incoming or outgoing particle amplitude associated with a given final or incident direction (or angular momentum).

However, when the potential at long distances is singular, V(k) diverges at k = 0. In this case, the arguments in the preceding paragraph must be modified. For simplicity, let us suppose that in coordinate space, V diverges like αr^2 as $r^2 \longrightarrow \infty$. We may then separate V into two parts:

$$V(r) = V_{SR}(r) + \alpha r^2 \quad , \tag{2.2}$$

where we assume that $V_{SR}(r)$ falls exponentially as $r \longrightarrow \infty$. At short distances, V and V_{SR} are identical. In this case, in momentum space,

$$V(k) = V_{SR}(k) - \alpha (2\pi)^3 \nabla_k^2 \, \delta^{(3)}(k) \tag{2.3}$$

where $V_{SR}(0)$ is finite. Equation (2.1) becomes

$$\left(E - \frac{p^2}{2m}\right)\psi_E(p) = -\alpha \nabla_p^2 \, \psi_E(p) + \int \frac{d^3q}{(2\pi)^3} V_{SR}(p-q)\, \psi_E(q) \quad . \tag{2.4}$$

We now see that there is no reason for a singularity to develop in $\psi_E(p)$ when $p^2 = 2mE$, since the right-hand side of (2.4) can adjust to the vanishing of the left-hand side. We find, instead, that since (2.4) is a differential equation in p, boundary conditions at $p = 0$ and $p = \infty$, the singular points of the operator ∇_p^2, are required for a solution. Near $p = 0$, the V_{SR} contribution in (2.4) is negligible but the momentum space centrifugal barrier contained in ∇_p^2 dominates. When we impose the boundary condition that ψ remains finite at $p = 0$, we eliminate one of the two solutions of (2.4). If we then trace this solution to $p = \infty$, we find in general an exploding (Gaussian) exponential solution, dominated by the singular part of ∇_p^2 and the kinetic energy. If we impose the finite boundary condition at $p = \infty$, it cannot be satisfied except for special values of E. This leads to the discrete spectrum of eigenvalues in the well. However, when E is so chosen, the actual form of the wave function as $p \longrightarrow \infty$ is governed by the second term in (2.4), the short range potential, because in this case the oscillator potential gives an exponentially small contribution. That is, the spectrum of states is fixed predominantly by the long range part of V while the actual form of the wave function as $p \rightarrow \infty$ is governed by the short range part of V.

We note, of course, that when $p^2 = 2mE$,

$$(E - p^2/2m)\,\psi_E(p) \equiv 0 \ .$$

Hence, there is no asymptotic free wave function in coordinate space; the transition matrix vanishes identically.

In the next section, we shall study how all of these ideas can be taken over directly in a relativistic context.

III. The Relativistic Well with Oscillator Walls

Let us consider a relativistic field theory model with q(x) standing for the quark field operator. Since our mechanism will be dynamical, let us suppress spin and internal quantum numbers for the moment. We shall discuss these in Section VII.

We shall suppose the dynamics is characterized by

$$(-\Box^2 + m^2)\, q(x) = I(x)\, q(x) \quad . \tag{3.1}$$

In (3.1), m stands for the physical mass of the quark. I(x) is an operator formed as a function of q(x), and the other fields to which the system is coupled. It will be taken as a local, relativistic operator, as is q(x). The ideas in this paper are meant to be completely conventional. Let us suppose that the eigenstates of the Hamiltonian of the system fall into two categories, "good" states $|G\rangle$ and "bad" states $|B\rangle$. (These will later be distinguished by internal quantum numbers.) The good states are the sorts of ordinary hadron states which are present in the world. The bad states, on the other hand, will contain ordinary hadrons and other things which do not exist in the world. In our model, it should be impossible to produce bad states through collisions of good states which are initially asymptotically separated. Thus, we shall try to insure that amplitudes like $\langle B, B' \,^{out}|G\rangle$ vanish. Here, B, B' are an asymptotically separated pair of bad states with a total "good" quantum number. The bad and good states will only be connected by the quark field operator q(x). We shall assume that

$$\langle B|q|G\rangle \neq 0$$

but

$$\langle G|q|G'\rangle \equiv 0 \quad \text{and} \quad \langle B|q|B'\rangle \equiv 0.$$

(The assumption that $\langle B|q|B'\rangle = 0$ will be modified in the more realistic model when we incorporate the internal symmetry explicitly.)

If we apply (3.1) to the calculation of

$$\langle B|q|G\rangle ,$$

we find the equation

$$(-u+m^2)\langle B|q|G\rangle = \langle B|Iq|G\rangle = \sum_{B'} \langle B|I|B'\rangle \quad \langle B'|q|G\rangle \tag{3.2}$$

where only $|B\rangle$ states occur as a consequence of the assumption that $\langle G'|q|G\rangle \equiv 0$. In (3.2), $u = -(B-G)^2$ is the virtual mass of the quark. In analogy with the situation in the nonrelativistic Schrödinger equation (2.1), we ask whether it is possible to have the vanishing of the left-hand side of (3.2) at the position of the physical quark mass compatible with the right-hand side. We conjecture that it will be if the operator I has matrix elements between bad states which are sufficiently singular so that the right-hand side involves differential operators in the variable u. This circumstance will be exactly analogous to the same situation in the nonrelativistic Schrödinger equation described in Section II. If we can accomplish this, then the matrix elements $\langle B, ^{out} q|G\rangle$ and $\langle B|G, ^{in} q\rangle$ will vanish identically. (They would be proportional to $(-u+m^2)\langle B|q|G\rangle|_{u=m^2} \equiv 0$.) Thus it will be impossible to generate free quarks or the bad eigenstates of the Hamiltonian by a process which involves an incoming good state. Naturally, the same considerations must also apply to the antiquark operator. For simplicity, in the naive model we assume total neutrality. We shall discuss the more realistic case which includes SU(3) in Section VII.

We now ask whether or not it is contradictory to assume that the operator I is singular enough between bad states so that the right-hand side of (3.2) contains a differential operator in u.

Clearly, the simplest possibility will result if the same channel is involved on the right-hand side of (3.2) as occurs on the left-hand side. Thus, we assume that the place where it is most reasonable to have a singularity is in the elastic matrix elements of I between bad states. Therefore, we suppose that $\langle B|I|B'\rangle$, when $m_B^2 = m_{B'}^2$ is so singular at $t = -(B-B')^2 = 0$ that the effect of the integration over B' in (3.2) produces a differential operator in u. Since, when $m_B = m_{B'}$, t is always negative in the integration over B', it becomes zero only when

$u' = -(B' - G)^2 \longrightarrow u$, that is, where the virtual quark scatters elastically from the "potential" I. Naturally, the singularity in the matrix element of I at $t = 0$ cannot be associated with a real threshold at $t = 0$, since zero mass particles can play no role in hadronic processes. Consequently, the singularity in I at $t = 0$ must be a consequence of compositeness, that is, it must be analogous to (but will not be) a so-called anomalous threshold. In our simple model, this compositeness will be associated with the fact that the bad states are composed of a virtual quark and good particles. As a consequence of the good particles being on infinitely rising Regge trajectories, the bad states will have singular form factors at $t = 0$ (see Section VI).

In summary, we conjecture that the role of the oscillator well in a relativistic theory is played by a set of states B, with singular (at $t = 0$) elastic matrix elements of the interaction I. We know of no work in relativistic quantum theory which says that such a singularity violates a sacred principle. That is, such a singularity is compatible with totally conventional ideas. The states with such singular form factors will then not appear in the real world, since a quark will be glued on them. This bound system will be a resulting physical state. It will be on an infinitely rising trajectory corresponding to the excitations of the quark in the well. In the next section, we shall study what some of the consequences of such an assumption will be, in the context of our simple model.

IV. A Simple Relativistic Model with Oscillator Walls

To illustrate the idea, we shall make a simple model calculation. We assume $q(x)$ is a spinless field and that we wish to solve (3.2) for those states G which are massive. We assume that these occur as a consequence of a $t = 0$

singularity in the elastic[10] matrix element,

$$\langle B|I|B'\rangle \tag{4.1}$$

where $m_B = m_{B'}$. We shall assume initially that there is only one spinless particle state $|B\rangle$.

We suppose that as $t \longrightarrow 0$, the matrix element (4.1) has the form,

$$\langle B|I|B'\rangle \approx 2\pi\gamma\, m_B^5 \lim_{\mu \to 0} \left(\frac{\partial}{\partial\mu}\right)^3 \frac{1}{\mu^2 - t} \tag{4.2}$$

$$= 4\pi^2 \gamma\, m_B^5 \frac{\partial}{\partial t}\, \delta''(\sqrt{-t}) \tag{4.3}$$

where $t = -(B-B')^2$. This form is motivated by the potential

$$r^2\, e^{-\mu r}/4\pi = -\left(\frac{\partial}{\partial\mu}\right)^3 \frac{e^{-\mu r}}{4\pi r}$$

where

$$\frac{e^{-\mu r}}{4\pi r} \longrightarrow \frac{1}{\mu^2 - t}$$

as a function of t.

In order to compute the resulting asymptotic spectrum, we omit all other matrix elements, since the mass of the asymptotic states should be dominated by the $t = 0$ singularity in I. We shall assume that the state $|G\rangle$ has spin J. Then the wave function $\langle B|q|G,J\rangle$ will have the form

$$\langle B|q|G,J\rangle = q^J(u)\, T^J_{\mu_1 \cdots \mu_J}(\hat{B}_G) \quad , \tag{4.4}$$

where $T^J_{\mu_1 \cdots\cdots \mu_J}(\hat{B}_G)$ is a totally symmetric traceless tensor of rank J formed using the unit vector

$$\hat{B}^\mu_G = \left(B^\mu + \frac{B \cdot G G^\mu}{m_G^2}\right) \frac{1}{\left\{-m_B^2 + \frac{(B \cdot G)^2}{m_G^2}\right\}^{1/2}} \quad . \tag{4.5}$$

Further, $G_{\mu_i} T^J_{\mu_1 \cdots\cdots \mu_i \cdots\cdots \mu_J} = 0$. We normalize T^J so that

$$T^J_{\mu_1 \cdots\cdots \mu_J}(\hat{B}_G) \cdot T^J_{\mu_1 \cdots\cdots \mu_J}(\hat{B}'_G) = P_J(\hat{B}_G \cdot \hat{B}'_G) \tag{4.6}$$

where $P_J(z)$ is the ordinary Legendre polynomial. We insert (4.4) into (3.2), written in the form:

$$(-u + m^2)\langle B|q|G\rangle = \int \frac{d^4B'}{(2\pi)^3}\, \theta(B'^0)\, \delta\left(B'^2 + m_B^2\right) \langle B|I|B'\rangle \langle B'|q|G\rangle \tag{4.7}$$

$$+ \text{ channels with } m_{B'} \neq m_B,$$

and then drop other channels and the matrix elements of I for $t \neq 0$. Thus we obtain with (4.3) for I,

$$(-u + m^2)\, q^J(u) = m_B^2 \gamma \left\{ \left[(m_G + m_B)^2 - u\right]\left[(m_G - m_B)^2 - u\right]\left(\frac{\partial}{\partial u}\right)^2 \right.$$
$$\left. -3\left(m_G^2 + m_B^2 - u\right)\frac{\partial}{\partial u} + \frac{3}{4} - \frac{4m_B^2 m_G^2 J(J+1)}{\left[(m_B + m_G)^2 - u\right]\left[(m_B - m_G)^2 - u\right]} \right\} q^J(u) \tag{4.8}$$

Since the right-hand side of (3.2) now contains a differential operator in u, $(-u+m^2)\langle B|q|G\rangle$ vanishes identically at $u = m^2$. We might say that the differential operator has made the free quark pole ineffectual. Therefore, $\langle B,^{out} q|G\rangle$ and $\langle B|\bar{q},^{in} G\rangle$ vanish identically. The quarks cannot get out.

The range of u relevant to the calculation of the matrix element $\langle B|q|G\rangle$ is

$$-\infty < u \leq (m_B - m_G)^2 \quad . \tag{4.9}$$

Another range of u is physical, namely,

$$(m_G + m_B)^2 \leq u < \infty \quad . \tag{4.10}$$

We shall postpone for the moment the discussion of the second "crossed" range.

The differential equation (4.8) has two regular singular points, $u = (m_G \pm m_B)^2$, and an irregular singular point at $u = \infty$. Nothing special occurs at the position of the quark mass $u = m^2$. As $u \longrightarrow -\infty$, the solutions of (4.8) become

$$e^{\pm \frac{2}{mB} \sqrt{-\frac{u}{\gamma}}} \quad .$$

We must choose the bounded solution,

$$e^{-\frac{2}{mB} \sqrt{-\frac{u}{\gamma}}} \quad .$$

If we integrate this form back to the singular point, $u = (m_B - m_G)^2$, we will find, in general, an unbounded function. We quantize the mass spectrum of the states G, so that q^J is finite at $u = (m_G - m_B)^2$. This is exactly analogous to the situation in the nonrelativistic Schrödinger equation. Naturally, we may also study (4.8) in the range (4.10). These solutions may be interpreted as the "crossed" matrix elements, $\langle 0|q|\bar{B}, G\rangle$, for example. In this domain, both asymptotic solutions,

$$e^{\pm \frac{2i}{mB}\sqrt{\frac{u}{\gamma}}} ,$$

are allowed. We must choose the linear combination which remains finite at the other regular singular point, $u = (m_G + m_B)^2$. No quantization is required since both solutions at the irregular singular point $u = \infty$ are allowed. However, the solution that we obtain in this fashion will not, in general, be the "crossed" version of the solution obtained in the other range (4.9). Crossing symmetry will, therefore, in general, be violated when the crossing involves the B and G channels.

V. Asymptotic Spectrum

To calculate the mass spectrum which results from the boundary conditions on q^J described above, it is convenient to use the dimensionless variable z,

$$2\, m_B m_G z = m_G^2 + m_B^2 - u \quad , \tag{5.1}$$

which runs over a range independent of m_G, namely, for the bound-state channel, $u \longrightarrow -\infty$, $1 \leq z \leq \infty$ and for the crossed channel, $-\infty \leq z \leq -1$. The differential equation (4.8) becomes

$$\left\{\left(\frac{m}{m_B}\right)^2 - 1 - \left(\frac{m_G}{m_B}\right)^2 + 2\,\frac{m_G}{m_B} z\right\} q^J = \gamma \left\{(z^2 - 1)\left(\frac{\partial}{\partial z}\right)^2 + 3z\,\frac{\partial}{\partial z} + \frac{3}{4} - \frac{J(J+1)}{z^2 - 1}\right\} q^J \quad . \tag{5.2}$$

If we let

$$q^J = \frac{1}{(z^2 - 1)^{3/4}}\, Q^J \quad ,$$

we eliminate the $\frac{\partial}{\partial z}$ term in (5.2) to obtain the equation,

$$\left\{\left(\frac{m}{m_B}\right)^2 - 1 - \left(\frac{m_G}{m_B}\right)^2 + 2\,\frac{m_G}{m_B} z\right\} Q^J = \gamma \left\{(z^2 - 1)\left(\frac{\partial}{\partial z}\right)^2 - \frac{J(J+1) - \frac{3}{4}}{z^2 - 1}\right\} Q^J \quad . \tag{5.3}$$

This differential equation has resisted being related to one of the standard ones in mathematical physics. However, since (5.3) can be reasonably expected to describe correctly only the massive states which are most sensitive to the $t = 0$ singularity in I, a W.K.B. quantization should give all the accuracy one could reasonably expect to make any sense.

The solution in the domain $1 \leq z \leq \infty$ is governed by an exponentially falling wave function at $z = \infty$ and a solution finite at $z = 1$. If we apply the standard W.K.B. method to join these, we find as a consequence the eigenvalue condition on m_G,

$$\left(n+\frac{1}{2}\right)\pi=\int_{z_-}^{z_+} dz\left[-\frac{J(J+1)-\frac{3}{4}}{\left(z^2-1\right)^2}+\frac{1}{\gamma}\left\{\frac{-\left(\frac{m}{m_B}\right)^2+1+\left(\frac{m_G}{m_B}\right)^2-2\,\frac{m_G}{m_B}\,z}{z^2-1}\right\}\right]^{1/2} \tag{5.4}$$

$z_\pm$ labels the turning points. This rule should give reliable results for the limiting cases, J large, n arbitrary, or n large, J arbitrary where arbitrary means large or small. In Regge terminology, $n = 0$ labels the leading trajectory, $n \geq 1$ labels the daughters. The case where both n and J are large, which will be important to obtain quantitative results in the next section, has not been dealt with so far. However, when n is fixed and J is asymptotic or when J is fixed and n is asymptotic, approximations may be made to the integrand of (5.4) which allow a simple analytic evaluation. The results[11] are

$$\frac{m_G^2}{m_B^2} = \sqrt{27\gamma}\, J\left(1+(3\gamma)^{\frac{1}{4}}\,\frac{n+\frac{1}{2}}{\sqrt{J}}\right) \qquad J \gg n \tag{5.5a}$$

and

$$\frac{m_G^2}{m_B^2} = \frac{\gamma \pi^2 n^2}{\left\{\log\left(\frac{8\sqrt{\gamma}\,\pi^2 n^2}{J}\right) + 2 + 0(\log\log n)\right\}^2} \quad . \qquad n \gg J > 1 \tag{5.5b}$$

It is encouraging to note that the leading Regge trajectory, as well as that of all the daughters, becomes linear as $J \longrightarrow \infty$. However, the linearity occurs in the asymptotic region when all the daughters merge with the mother.

If we study the other region, $z \leq -1$, which corresponds to the crossed channels $\langle 0|q|\bar{B}, G\rangle$ (or $\langle B, \bar{G}|q|0\rangle$), we see that the solution obtained for the region $z \geq 1$ is not relevant, since it is not finite at the singular point $z = -1$ in general. In this channel, we solve the differential equation with the finite solution at $z = -1$. The asymptotic form for large z is then of the form, $a \sin(\sqrt{-z}) + b \cos \sqrt{-z}$. Therefore, no quantization condition is required. The amplitude corresponds to an asymptotic state consisting of a bad particle and an incoming (or outgoing) physical hadron. There will be an infinite set of such bad states corresponding to all possible physical hadrons which result from the quantization in the negative z channel. Approximate wave functions may again be obtained with the W.K.B. method. This channel is relevant for the calculation of contributions to the absorptive part of the quark field propagator,

$$\langle 0|T\big(q(x)\,q(0)\big)|0\rangle \quad .$$

The asymptotic form of the absorptive part of this for large mass can be computed using the W.K.B. wave functions. It is, of course, natural to assume that the lowest mass bad particle has the quantum numbers of the quark, i.e., is the "physical" quark. This work has not yet been carried out. It is not yet clear of what utility the quark propagator would be.

VI. Self-Consistency

We shall now try to determine to what extent the ansatz (4.2) concerning the t = 0 singularity in the elastic matrix elements of I is self-consistent. Thus, we now will calculate the matrix elements of I. To begin, for simplicity, we suppose the interaction I is a contact interaction of the form

$$I = \lambda\left(q(x)\right)^2 + \text{counter terms}.$$

Let us first calculate the matrix elements of q^2 between the "good" hadron states,

$$\left\langle G\middle|q(0)^2\middle|G'\right\rangle = \sum_B \left\langle G\middle|q(0)\middle|B\right\rangle\left\langle B\middle|q(0)\middle|G'\right\rangle \tag{6.1}$$

We consider in particular the elastic matrix elements between states of the same spin. We use the wave functions (4.4) to find

$$\left\langle G, J, \mu_1 \cdots \middle|q(0)^2\middle|G', J, \mu_1' \cdots\right\rangle = \int \frac{d^4B}{(2\pi)^3}\, \delta\left(B^2 + m_B^2\right)\theta(B^0)\, q^J(B\cdot G)^* q^J(B\cdot G')\cdot$$
$$\cdot\, T^J_{\mu_1\cdots}\left(\hat{B}_G\right) T^J_{\mu_1'\cdots}\left(\hat{B}_{G'}\right) \tag{6.2}$$

where it will be helpful to denote the argument of the wave function as $B\cdot G$ rather than $u = -(B-G)^2$. It will be useful to define a single invariant form factor by multiplying the above by a tensor[12]

$$M^{\mu_1\cdots\mu_J,\ \mu_1'\cdots\mu_J'}(G, G') \tag{6.3}$$

and contracting on $\mu_1 \cdots \mu_J'$. As $t \longrightarrow 0$, (6.3) will reduce to $-g^{\mu_1\mu_1'}\ldots$, so the corresponding form factor is equivalent to an average over the spin states. We then obtain

$$F_G^J(t) = \int \frac{d^4B}{(2\pi)^3}\,\delta\left(B^2+m_B^2\right)\theta(B^o)q^{J*}(B\cdot G)q^J(B\cdot G')P_J\left[\frac{G\cdot G' m_B^2 + B\cdot G' B\cdot G}{m_G^2\left\{-m_B^2+\frac{(B\cdot G)^2}{m_G^2}\right\}^{\frac{1}{2}}\left\{-m_B^2+\frac{(B\cdot G')^2}{m_G^2}\right\}^{\frac{1}{2}}}\right]. \tag{6.4}$$

Since the matrix elements $\langle B|q|G\rangle$ were obtained from a linear differential equation in u, they are defined up to an overall normalization factor. Consequently, $F_G^J(0) > 0$ but is otherwise arbitrary at this point. We also note that F_G^J is smooth near $t = 0$, so the virtual quark in one hadron will feel no singular potential with respect to another.

We can now calculate the corresponding matrix element of I between the states $|B\rangle$.

$$F_B(t) = \langle B|q^2|B'\rangle = \sum_{G,J} \langle B|q|G,J\rangle \langle G,J|q|B'\rangle =$$

$$= \sum_{G,J} \int \frac{d^4G}{(2\pi)^3}\delta\left(G^2+m_G^2\right)\,\theta(G^o)q^{J*}(B\cdot G)q^J(B'\cdot G)\;P^J\left(\hat{B}_G\cdot\hat{B}'_G\right) \tag{6.5}$$

where

$$\hat{B}_G\cdot\hat{B}'_G = \frac{B\cdot B' + \frac{B'\cdot G\, B\cdot G}{m_G^2}}{\left\{-m_G^2+\frac{(B\cdot G)^2}{m_G^2}\right\}^{\frac{1}{2}}\left\{-m_B^2+\frac{(B'\cdot G)^2}{m_G^2}\right\}^{\frac{1}{2}}} . \tag{6.6}$$

If, in the integrations in (6.5), we let $G \longrightarrow \frac{m_G}{m_B} \overline{B}$, we see that each term in (6.5) becomes identical to (6.4). Therefore, we find

$$F_B(t) = \sum_{G,J} \frac{m_G^2}{m_B^2} F_G^J\left(\frac{m_G^2}{m_B^2} t\right) . \tag{6.7}$$

Consequently, we see that as $t \longrightarrow 0$

$$F_B(t) \rightarrow \sum_{G,J} \frac{m_G^2}{m_B^2} F_G^J(0) \tag{6.8}$$

where $I_B = \lambda F_B$.

We note that, as a consequence of our initial ansatz, we have obtained a mass spectrum which extends over an infinite range. Therefore, the sum over this spectrum in (6.8) extends to infinity. Further, the constants $F_G^J(0)$ are, at this point, arbitrary positive numbers. Therefore, we should choose them to behave for large m_G and J so that the sum in (6.8) diverges as $t \rightarrow 0$ in the way to represent the singularity in our initial ansatz, (4.2). To compute the necessary constants, we require a calculation of the spectrum, m_G^2, as a function of (n, J) when they are both large. This requires an evaluation of (5.4) for general, large values of n and J. To study the form of the singularity near $t = 0$, we also require the W.K.B. wave functions. Neither of these calculations has been made so far. However, to the extent that the diverging mass spectrum with positive $F_G^J(0)$ makes an arbitrary singularity at $t = 0$ possible, consistency may be achieved.

The physical picture is, however, clear. The heavy hadrons are larger and larger, since they correspond to a virtual quark rattling in the oscillator well provided by the bad particle. This allows self-consistency, the generation of the walls of the well by the interaction of the quark with a virtual quark present in the bad particle regarded as formed from a hadron and the quark. Since there are an infinite number of ways that this can occur, we get the infinitely high walls. The quark force need not be large. Its strength only mainly affects the hadron level spacing.

VII. Internal Symmetry

The picture we have developed now allows us to discuss the consistency of our dynamical mechanism with the internal quantum numbers of quarks. We shall begin by assuming that the quarks are coupled through a neutral vector meson glue, that is, we assume that we have a field equation of the form

$$\left(\gamma^{\mu}\left(\frac{1}{i}\,\partial_{\mu}\right)+m\right)q(x)=g\gamma\cdot A(x)\,q(x)$$

and

$$\left(-\Box^{2}+\lambda^{2}\right)A^{\mu}(x)=g\left[\bar{q}(x)\,\gamma^{\mu},\ q(x)\right] \quad . \tag{7.1}$$

Then (7.1), taken between B and G states, becomes

$$(\gamma\cdot(G-B)+m)\langle B|q|G\rangle=g\sum_{B'}\langle B|A^{\mu}|B'\rangle\,\gamma_{\mu}\langle B'|q|G\rangle \ . \tag{7.2}$$

Further,

$$\left((B-B')^{2}+\lambda^{2}\right)\langle B|A^{\mu}|B'\rangle=g\sum_{s}\langle B|\bar{q}|s\rangle\,\gamma^{\mu}\,\langle s|q|B'\rangle$$

$$-g\sum_{G}\langle B|q|G\rangle\,\langle G|\bar{q}|B'\rangle\,\gamma_{\mu} \quad . \tag{7.3}$$

We imagine inserting the solution of (7.3) into (7.2). We then will draw pictures to represent the terms in this formula. The picture of (7.2) and (7.3) is given in Fig. 1. The analogous equations for the antiquark operator $\bar{q}$ are represented in Fig. 2. These figures are not Feynman diagrams but merely a representation of the above formulas. The qualitative feature provided by the vector meson glue is of an attraction in the $q\bar{q}$ interaction, repulsion in the qq or $\bar{q}\bar{q}$ interaction.

We denote with s a state which, at this point, may be good or bad. If our mechanism is to be self-consistent, we should have a deep well in the channel with attraction, and, as a consequence, a linearly rising Regge spectrum of states for the hadrons described as an excited quark moving in the well. It is clear that this is self-consistent with the first terms in Fig. 1 and 2, that is, if we input a singular well, we will obtain a spectrum which leads to a singular well as discussed in Sections IV, V, and VI. If, however, the mechanism is to be stable, we should not have to worry about cancellations which could come from the repulsive second term in Fig. 1 and 2. Hence the states s_1 and s_2 must not lie on infinitely rising Regge trajectories, that is, they must be bad. We shall discuss in a moment the dynamical consistency of this assumption. Let us suppose that the quark carries a baryon number b_q. The baryon numbers of the states G, B_1, etc., we call b_G, b_{B_1}, ... We then have the relations which follow from conservation of baryon numbers in Fig. 1 and 2,

$$\begin{aligned} b_G &= b_q + b_{B_1} \\ b_G &= -b_q + b_{B_2} \\ b_{B_1} &= b_{s_1} + b_q \\ b_{B_2} &= b_{s_1} - b_q \quad . \end{aligned}$$

If we express the baryon numbers of the bad states coupled to G in terms of b_G

and b_q, we have

$$b_{B_1} = b_G - b_q$$
$$b_{s_1} = b_G - 2b_q$$
$$b_{B_2} = b_G + 2b_q$$
$$b_{B_2} = b_G + b_q \quad .$$

Therefore, a good channel with baryon number b_G must be coupled to four neighboring bad channels with baryon numbers $b_G - 2b_q$, $b_G - b_q$, $b_G + b_q$, $b_G + 2b_q$. We see that this pattern will be consistent with no good particles in bad channels and with TCP only if $b_G = 0, \pm 3b_q, \pm 6b_q$, etc. It is natural to put the arbitrary unit $b_q = \frac{1}{3}$, so $b_G = 0, \pm 1, \pm 2$, etc. Further, if the quarks are SU(3) triplets with triality $t = +1$, the above rule is equivalent to the statement that all good hadrons must have zero triality. Further, $t_{B_1} = t_{s_1} = -1$ and $t_{B_2} = t_{s_1} = +1$ will be the trialities of the four neighboring bad channels, which are coupled by the quark field operator to the good channel. Naturally, the B_2 with $b_q = 1/3$ is the physical quark.

We must finally check whether or not the hypothesis that the states s_1 and s_2 are bad is dynamically self-consistent. Therefore, we must study the equations for the matrix elements,

$$\langle s_1 | q | B_1 \rangle \qquad (\text{and } \langle s_2 | \bar{q} | B_2 \rangle).$$

We study the first of these. The calculation for the other is the same. The equation for $\langle s_1 | q\, B_1 \rangle$ is pictorially represented in Fig. 3, where B_1 couples to the good hadrons, G', with baryon number $b_G - 1$. We see, however, that the effect is one of a repulsion in the channel with the singular potential. We might worry that the coupling to a singular repulsion would lead to a catastrophe of the sort which occurs in the nonrelativistic Schrödinger equation with a potential $-r^2$.

If we return to (5.2), we find that this is not so. If we reverse the sign of γ in (5.2), we find that it has perfectly acceptable solutions corresponding to an open channel, i.e., particles which can separate. (G is replaced by B_1, B by s_1.) The wave function has an asymptotic form for large z, $e^{\pm 2(i/\gamma)\sqrt{+z}}$. There is, therefore, no discrete quantization in the channel B_1. However, in the crossed channel, $|\bar{s}_1, B\rangle$, there can be quantization, but only with a finite number of states produced. We see this if we look at (5.2), with $m_G \rightarrow m_{B_1}$, $m_B \rightarrow m_s$, and with $\gamma \longrightarrow -\gamma$, corresponding to the singular repulsion in the second term in Fig. 2, and with $z \leq -1$, corresponding to the crossed channel. We have in this case the equation (we take the form (5.3)):

$$\left\{\left(\frac{m}{m_{s_1}}\right)^2 - 1 - \left(\frac{m_{B_1}}{m_{s_1}}\right)^2 + 2\,\frac{m_{B_1}}{m_{s_1}} z\right\} Q_J = -\gamma \left\{\left(z^2 - 1\right)\left(\frac{\partial}{\partial z}\right)^2 + \frac{\frac{3}{4} - J(J+1)}{z^2 - 1}\right\} Q^J$$

or

$$\left(-\left(\frac{\partial}{\partial z}\right)^2 + \left\{\frac{J(J+1) - \frac{3}{4}}{\left(z^2 - 1\right)^2} + \frac{1}{\gamma}\left(\left(\frac{m_{B_1}}{m_{s_1}}\right)^2 + 1 - \left(\frac{m}{m_{s_1}}\right)^2 - 2\,\frac{m_{B_1}}{m_{s_1}} z\right)\right\}\right) Q^J = 0 \tag{7.4}$$

When $z \longrightarrow -\infty$, the bounded solution is

$$Q \approx e^{-\frac{2}{\gamma}\sqrt{\frac{m_{B_1}}{m_s}(-z)}}$$

which leads to the possibility of bound states. However, we see that when $J \geq 1$, the "potential" is always of one sign. Therefore, there is no value of m_{B_1} which leads to an eigenvalue of (7.4). If $J = 0$, there is at most a finite number. Therefore, at best, we could find a few discrete channels. These would depend on the details of the other effects not contained in (7.4). The asymptotically high spectrum would be continuous. Therefore, again our mechanism is consistent with the absence of a linearly rising trajectory for the bad particles.

VIII. Discussion and Conclusions

We have illustrated how it is possible to keep quarks inside of hadrons, if we allow an interaction operator (that is, a relativistic potential) to have matrix elements between certain states which are sufficiently singular at zero momentum transfer so that it becomes equivalent to a differential operator in the virtual quark mass. We have seen that if this singularity is in elastic matrix elements, it need not violate any sacred relativistic law. On the other hand, since the ordinary hadrons have matrix elements of this operator with smooth behavior, there will be no unusual long range effects expected to act between them. We have seen that the singularity at $t = 0$ could be expected to be self-consistent, that is, a consequence of itself. Further, the states with the peculiar form factors could not be produced in any process involving incoming ordinary hadrons.

The mechanism we have provided is analogous to that which produces the distorted potential which acts between an electron and its image charge in a highly charged atom. Indeed the analogy might be even closer if we did not take the limit $\mu \longrightarrow 0$ in (4.2), that is, if we let the well for the quark be deep but finite. In this case, the uniform spacing of hadron levels would be a low mass approximation. As the mass increases, the levels would begin to converge to an "ionization" limit. In this case, one would forecast the appearance of real quarks on the "outside." The problem would be one analogous to barrier penetration with the rise of the cross section for quark production related to the curvature of the Regge trajectory.

We remark in conclusion that the idea presented here is most attractive because it is fairly simple. A few moments reflection will convince one that if it turns out to be correct, it might allow one to reconcile many of the seemingly contradictory approaches to problems in strong interactions.

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REFERENCES

1. G. F. Chew and S. C. Frautschi, Phys. Rev. Letters 7, 394 (1961).
2. F. Gursey and L. Radicati, Phys. Rev. Letters 13, 173 (1964).
3. R. Dalitz, Proceedings of the XIIIth International Conference on High Energy Physics, Berkeley, 1966 (University of California Press, Berkeley, 1967); G. Morpurgo, Proceedings of the XIVth International Conference on High Energy Physics, Vienna, 1968 (CERN, Scientific Information Service, Geneva, 1968).
4. M. Gell-Mann, Physics 1, 63 (1964).
5. J. S. Ball and F. Zachariasen, Phys. Rev. 170, 1541 (1968); D. Amati, L. Caneschi and R. Jengo, Nuovo Cimento 58, 783 (1968); D. Amati, R. Jengo, H. R. Rubenstein, G. Veneziano and M. A. Virasoro, Phys. Letters 27B, 38 (1968); M. Ciafaloni and P. Menotti, Phys. Rev. 173, 1575 (1968); M. Ciafaloni, Phys. Rev. 176, 1898 (1968).
6. S. D. Drell and T. D. Lee, Phys. Rev. (to be published), [CO-3067(21-1, SLAC-PUB-997, December 1971].
7. L. Van Hove, Phys. Letters 24B, 183 (1967); L. Durand, Phys. Rev. 161, 1610 (1967); S. Mandelstam, Phys. Rev. 166, 1539 (1968).
8. G. Veneziano, Nuovo Cimento 57A, 190 (1968).
9. J. Kuti and V. F. Weisskopf, Phys. Rev. D 4, 3418 (1971).
10. Since the singularity is only in the elastic matrix element, t will always be negative in (3.2), and we thereby avoid unphysical, "time-like" excitations in the oscillator well.
11. See Appendix A.
12. See Appendix B.

Appendix A

We first wish to evaluate (5.4), when J is large. In this case m_G/m_B becomes large so we drop the terms $1-(m/m_B)^2$ inside the root in (5.4). Let

$$(m_B/m_G)^2 = x \quad .$$

We then have for (5.4),

$$\left(n+\frac{1}{2}\right)\pi = \int dz \left[\frac{1}{\gamma}\left\{\frac{x^2-2\,xz}{z^2-1}\right\} - \frac{J^2}{(z^2-1)^2}\right]^{1/2} \tag{A.1}$$

The real positive roots of the cubic equation in z

$$(x^2-2\,xz)(z^2-1) - \gamma J^2 = 0$$

define the range in (A.1). Let us put

$$z = x\zeta \quad .$$

Then with x large we find

$$\frac{\sqrt{\gamma}\left(n+\frac{1}{2}\right)\pi}{x} = \int \frac{d\zeta}{\sqrt{\zeta}} \left\{-2+\frac{1}{\zeta}-\frac{\gamma J^2}{x^4}\cdot\frac{1}{\zeta^3}\right\}^{1/2} \quad . \tag{A.2}$$

As x increases for fixed n, the integral should vanish. As $\gamma J^2/x^4 \rightarrow 1/27$, the range of ζ becomes a vanishing interval around $\zeta = 1/3$. That is, x and J are both consistently large. If we evaluate (A.2) in this limit, taking the leading and, next to leading terms in (A.2) we obtain the first result in (5.5). Similar analysis yields the second form in (5.5).

Appendix B

It will be convenient to define unit time-like vectors,

$$g^\mu = G^\mu/m_G\,, \qquad g'^\mu = G'^\mu/m_G\,, \qquad b^\mu = B^\mu/m_B$$

so that

$$\hat{B}_G = (b^\mu + g^\mu\, b\cdot g)\frac{1}{\left((b\cdot g)^2 - 1\right)^{1/2}} = b_g^\mu \quad .$$

Likewise

$$g_b^\mu = (g^\mu + b^\mu\, b\cdot g)\,\frac{1}{\left((b\cdot g)^2 - 1\right)^{1/2}}$$

is a unit space-like vector.

Then we want to find a tensor $M^{\mu_1\ldots\mu_J,\nu_1\ldots\nu_J}(g,g')$ so that

$$T^J_{\mu_1\ldots\mu_J}(b_g)\, T^J_{\nu_1\ldots\nu_J}(b_{g'})\cdot M^{\mu_1\ldots\mu_J,\nu_1\ldots\nu_J}(g,g')$$

$$= P^J(g_b\cdot g_b') \tag{B.1}$$

where

$$g_b\cdot g_b' = (g\cdot g' + g\cdot b\; g'\cdot b)\,\frac{1}{\left((g\cdot b)^2 - 1\right)^{1/2}\left((g'\cdot b)^2 - 1\right)^{1/2}} \quad . \tag{B.2}$$

Let us suppose L^μ_ν has the properties,

$$\begin{aligned} g^\mu &= L^\mu{}_\nu\, g'^\nu \\ L^\mu{}_\alpha L^\beta{}_\mu &= \delta^\beta_\alpha \\ L^{\mu\nu} b_\mu b_\nu &= g\cdot g' \end{aligned} \tag{B.3}$$

then

$$L_{\mu_1}{}^{\nu_1}\ldots L_{\mu_J}{}^{\nu_J}\; T^J_{\nu_1\ldots\nu_J}(b_{g'}) = T^J_{\mu_1\ldots\mu_j}\left((Lb)_g\right)$$

where

$$(Lb)_g = \left((Lb) + g(b\cdot g')\right) \frac{1}{\left((b\cdot g')^2-1\right)^{1/2}} \quad .$$

Then if we use (A.6) and (B.3) we find

$$T^J_{\mu_1\cdots\mu_J}(b_g) \left(L^{\mu_1\nu_1}\cdots L^{\mu_J\nu_J}\right) T^J_{\nu_1\cdots\nu_J}(b_{g'}) = P^J(g_b\cdot g_b') \quad . \tag{B.4}$$

We note that

$$L_{\mu\nu} = -g_{\mu\nu}\, g\cdot g' + g'_\mu g_\nu - g'_\mu g_\nu + i\, \epsilon_{\mu\nu\alpha\beta} g'^\alpha g^\beta \tag{B.5}$$

satisfies (B.3). Further in (B.4), we can drop all terms in $L \ldots L$ of the form g^{μ_i} or g'^{ν_i}, since they give no contribution. In this way we obtain M. We see that as $g \to g'$, $M^{\mu_1\cdots\mu_J,\nu_1\cdots\nu_J}$ becomes $g^{\mu_1\nu_1}\cdots g^{\mu_J\nu_J}$.

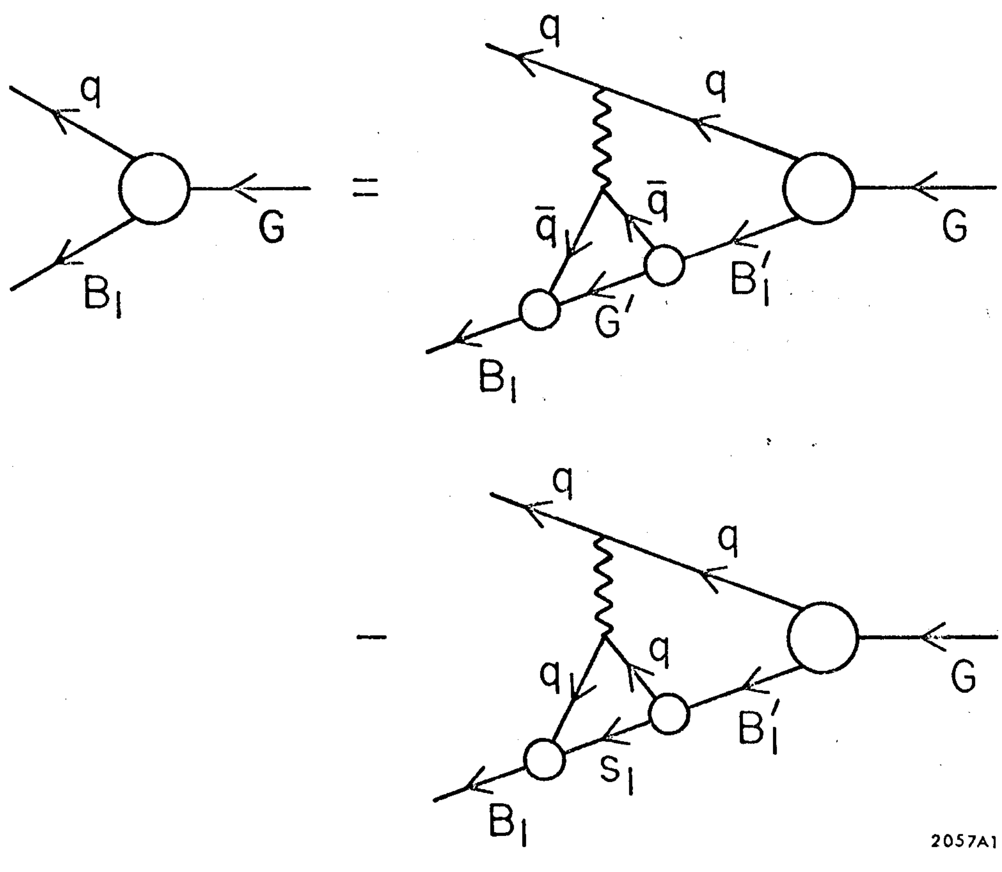

Fig. 1

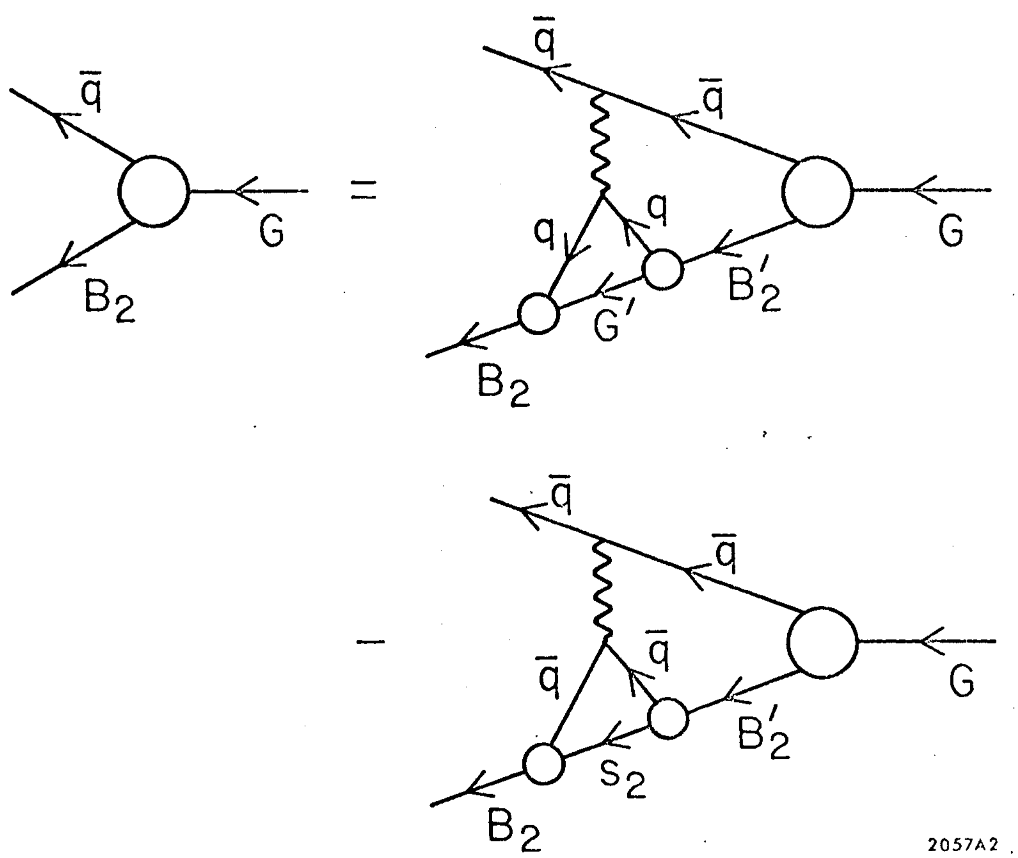

Fig. 2

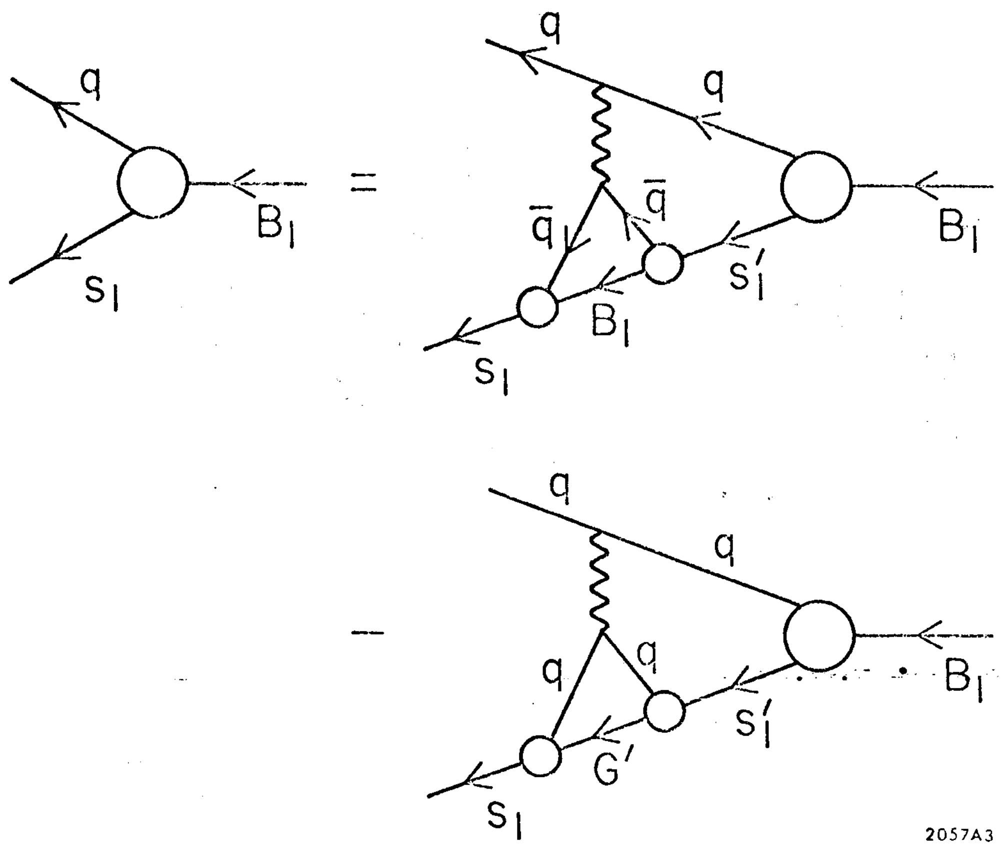

Fig. 3