SLAC-PUB-1031
(TH) and (EXP)
February 1972

UNITARY PARAMETERIZATION OF THREE-BODY OBSERVABLES*

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Abstract

The three-particle unitarity relations are solved explicitly in order to derive a general representation for the measurable three-body amplitudes. The representation is characterized by a well-defined set of real parameters which are the equivalent of a phase shift analysis for the three-body problem. The resulting parameterization is suitable for both the data analysis of three-particle final states, and as a starting point for a three-body phenomenology.

*Work supported by the U. S. Atomic Energy Commission.

(To be published in Phys. Rev. D)

I. Introduction

In this article, we propose a scheme for parameterizing all measurable amplitudes of a three-particle system in an explicitly unitary fashion. As in the familiar phase-shift analysis of two-body scattering, such a scheme has the advantage of specifying the amplitudes in terms of the truly arbitrary parameters which remain after the constraints of unitarity are satisfied. One obvious motivation for such a development is to provide a systematic framework for the data analysis of three-particle final states. Additionally, one would expect this to be a useful starting point in phenomenological approaches to the three-body problem, such as that proposed recently by Noyes.[1]

The procedure we shall follow is based on the explicit solution of the three-particle unitarity relations, the key step being an expansion of the breakup and three-free-particle amplitudes in terms of an advantageous complete set. The problem then reduces formally to a system of coupled channels, the solution to which can be obtained via a straightforward application of well-known techniques.[2] The development is completely general; although we will employ non-relativistic kinematics, the necessary modifications for the relativistic case will be obvious.

II. General Solution of the Unitarity Constraints

In what follows, we shall assume that the amplitudes under consideration are characterized by a set of conserved quantum numbers $\mathscr{A}$, a set of auxiliary quantum numbers σ, the channel index α, and the continuous momentum variables p,q. For example, $\mathscr{A}$ may consist of the total angular momentum, total isospin, their z-components, etc.; σ contains the angular momentum of two of the particles, the angular momentum of the third particle relative to the c.m. of this pair, and similarly for the spins and isospins allowed by the set $\mathscr{A}$. We will assume that in practice, for a given $\mathscr{A}$, the set σ is restricted to a finite number of different combinations, and hence can be characterized by a single index n, such that $1 \leq n \leq N_{\mathscr{A}}$. This would be the case, for example, if the interactions between the particles are effectively zero in all but a finite number of partial-waves. One may then describe the three-body system via the states $|\alpha\, pq\, \mathscr{A} n\rangle$.

If, for simplicity, we assume that there is but one bound state in each two-body channel, the measurable three-body amplitudes are as follows:

(1) An amplitude describing the scattering at c.m. energy $W > 0$ from an initial state of three free particles to a final state of three free particles; we denote this by $F^{\mathscr{A}}_{nn'}(q, q';W)$, where the continuous variables q,q' are restricted to the finite domain $0 \leq q,q' \leq Q_\alpha$, with $Q_\alpha = (2M_\alpha W)^{1/2}$, M_α being the reduced mass appropriate to the variable q.

(2) An amplitude $B^{\mathscr{A}}_{n;\,\delta' m'}(q, W)$ describing break-up of the bound state of particles β and γ, $\delta' \neq \beta \neq \gamma$. Here the index m' ranges over the subset of σ which is compatible with $\mathscr{A}$ and the bound state, i.e., $m' \leq M_{\delta'} \leq N_{\mathscr{A}}$.

(3) The elastic scattering and bound state rearrangement amplitudes $E^{\mathscr{A}}_{\delta m;\,\delta' m'}(W)$; these are non-vanishing when W is greater than

the thresholds $-E_\delta$, $-E_{\delta'}$, corresponding to the two-body binding energies E_δ, $E_{\delta'}$.

It is well known that these amplitudes satisfy the unitarity constraints[3]

$$
F^{\mathscr{D}}_{nn'}(q,q';W) - F^{\mathscr{D}}_{nn'}(q,q';W)^* = -i\pi \sum_{n''} 2\mu_\alpha \int_0^{Q_\alpha} dQ Q^2 K_\alpha F^{\mathscr{D}}_{nn''}(q,Q;W) F^{\mathscr{D}}_{n''n'}(Q,q';W)^*
$$

$$
-i2\pi \sum_{\gamma,m} M_\gamma q_\gamma B^{\mathscr{D}}_{n;\gamma m}(q,W) B^{\mathscr{D}}_{n';\gamma m}(q',W)^* ,
$$

$$
B^{\mathscr{D}}_{n;\delta' m'}(q,W) - B^{\mathscr{D}}_{n;\delta' m'}(q,W)^* = -i\pi \sum_{n''} 2\mu_\alpha \int_0^{Q_\alpha} dQ Q^2 K_\alpha F^{\mathscr{D}}_{nn''}(q,Q;W) B^{\mathscr{D}}_{n'';\delta' m''}(Q,W)^*
$$

$$
-i2\pi \sum_{\gamma,m} M_\gamma q_\gamma B^{\mathscr{D}}_{n;\gamma m}(q,W) E^{\mathscr{D}}_{\gamma m;\delta' m'}(W)^* , \tag{1}
$$

$$
E^{\mathscr{D}}_{\delta m;\delta' m'}(W) - E^{\mathscr{D}}_{\delta m;\delta' m'}(W)^* = -i2\pi \sum_{\gamma,m''} M_\gamma q_\gamma E^{\mathscr{D}}_{\delta m;\gamma m''}(W) E^{\mathscr{D}}_{\gamma m'';\delta' m'}(W)^*
$$

$$
-i\pi \sum_{n''} 2\mu_\alpha \int_0^{Q_\alpha} dQ Q^2 K_\alpha B^{\mathscr{D}}_{n'';\delta m}(Q,W) B^{\mathscr{D}}_{n'';\delta' m'}(Q,W)^* .
$$

Here μ_α is the reduced mass appropriate to the variable p, while

$$
K_\alpha = \left[2\mu_\alpha\left(W - Q^2/2M_\alpha\right)\right]^{1/2} ,
$$

$$
q_\gamma = \left[2M_\gamma\left(W + E_\gamma\right)\right]^{1/2}. \tag{2}
$$

Recently, Noyes[1] has suggested that one replace the dependence of these amplitudes on the continuous variable q in terms of an expansion in some complete set of functions on $(0, Q_\alpha)$. This approach takes into account the realities of data analysis, in which one really only has information for a finite set of points q_k, and is clearly analogous to the familiar partial-wave analysis of two-particle amplitudes. As in this analogy, we wish to express the coefficients of this expansion in terms of an arbitrary set of real parameters in such a way that the amplitudes manifestly satisfy the unitarity constraints. The parameters thus defined are the equivalent of a phase shift analysis for the three-body problem. We therefore introduce the complete orthonormal set of functions (in channel α)

$$\phi_k^\alpha(q, W) = \left[\frac{2k+1}{2\mu_\alpha K_\alpha Q_\alpha}\right]^{1/2} P_k(-1 + 2q/Q_\alpha)/q \quad , \tag{3}$$

where $P_k(x)$ is the usual Legendre polynomial. It follows that

$$2\mu_\alpha \int_0^{Q_\alpha} dQ\, Q^2 K_\alpha\, \phi_k^\alpha(Q, W)\, \phi_{k'}^\alpha(Q, W) = \delta_{kk'} \quad . \tag{4}$$

Defining the expansions

$$\begin{aligned} F_{nn'}^{\mathscr{A}}(q, q'; W) &= \sum_{k,k'} f_{nk;n'k'}^{\mathscr{A}, W}\, \phi_k^\alpha(q, W)\, \phi_{k'}^\alpha(q', W) \ , \\ B_{n;\delta' m'}^{\mathscr{A}}(q, W) &= \sum_k \frac{b_{nk;\delta' m'}^{\mathscr{A} W}}{\left(2 M_{\delta'} q_{\delta'}\right)^{1/2}}\, \phi_k^\alpha(q, W) \ , \end{aligned} \tag{5}$$

and renormalizing the elastic and rearrangement amplitudes in the form

$$E_{\delta m;\delta' m'}^{\mathscr{A}}(W) = \left(4 M_\delta q_\delta M_{\delta'} q_{\delta'}\right)^{-1/2} e_{\delta m;\delta' m'}^{\mathscr{A}, W} \ , \tag{6}$$

Eq. (1) implies the following set of constraints among the coefficients:

$$f^{\ell,W}_{nk;n'k'} - f^{\ell,W*}_{nk;n'k'} = -i\pi \sum_{n'',k''} f^{\ell,W}_{nk;n''k''} f^{\ell,W*}_{n''k'';n'k'} - i\pi \sum_{\gamma,m} b^{\ell,W}_{nk;\gamma m} b^{\ell,W*}_{n'k';\gamma m} ,$$

$$b^{\ell,W}_{nk;\delta' m'} - b^{\ell,W*}_{nk;\delta' m'} = -i\pi \sum_{n'',k''} f^{\ell,W}_{nk;n''k''} b^{\ell,W*}_{n''k'';\delta' m'} - i\pi \sum_{\gamma,m} b^{\ell,W}_{nk;\gamma m} e^{\ell,W*}_{\gamma m;\delta' m'} , \quad (7)$$

$$e^{\ell,W}_{\delta m;\delta' m'} - e^{\ell,W*}_{\delta m;\delta' m'} = -i\pi \sum_{\gamma,m''} e^{\ell,W}_{\delta m;\gamma m''} e^{\ell,W*}_{\gamma m'';\delta' m'} - i\pi \sum_{n'',k''} b^{\ell,W}_{n''k'';\delta m} b^{\ell,W*}_{n''k'';\delta' m'} .$$

Formally, the indices k, k', k'' take on an infinite number of values in the above equations. However, we would expect that the expansions of Eq. (5) may be safely truncated at a sufficiently large index K_M; in practice, K_M is determined by the number of data points available. With this assumption, it is clear that for a given ℓ, the sets of indices labeling the above coefficients can be put into a unique correspondence with the single indices $j = j(n,k) \leq N$, $\mu = \mu(\delta, m) \leq \bar{N}$; where $N = N_\delta \times K_M$, $\bar{N} = M_1 + M_2 + M_3$. Dropping the explicit ℓ, W labels for simplicity, the above coefficients can be written as $f_{jj'}$, $b_{j\mu}$, $e_{\mu\mu'}$, i.e., as matrix elements of the finite matrix operators f, b, e. One then obtains the following compact representation of Eq. (7):

$$\begin{aligned} f - f^\dagger &= -i\pi f f^\dagger - i\pi b b^\dagger , \\ b - b^* &= -i\pi f b^* - i\pi b e^\dagger , \\ e - e^\dagger &= -i\pi e e^\dagger - i\pi b^\dagger b , \end{aligned} \quad (8)$$

where we have invoked time-reversal invariance to obtain such relations as $e^* = e^\dagger$, $e^* e = e e^*$, etc. Before considering the most general realization of this equation, we note that in the theoretical analysis of the three-body problem, it may be more convenient to work instead with the channel amplitudes $T_{\beta\alpha}$ such that $T = \sum_{\beta} T_{\beta\alpha}$, T being the three-body t-matrix. If one then determines the constraints on $T_{\beta\alpha}$ such that the physical amplitudes derived from T satisfy the above unitarity relations, the resulting equations are identical to Eq. (8), except that the channel index β must also be incorporated into the overall index j. With this interpretation, Eq. (8) embodies the full content of three-particle unitarity for either experimental analysis or three-body phenomenology.

In order to obtain the general solution of the coupled matrix equations given in Eq. (8), we consider first the $e - e^{\dagger}$ relation, introducing the matrix $\sigma = b^{\dagger} b$, which is clearly Hermitian. Moreover, the time-reversal properties noted above imply that σ is also real, and hence symmetric. If we represent the complex symmetric e by the real symmetric matrices A and B, $e = A + iB$, Eq. (8) implies that

$$2iB = i\pi\left(A^2 + B^2 + \sigma + i\,[B,A]\right) ; \tag{9}$$

the reality of σ thus requires that A and B commute. It follows that there exists a real orthogonal matrix U, such that

$$\begin{aligned} U U^T &= 1 , \\ e &= U e' U^T , \end{aligned} \tag{10}$$

where e' is diagonal. Substituting this result into Eq. (8), we have that

$$e' - e'^* + i\pi e' e'^* = -i\pi U^T \sigma U \quad , \tag{11}$$

so that U must also diagonalize σ. Hence $\sigma = U \sigma' U^T$ and Eq. (11) becomes a scalar equation, with the solution

$$e'_\mu = \frac{-1}{i\pi}\left(\eta_\mu e^{2i\delta_\mu} - 1\right) , \tag{12}$$

$$\sigma'_\mu = \left(1 - \eta_\mu^2\right)/\pi^2 \quad .$$

Thus e can be parameterized in terms of "eigen" phase-shifts and elasticities, as well as the "mixing" matrix U.

The equation for $f - f^\dagger$ is precisely identical, except that we must work with the matrix $\bar{\sigma} = b b^\dagger$. There is thus a real orthogonal matrix V such that $f = V f' V^T$, $\bar{\sigma} = V \bar{\sigma}' V^T$; with

$$f'_j = \frac{-1}{i\pi}\left(\bar{\eta}_j e^{2i\bar{\delta}_j} - 1\right), \tag{13}$$

$$\bar{\sigma}'_j = \left(1 - \bar{\eta}_j^2\right)/\pi^2 \ .$$

The parameters determining the e , f matrices are not independent, however, but are related by the equation for $b - b^*$. To solve the latter, we let $b = V b^o U^T$; thus

$$b^o - b^{o*} = -i\pi f' b^{o*} - i\pi b^o e'^* \quad . \tag{14}$$

Substituting the expressions given in Eq. (12) and (13) for e' and f', Eq. (14) becomes

$$\bar{\eta}_j e^{2i\bar{\delta}_j} b^{o*}_{j\mu} = \eta_\mu e^{-2i\delta_\mu} b^o_{j\mu} \quad . \tag{15}$$

It is therefore clear that either $\bar{\eta}_j = \eta_\mu$, or that $b^o_{j\mu} = 0$. In the former case, we let $b^o_{j\mu} = R_{j\mu} e^{i\lambda_{j\mu}}$, where $R_{j\mu}$ is real. It then follows that

$$\lambda_{j\mu} = \bar{\delta}_j + \delta_\mu , \tag{16}$$

up to an integral multiple of π which we incorporate into the sign of $R_{j\mu}$.

In order to determine the $R_{j\mu}$, we first observe that although in principle there are N arbitrary diagonal elements $\bar{\sigma}'_j$, and $\bar{N} < N$ diagonal elements ϵ'_μ, the nature of $\bar{\sigma}$ restricts all but $\bar{N}$ of the $\bar{\sigma}'_j$ to be zero. To see this, we form an $N \times N$ matrix Z such that

$$\begin{aligned} Z_{jk} &= b_{jk} , && k \leq \bar{N} , \\ &= 0 , && \bar{N} < k \leq N , \end{aligned} \tag{17}$$

for $j \leq N$. It follows that $\bar{\sigma} = Z Z^\dagger$. However, all but the first $\bar{N}$ rows of $Z^\dagger$ vanish identically, and hence $\bar{\sigma}$ annihilates a subspace of dimension $N - \bar{N}$. It follows that $\bar{\sigma}$ may have at most $\bar{N}$ non-zero eigenvalues (the $\bar{\sigma}'_j$). From the above, we know that the $\bar{N}$ non-vanishing $\bar{\sigma}'_j$ must be identical with the $\bar{N}$ elements σ'_μ. One can easily verify that there is no loss in generality in taking

$$\begin{aligned} \bar{\sigma}'_k &= \sigma'_k , && k \leq \bar{N} , \\ &= 0 , && k > \bar{N} ; \end{aligned} \tag{18}$$

any other permutation of the $\bar{\sigma}'_k$ can be realized by suitably choosing V.

At this point, it is necessary to distinguish between several possibilities concerning the set σ'_μ. If we first assume that each element is distinct, the discussion following Eq. (15) implies that

$$\begin{aligned} b^o_{j\mu} &= \delta_{j\mu} R_{\mu\mu} e^{i\left(\bar{\delta}_\mu + \delta_\mu\right)} , && j \leq \bar{N} ; \\ &= 0 , && j > \bar{N} . \end{aligned} \tag{19}$$

The $\bar{N}$ quantities $R_{\mu\mu}$ determine the set σ'_μ via the relation $\sigma' = b^{o+} b^o$, or $R_{\mu\mu} = (\sigma'_\mu)^{1/2}$. Thus specifying the set η_μ determines the $R_{\mu\mu}$ completely.[4]

If, however, some of the η_μ are degenerate, the situation becomes more complicated, although in a trivial way. For example, suppose that η_1 and η_2 are identical; Eq. (19) then holds for $\mu > 2$. For $\mu \leq 2$, $b^o_{j\mu} \neq 0$ for $j = 1,2$, and these elements must satisfy

$$\delta_{\mu\mu'}\sigma_1' = \sum_{j=1}^{2} b^{o*}_{j\mu} b^o_{j\mu'} = \sum_{j=1}^{2} R_{j\mu} R_{j\mu'} ; \quad \mu = 1,2 \tag{20}$$

The solution of this is $R_{j\mu} = (\sigma_1')^{1/2} O_{j\mu}$, where O is an arbitrary orthogonal matrix. Therefore, instead of the two real parameters R_{11}, R_{22} which we required in the non-degenerate case, we must specify the scale $(\sigma_1')^{1/2}$ and the one real parameter which characterizes O. The extension to degeneracy greater than two is obvious; in practice, however, we would expect the non-degenerate case to be most common.

III. Conclusion

Assuming the non-degenerate case for definiteness, we can summarize our results as follows: The solution of the unitarity constraints represented in Eq. (8) can be totally specified by the N phases $\bar{\delta}_j$, the $\bar{N}$ phases δ_μ, the $\bar{N}$ real numbers η_μ, an $N \times N$ orthogonal matrix V, and an $\bar{N} \times \bar{N}$ orthogonal matrix U. Since the arbitrary orthogonal matrix of dimension n is characterized by $n(n-1)/2$ real parameters, we conclude that a total of $\mathcal{N}$ real parameters is required to completely specify the amplitudes e, b, f, where

$$\mathcal{N} = \frac{1}{2}\left(N^2 + \bar{N}^2 + N + 3\bar{N}\right). \tag{21}$$

This is to be compared with the $2\left(N^2 + \bar{N}^2 + N\bar{N}\right)$ real parameters which would be required in the absence of unitarity constraints.

We have thus achieved a parameterization of three-body observables which is well defined and explicitly unitary. It is thus suitable for both the analysis of experimental data and as a starting point for a three-particle phenomenology. As an example of the usefulness of such a scheme, we note that we may employ it to clarify the extent to which the amplitude f (which at present cannot be measured in practice) is determined by measurements of e and b. For simplicity, suppose that our system consisted of three scalar bosons interacting via two-body s-wave forces only, with a single ($\ell = 0$) bound state between each pair. In terms of the above notation, we then have $\mathscr{A} = \{J, M\}$, σ is superfluous; thus $N_{\mathscr{A}} = 1$, $M_\delta = 1$, $N = K_M$, $\bar{N} = 3$. By measuring the e and b amplitudes, we could determine all of the $\bar{\delta}_j$, δ_μ, η_μ and the matrix U, as well as V_{j1}, V_{j2}, V_{j3} (the first three columns of V). To the extent that these columns determine V, we can explicitly determine f in terms of e and b. In fact, the knowledge of these three columns imposes $3N - 6$ constraints on the $N(N-1)/2$ parameters defining V, and hence f is completely determined in the approximation $K_M = 4$, is determined up to a single real parameter for $K_M = 5$, etc.

References

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3. The unitarity conditions follow, of course, from quite general principles. For a discussion within the context of three-particle dynamical formalisms, see E. O. Alt, P. Grassberger and W. Sandhas, Nucl. Phys. B2, 167 (1967); and D. Z. Freedman, C. Lovelace and J. M. Namyslowski, Nuovo Cimento 43A, 258 (1966).

4. In principle, $R_{\mu\mu}$ may be negative. However, the sign of $R_{\mu\mu}$ may be incorporated into the matrix V by changing the sign of the μ-th column. Since such a change leaves both f and b invariant, there is no loss in generality in taking $R_{\mu\mu}$ positive.