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AN IMPROVED WEIZSACKER WILLIAMS METHOD AND PHOTOPRODUCTION OF LEPTON PAIRS*

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ERRATA

1. p. 5, Eq. (11): change $\frac{dt}{t}$ to read $\frac{dt}{t^2}$.

2. p. 8, 6th line from the bottom: "Be proton" should be changed to "proton inelastic."

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ABSTRACT

An improved Weizsacker Williams method which properly handles the atomic and nuclear form factors is given. The method is applied to calculate the energy-angle distributions of photoproduced lepton pairs: 1) An electron pair from an atom with screening effects; 2) a muon pair from a nucleus with an elastic form factor. The numerical results are compared with the exact calculation using the Born approximation, and the agreements are found to be better than a few percents. The numerical results of production of heavy leptons of masses m = 0.5, 1, 2, 4, and 6 GeV from photons of energies 20, 40, 100, and 200 GeV are also given.

In the lowest order Born approximation (Fig. 1), the photoproduction of a pair of nonstrongly interacting charged particles can be calculated in terms of two form factors[1] $W_1(q^2, M_f^2)$ and $W_2(q^2, M_f^2)$ which appear in the electron scattering (see Eq. (1)). However if one wants to calculate the energy-angle distribution, $d\sigma/dpd\Omega$, of one of the particles or the total cross section of the process, it often requires some tedius analytical and computer work not only because the number of terms involved is large and integrations are complicated but also

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because of the occurrence of intricate cancellations among various terms. For particles which are as common as electrons and muons it is desirable to have simple and reliable formulas. They are given by Eqs. (10), (12) and (10), (13) respectively. Whether leptons heavier than muon exist in nature is a very interesting question.[2] We have done extensive calculations by the Born approximation and the detailed results of the energy-angle distributions of heavy leptons will be published elsewhere shortly. In Table II we give the results of the calculation for the total cross sections. In the usual application of the Weizsacker Williams method[3] to the pair production (or bremsstrahlung problem), the form factors of the target particle are not taken into account. Also, it is used only to calculate the energy distribution or total cross section but not the energy-angle distribution. In this note we demonstrate that the form factors can be taken into account and that the method can be used to calculate the energy-angle distribution as well as the energy distribution or the total cross section.

In the Born approximation, the pair production cross section can be written as (see Fig. 1 for notations)

$$d\sigma = \frac{e^6}{(2\pi)^5} \frac{M_i}{4(k\cdot p_i)} \frac{d^3p}{E} \int \frac{d^3p_+}{E_+} \frac{1}{q^4} \left(-L^{\mu\nu} W_{\mu\nu}\right) , \tag{1}$$

where

$$W_{\mu\nu} = M_i^{-2}\left(p_{i\mu} - q_\mu (q\cdot p_i)/q^2\right)\left(p_{i\nu} - q_\nu (q\cdot p_i)/q^2\right) W_2$$
$$- (g_{\mu\nu} - q_\mu q_\nu / q^2) W_1 \quad . \tag{2}$$

For pair production, the tensor $L^{\mu\nu}$ can be written as

$$L^{\mu\nu} = \sum_{j=1}^{4} L_j^{\mu\nu} T_j \quad , \tag{3}$$

where

$$L_{1\mu\nu} = (q^2/p\cdot q)\, p_\mu p_\nu + p\cdot q\, g_{\mu\nu} - p_\mu q_\nu - p_\nu q_\mu \quad ,$$

$$L_{2\mu\nu} = (q^2/p_+\cdot q)\, p_{+\mu} p_{+\nu} + p_+\cdot q\, g_{\mu\nu} - p_{+\mu} q_\nu - p_{+\nu} q_\mu \quad ,$$

$$L_{3\mu\nu} = \left[k\cdot p\, p_\mu - k\cdot p_+ p_{+\mu} + \frac{1}{2} q_\mu \quad (k\cdot p - k\cdot p_+)\right][\mu \to \nu] \quad ,$$

and

$$L_{4\mu\nu} = q^2 g_{\mu\nu} - q_\mu q_\nu \quad .$$

For the production of a pair of spin 1/2 particles, each of mass m, T_j can be written as

$$T_1 = \frac{-k\cdot p_+ + \frac{1}{2}t}{(k\cdot p)(k\cdot p_+)} \quad , \quad T_2 = \frac{-k\cdot p + \frac{1}{2}t}{(k\cdot p)(k\cdot p_+)} \quad , \quad T_3 = -\frac{2m^2}{(k\cdot p)^2 (k\cdot p_+)^2}$$

and

$$T_4 = -\frac{m^2}{2}\left(\frac{1}{k\cdot p} + \frac{1}{k\cdot p_+}\right)^2 \quad . \tag{4}$$

For the production of particles of other spins, only the expression of T_j's are different. The important thing to notice is that all these functions are smoothly varying functions of $t \equiv -q^2$, when t is small.

The integration with respect to the undetected particles p_+ and p_f can be carried out in the rest frame of $u \equiv p_+ + p_f = k + p_i - p$ as shown in Fig. 2. In this frame the integration can be cast into a convenient form:

$$I \equiv \int \frac{d^3 p_+}{E_+} \frac{1}{t^2}\left(-L^{\mu\nu} W_{\mu\nu}\right) = \frac{1}{4M_i\, |\vec{k} - \vec{p}|} \int_{t_{min}}^{t_{max}} \frac{dt}{t^2} \int_{M_i^2}^{(u-m)^2} dM_f^2 \int_0^{2\pi} d\phi \left(-L^{\mu\nu} W_{\mu\nu}\right) \tag{5}$$

where

$$u = \left[(k+p_i-p)^2\right]^{1/2} = \left[M_i^2 + m^2 + 2(k-E)\, M_i - 2(k\cdot p)\right]^{1/2} \quad .$$

In the derivation of our modified W.W. formula, we shall assume the following kinematical conditions:

$$E, \quad k-E \gg m, \quad k\cdot p/M_i, \quad (k\cdot p)^{1/2}, \quad (M_f^2 - M_i^2)/(2M_i) \quad . \tag{6}$$

Under these conditions we obtain the following:

1) The integrand in Eq. (5) is dominated by the region of t small compared with m^2. When $t \ll m^2$, $k\cdot p_+$ must necessarily be very close to its value evaluated at $\theta_+ = 0$ (see Fig. 2), where

$$k\cdot p_+ \approx (k\cdot p)\, E/(k-E) \;.$$

2) $t_{\substack{min\\max}}$ can be written as

$$t_{min} \approx (k\cdot p)^2/(k-E)^2 + (k\cdot p)\,(M_f^2 - M_i^2)/\left[M_i(k-E)\right] \tag{7}$$

and

$$t_{max} \approx 4M_i^2(k-E)^2/u^2 \quad .$$

3) W_1 and $L_4^{\mu\nu}$ in Eq. (3) can be dropped and after some manipulations we can show[4]

$$\frac{-1}{2\pi}\int_0^{2\pi} d\phi\, L_j^{\mu\nu}\,\frac{p_{i\mu}p_{i\nu}}{M_i^2} \approx \frac{1}{2}\left(g_{\mu\nu}\, L_j^{\mu\nu}\right)_{\theta_+=0}\frac{1}{t_{min}}(t-t_{min}) \quad . \tag{8}$$

From this equation, the factor I given in Eq. (5) can be approximated by

$$I_{W.W.} = \frac{\pi\left(g_{\mu\nu}L^{\mu\nu}\right)_{\theta_+=0}}{4M_i\,|\vec{k}-\vec{p}\,|\,t_{min}}\int_{t_{min}}^{m^2(1+\ell)^2}\frac{dt}{t^2}\int_{M_i^2}^{(u-m)^2} dM_f^2\,(t-t_{min})\,W_2 \quad , \tag{9}$$

where

$$\ell = (E/m)^2\,\theta^2 \quad .$$

The upper limit t_{max} in Eq. (5) is replaced by $m^2(1+\ell)^2$ for the following reasons: 1) Eq. (8) is true only if $t \ll m^2$. When $t \geq m^2$, the coefficient in the right-hand side of Eq. (8) becomes much smaller so we need a cutoff of order m^2. 2) When the scattering is from a point or a screened Coulomb charge (see Eq. (12)) an approximate analytical expression for $d\sigma/d\Omega dp$ can be obtained[5] from the first Born approximation. This choice of the upper limit gives the correct expression. 3) For production of muons or heavy particles, the nuclear form factor automatically insures the convergence of Eq. (9). However this choice of upper limit improves the numerical value by a few percent compared with the choice of $t_{max} = \infty$.

From all the equations given so far, we obtain finally the energy-angle distribution:

$$\frac{d\sigma}{d\Omega dp} = \frac{2\alpha^3}{\pi k}\left(\frac{E^2}{m^4}\right)\left[\frac{2x^2 - 2x + 1}{(1+\ell)^2} + \frac{4x(1-x)\,\ell}{(1+\ell)^4}\right]\chi \tag{10}$$

where

$$\chi = \frac{1}{2M_i}\int_{t_{min}}^{m^2(1+\ell)^2}\frac{dt}{t}\int_{M_i^2}^{(u-m)^2} dM_f^2 (t-t_{min})\, W_2 \quad , \tag{11}$$

$$x = E/k, \qquad \gamma = E/m, \qquad \text{and} \qquad \ell = \gamma^2\theta^2 \quad .$$

A. Electron Pair. In this case t_{min} is so small that only the atomic screening and the scattering from electrons in the atom are important while the nuclear form factors can be ignored. The atomic form factor for the scattering from a screened nuclear charge can be written as[6] (M_i = mass of nucleus)

$$W_2(\text{screened nucleus}) = 2M_i\,\delta\,(M_f^2 - M_i^2)\, Z^2 \,\frac{a^4 t^2}{(1+a^2 t)^2}$$

where a is the atomic radius given by $a = 111/(m_e Z^{1/3})$. The form factor for the scattering from atomic electrons screened by the nuclear charge can be written as

$$W_2(\text{atomic electrons}) = 2M_i\ \delta(M_f^2 - M_i^2)\, Z\, \frac{a'^4 t^2}{(1+a'^2 t)^2}$$

where $M_i = m_e$ and $a' = 1440 \times (2.718)^{-1/2}/(m_e Z^{2/3})$. Substituting

$$W_2 = W_2(\text{screened nucleus}) + W_2(\text{atomic electrons})$$

into Eq. (11), we obtain

$$\chi(\text{atom}) = Z^2 \left(\ln \frac{a^2 m^2 (1+\ell)^2}{a^2 t_{min} + 1} - 1 \right) + Z \left(\ln \frac{a'^2 m^2 (1+\ell)^2}{a'^2 t_{min} + 1} - 1 \right) \tag{12}$$

Integrating Eq. (10) with χ given by Eq. (12), we obtain an expression for $d\sigma/dp$ which agrees with the formulas given by Bethe and Ashkin[7] for complete screening ($a^2 t_{min} \ll 1$) and for no screening ($a^2 t_{min} \gg 1$) cases.

B. Muon Pair. In this case the magnitude of t_{min} is such that the presence of atomic electrons can be completely ignored but we must take into account the elastic nuclear form factor which can be written as:

$$W_2(\text{coherent}) = 2M_i\ \delta(M_f^2 - M_i^2)\, Z^2/(1+t/d)^2 \ ,$$

where $d = 6/(1.2 \text{ fermi } A^{1/3})^2 \approx 0.164\ A^{-2/3}\ \text{GeV}^2$. Substituting this W_2 into Eq. (11), we obtain

$$\chi(\text{coherent}) = Z^2 \left[(1+2b) \ln \frac{1+b^{-1}}{1+c^{-1}} - \left(1 - \frac{b}{c}\right) \frac{1+2c}{1+c} \right] , \tag{13}$$

where $b = t_{min}/d$ and $c = m^2(1+\ell)^2/d$. The numerical results of this equation for a Be nucleus and various masses of leptons are given in Table I.A. Compared with the exact Born approximation calculation, Eq. (1), which is about 100 times more complicated to handle, the agreements are impressive. From the total cross sections given in Table II, we see that the contributions from other processes are negligible for production of muons.

C. Heavy Lepton Pair. When $(t_{min})^{1/2}$ is larger than or comparable to the internucleon distance within the nucleus, we have to consider the incoherent production in addition to the coherent production given by Eq. (13). The form factors for the incoherent production consist of two parts: quasi-elastic and meson production. The suppression due to the Pauli exclusion principle is important in the quasi-elastic form factors but is negligible in the meson production. Using the dipole approximation, the elastic proton and neutron form factors can be written as: $(M_i = M_p)$

$$\begin{bmatrix} W^{el}_{1p} \\ W^{el}_{1p} \\ W^{el}_{2n} \\ W^{el}_{1n} \end{bmatrix} = \frac{2M_p\,\delta(M_f^2 - M_p^2)}{(1+t/.71)^4} \begin{bmatrix} (1+2.79^2\ t/4M_p^2)/(1+t/4M_p^2) \\ 2.79^2\ t/4M_p^2 \\ (1.91^2\ t/4M_p^2)/(1+t/4M_p^2) \\ 1.91^2\ t/4M_p^2 \end{bmatrix} \tag{14}$$

If we approximate the quasi-elastic bump by a δ function, then the quasi-elastic form factors from a nucleus can be written as

$$\begin{aligned} W_2^{\text{quasi-elastic}} &= C(t)\left[Z\,W^{el}_{2p} + (A-Z)\ W^{el}_{2n}\right] \\ W_1^{\text{quasi-elastic}} &= C(t)\left[Z\,W^{el}_{1p} + (A-Z)\ W^{el}_{1n}\right] \end{aligned} \tag{15}$$

where $C(t)$ is the Pauli suppression factor given by $C(t) = 1$ if $Q > 2P_F = 0.5$ GeV and

$$C(t) = \frac{3}{4}\,\frac{Q}{P_F}\left[1 - \frac{1}{12}\left(\frac{Q}{P_F}\right)^2\right] \quad \text{if } Q < 2P_F \ ,$$

where $Q^2 = t^2/(2M_p)^2 + t$. Omitting the Pauli suppression factor, the integrations in Eq. (11) can be carried out readily if we ignore the factor $(1+t/4M^2)^{-1}$. The numerical results are given in Table I.B. The agreement with the Born approximation calculation seem to be excellent. However when the target is a proton, one of the condition given in (6), i.e., E and $(k-E) \gg k\cdot p/M_i$, is not satisfied if m is large, θ is large and E or (k-E) is small. When this happens, the expression for t_{min} given by Eq. (7) can be a factor of two too small compared with the exact value, hence the exact expression for t_{min} was used in our calculation. Inspection of the Born approximation results show that W_1 can be as important as W_2 in such cases. Since we ignored W_1 in Eq. (9), the agreements between "Born" and "W.W." in Table I.B must be regarded as accidental under this condition.

In Table II, the results of the calculation using the Born approximation for the total cross section are given. The column "Be coherent" means the total cross section obtained by using Eq. (1) with W_2 given by W_2(coherent), Z=4 and A=9. Similarly for "proton elastic" and "neutron elastic" we use $(W^{el}_{1p}, W^{el}_{2p})$ and $(W^{el}_{1n}, W^{el}_{2n})$ respectively given by Eq. (14). For "Be quasi-elastic" we use Eq. (15). For "proton inelastic" we use the expressions of W_1 and W_2 given by suri and Yennie[8] which parameterize the results of SLAC-MIT ep inelastic scattering data rather well except in the resonance region, where the fits go smoothly through the average of the bumps. "Be total" is the sum "Be coherent" + "Be quasi-elastic" + 9 × "Be proton". We are surprised by the fact that "proton inelastic" was not very important compared with "proton elastic" even for the production of particles with m = 6 GeV. The reason is due to the expression for t_{min}, see Eq. (7), which suppresses the inelastic contributions. We hope Tables I and Table II would be useful for people who are planning to produce heavy leptons by photons.

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TABLE I

A. Coherent Production $d\sigma/dpd\Omega$ from Be

	m = 0.1056, k = 20, P = 8, 10^{-31} cm^2/GeV/sr		m = 0.5, k = 100, P = 40, 10^{-33} cm^2/GeV/sr		m = 4.0, k = 200, P = 80, 10^{-38} cm^2/GeV/sr		m = 6.0, k = 200, P = 80, 10^{-40} cm^2/GeV/sr	
$\gamma\theta$	Born	W.W.	Born	W.W.	Born	W.W.	Born	W.W.
0.0	1584	1543	1068	1069	1136	1162	1590	1682
0.6	1032	1088	686	695	337	348	360	387
1.2	310	317	174	173	15	15	11	12
1.8	84	81	38	37				

B. Elastic Production from a Proton $d\sigma/dpd\Omega$

	10^{-32} cm^2/GeV/sr		10^{-34} cm^2/GeV/sr		10^{-39} cm^2/GeV/sr		10^{-40} cm^2/GeV/sr	
$\gamma\theta$	Born	W.W.	Born	W.W.	Born	W.W.	Born	W.W.
0.0	1116	1022	950	952	6485	7112	2805	2341
0.6	728	745	619	663	3062	3183	586	472
1.2	231	242	181	188	271	212	0	0
1.8	70	71	47	47	2	1		

TABLE II

Total Heavy Lepton Production Cross Section

GeV k	Be Coherent	Proton Elastic	Neutron Elastic	Be-Quasi-Elastic	Proton Inelastic	Be Total
m = 0.105	10^{-30}	10^{-31}	10^{-33}	10^{-31}	10^{-33}	10^{-30}
20	1.611	1.267	1.546	1.081	6.114	1.774
40	2.047	1.551	1.557	1.134	6.336	2.238
100	2.579	1.926	1.563	1.171	6.044	2.750
200	2.787	2.177	1.565	1.184	5.683	2.956
m = 0.5	10^{-32}	10^{-33}	10^{-34}	10^{-33}	10^{-34}	10^{-32}
20	0.902	1.607	1.342	4.443	3.559	1.666
40	1.913	2.604	1.536	5.895	5.355	2.984
100	3.784	4.122	1.672	7.324	6.846	5.133
200	5.487	5.352	1.717	8.034	7.161	6.934
m = 1.0	10^{-33}	10^{-34}	10^{-35}	10^{-33}	10^{-34}	10^{-33}
20	0.170	0.923	1.958	0.410	0.288	0.839
40	0.797	2.293	3.070	0.814	0.728	2.266
100	3.014	5.063	4.014	1.358	1.343	5.578
200	5.857	7.698	4.442	1.703	1.664	9.057
m = 2.0	10^{-34}	10^{-35}	10^{-36}	10^{-34}	10^{-35}	10^{-34}
40	0.053	0.634	2.085	0.350	0.234	0.614
100	0.764	3.404	6.293	1.420	1.290	3.345
200	2.963	7.396	8.781	2.472	2.353	7.553
m = 4.0	10^{-36}	10^{-36}	10^{-37}	10^{-35}	10^{-36}	10^{-35}
100	0.243	0.371	1.498	0.223	0.140	0.374
200	2.856	2.758	7.990	1.432	1.131	2.735
m = 6.0	10^{-38}	10^{-38}	10^{-38}	10^{-37}	10^{-38}	10^{-36}
100	0.376	0.006	0.003	0.004	0	0
200	6.932	9.975	4.178	6.079	3.826	1.021

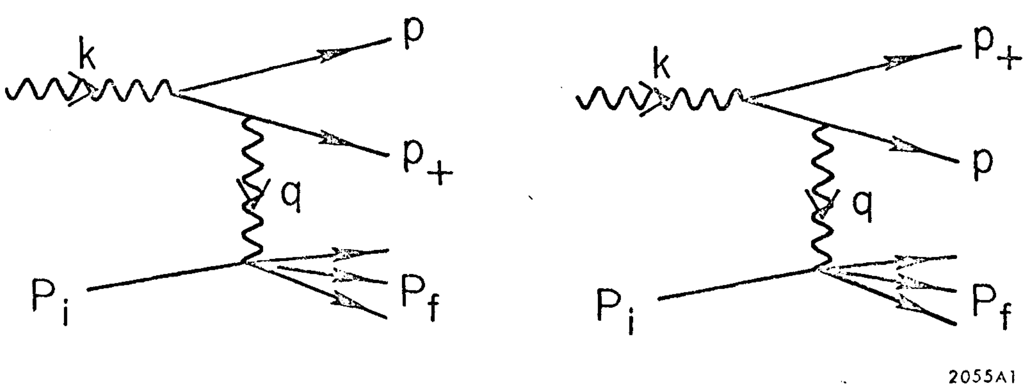

Fig. 1

Feynman diagram for production of spin 1/2 particles. For production of integer spin particles, a seagull diagram must be added.

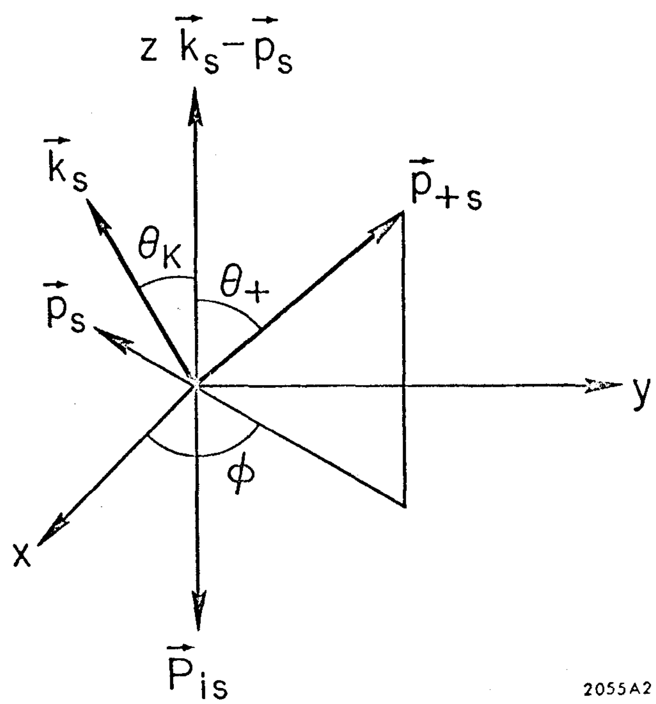

Fig. 2

A special frame used to integrate the unobserved lepton p_+. This is the rest frame of $p_+ + p_f$. In this frame $\vec{p}_{is}$ is antiparallel to $\vec{k}_s - \vec{p}_s$, both q^2 and M_f^2 are independent of ϕ.