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UNITARY MODELS OF MULTI-PARTICLE AMPLITUDES*

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ABSTRACT

A class of models of multi-particle scattering and production amplitudes is constructed for which the S-matrix is exactly unitary at high energies. Two specific models are studied in detail. One leads to a constant total cross-section, the other to a logarithmically increasing one. Particle production in inclusive and exclusive experiments is considered for both models.

I. INTRODUCTION

One of the central problems in strong interaction dynamics is to construct a realistic model of multi-particle scattering and production amplitudes. Certainly such a model must satisfy the constraints of multi-particle unitarity. Ideally one would like to construct models which automatically satisfy unitarity independent of any other physical input due to their structure.[1] As a first step in this direction we present a class of models for which the S-matrix elements satisfy all the multi-particle unitarity relations at high energies. To our knowledge this is the first example of a solvable multi-particle model with a unitary S-matrix. Although in some respects the present model is quite crude we believe that the ideas discussed here can be used to construct more sophisticated and hopefully more realistic models.

In Section II we combined ideas from the multiperipheral and eikonal models to write down a general S-matrix element in our model. Each matrix element is written in terms of a single function, W, associated with the creation or destruction of one secondary particle. In Section III we show that at high energies the S-matrix is unitary for a wide class of input functions, W. In Section IV we consider two specific forms for W. The first, which is suggested by the multiperipheral model, leads to a non-linear bootstrap equation for the elastic scattering amplitude. The solution to this equation corresponds to scattering from a black disc with a radius that grows logarithmically with energy. This form for the elastic amplitude has been arrived at recently in several different models.[2-5] The second simple form for W which we consider leads to a constant total cross-section, a non-shrinking diffraction peak for the elastic cross-section, and a multiplicity that increases logarithmically with energy. For both choices of W we give cross-sections for particle production in inclusive and exclusive experiments. In Section V we briefly discuss a generalization of the model based on the parton picture. Fragmentation effects, which are neglected in the earlier discussion, are included here. This model is also unitary and solvable. Section VI concludes with a brief resumé of our results.

II. DEFINITION OF THE MODEL

Let us start by defining the kinematics. A typical production diagram is shown in Figure 1. The particles whose momenta are labeled by p_a and p_b will be referred to as nucleons although we shall neglect spin and internal quantum numbers. The produced particles whose momenta are labeled by q_i will be referred to as pions. We work in the center of mass and take the z axis along the direction of the incoming particles. A general four-vector, q, will be written in terms of the transverse momentum, $\underset{\sim}{q}$, which is a two-dimensional vector in the x-y plane; and the longitudinal rapidity, y, defined by

$$y = \frac{1}{2} \ln[(q_o+q_z)/(q_o-q_z)] \ . \tag{1}$$

In particular

$$\begin{aligned} p_a &= m(\cosh y_a;\ 0,0,\ \sinh y_a) \\ p_b &= m(\cosh y_b;\ 0,0,\ \sinh y_b)\ , \end{aligned} \tag{2}$$

where m is the nucleon mass. At high energies

$$s = (p_a+p_b)^2 \simeq m^2\, e^{|y_a-y_b|} \equiv m^2\, e^Y \ . \tag{3}$$

It will usually be convenient to write our amplitudes as functions of the rapidity and impact parameters. The impact parameters are just the two-dimensional coordinates conjugate to the transverse momenta. For example, we write the elastic scattering amplitude in the form

$$M[Y,\tfrac{1}{2}(\underset{\sim}{p}'_a-\underset{\sim}{p}'_b)] = \int d^2B e^{i\frac{1}{2}(\underset{\sim}{p}'_a-\underset{\sim}{p}'_b)\cdot\underset{\sim}{B}} M(Y,\underset{\sim}{B})$$

$$\equiv 2is\int d^2B\, e^{i\frac{1}{2}(\underset{\sim}{p}'_a-\underset{\sim}{p}'_b)\cdot\underset{\sim}{B}} [1-e^{\frac{i}{2s}\delta(Y,\underset{\sim}{B})}] \,. \qquad (4)$$

The next step is to write down an explicit expression for the production amplitude shown in figure 1. As in the multiperipheral model, we assume that particles are produced from chains; however, unlike the multiperipheral model, we shall include diagrams with more than one chain. The reason for including the multi-chain diagrams is as follows. One starts by writing the amplitude for production from a single chain in terms of a product of two-body amplitudes. If the two-body amplitude has a large s behavior of the form s^a then so will the amplitude for production from a single chain. However, by investigating simple Feynman diagram models one sees that the amplitude for production from n chains will then have the asymptotic form $s^{1+n(a-1)}$. Since we are interested in diffraction scattering where $a\approx 1$, the multi-chain diagrams would appear to be important.

Our model for multi-chain exchange is based on an analogy with the relativistic eikonal model.[5,6] In the eikonal model one ordinarily considers the elastic scattering of two high energy particles. One finds that if the incident particles exchange objects such as vector mesons or QED towers, multiple exchanges are uncorrelated in impact parameter space.[7] We shall make the same assumption for our chains. The interested reader will easily construct some Feynman diagram models for which this assumption holds, and others for which it does not. It is well known that the eikonal model is only valid for classes of Feynman diagrams for which it is possible to neglect the fragmentation of the incident particles. As a result, our model of production amplitudes is expected to be applicable only to the pionization region. Since our "nucleons" are point-like particles which do not fragment, the reader may prefer to think of them as partons. We shall discuss ways of introducing fragmentation into the model in Section V.

The wavy lines in figure 1 correspond to direct exchanges between the nucleons. In the eikonal model each exchange contributes a factor of $\delta(Y,\underset{\sim}{B})$ to the amplitude. $\delta(Y,\underset{\sim}{B})$ is the two-dimensional Fourier transform of the amplitude for single exchange. The $i/2s$ in the exponential function in Eq. (4) is the usual eikonal factor for the propagation of the two nucleons between exchanges. Unlike the elastic amplitude, the production amplitudes have diagrams in which there are no direct exchanges between the nucleons. As a result, the total contributions of direct exchanges to the production amplitudes is just a multiplicative factor of the two-body S-matrix

$$S(Y,\underset{\sim}{B}) = e^{\frac{i}{2s}\,\delta(Y,\underset{\sim}{B})} \quad . \tag{5}$$

We now turn to the amplitude for the exchange of a single chain. Let us start by considering the case in which all sub-energies along the chain are large. In the multiperipheral model one ordinarily takes into account only interactions between nearest neighbors on the chain. The amplitude can then be written in terms of a product of two-body amplitudes. In the present model interactions between non-nearest neighbors can not be neglected. The eikonal model suggests that the interaction between each pair of non-nearest neighbors will give rise to a factor of the corresponding two body S-matrix. The S-matrices of course depend on the difference of rapidities between the interacting particles and on their separation in impact parameter space. In Section IV we shall see that for all values of the impact parameter for which there is appreciable scattering, the two body S-matrix goes to zero faster than any power of s as s becomes large. As a result of this very strong final state absorption, when all sub-energies are large, all chains are strongly suppressed except those from which only one particle is produced. Therefore, we shall include in our model only the simple chain shown in figure 2. For simplicity we ignore those chains from which several particles emerge with small relative energies. We plan to return to the interesting class of models generated by such chains at a later time.

Our model is now completely determined by specifying the amplitude for exchange of the single chain shown in figure 2. We denote this amplitude by $W(Y,\mathbf{B};y,\mathbf{b})$, where Y and $\mathbf{B}$ have been defined in Eqs. (3) and (4). The rapidity of the produced particle is y and $\mathbf{b}$ is the two-dimensional coordinate conjugate to its transverse momentum. If we denote the transverse distances between the produced pion and nucleons a and b by $\mathbf{b}_1$ and $\mathbf{b}_2$ respectively, then

$$\begin{aligned} \mathbf{B} &= \mathbf{b}_1 + \mathbf{b}_2 \\ \mathbf{b} &= \tfrac{1}{2}(\mathbf{b}_2 - \mathbf{b}_1) \ . \end{aligned} \tag{6}$$

The multiperipheral model suggests that we write W as a product of two-body amplitudes

$$W(Y,\mathbf{B};y,\mathbf{b}) = gM(y_a - y, \tfrac{1}{2}\mathbf{B} - \mathbf{b})M(y - y_b, \tfrac{1}{2}\mathbf{B} + \mathbf{b}), \tag{7a}$$

or in momentum space

$$\begin{aligned} W(Y,\tfrac{1}{2}(\mathbf{p}_a' - \mathbf{p}_b');y,\mathbf{q}) &= \int d^2b\, d^2B\, e^{i\mathbf{q}\cdot\mathbf{b}}\, e^{i\frac{1}{2}(\mathbf{p}_a' - \mathbf{p}_b')\cdot\mathbf{B}} \\ &\quad \cdot W(Y,\mathbf{B};y,\mathbf{b}) \\ &= gM(y_a - y, \mathbf{p}_a' - \tfrac{1}{2}\mathbf{q})M(y - y_b, \tfrac{1}{2}\mathbf{q} - \mathbf{p}_b') \ . \end{aligned} \tag{7b}$$

However, for much of our discussion it will not be necessary to specify the functional form of W.

There is one restriction which we place on W for the sake of consistency. In the eikonal model, exchanges between the incident nucleons are uncorrelated in impact parameter space only if the nucleons retain a large fraction of their incident momenta. This can be guaranteed by requiring W to vanish unless y lies in the range

$$(1-\epsilon)y_b \le y \le (1-\epsilon)y_a \tag{8}$$

where ϵ is an arbitrarily small number. In order for the model to be internally consistent most of the production should take place in the pionization region, so our results should be essentially independent of ϵ. This will indeed turn out to be the case.

We are now in a position to write down an arbitrary S-matrix element in our model. We denote by M_{nm} the amplitude to go from an initial state with m pions and two nucleons to a final state with n pions and two nucleons. The diagram corresponding to M_{no} is shown in figure 3a. It should be emphasized that the diagrams we are drawing each correspond to a sum of Feynman graphs. The sum being over all possible ways of attaching the legs of the exchanged objects to the nucleon lines. In impact parameter space we have

$$\begin{aligned} M_{no}&(Y,\underset{\sim}{B};y_1,\underset{\sim}{b}_1,\ldots y_n,\underset{\sim}{b}_n) \\ &= S(Y,\underset{\sim}{B})(i/2s)^{n-1}\prod_{j=1}^{n} W(Y,\underset{\sim}{B};y_j,\underset{\sim}{b}_j) \quad . \end{aligned} \tag{9}$$

The factors of $i/2s$ arise from the propagation of the nucleons between exchanges of the chains. The corresponding factors associated with the direct exchanges are contained in the factor $S(Y,\underset{\sim}{B})$ as explained before. The momentum space amplitude can be obtained by taking two-dimensional Fourier transforms.

$$M_{no}(Y, \tfrac{1}{2}(\underset{\sim}{p}_a' - \underset{\sim}{p}_b');\ y_1, \underset{\sim}{q}_1, \ldots y_n, \underset{\sim}{q}_n)$$

$$= (i/2s)^{n-1} \int d^2B\ e^{i\frac{1}{2}(\underset{\sim}{p}_a' - \underset{\sim}{p}_b')\cdot \underset{\sim}{B}}\ S(Y, \underset{\sim}{B}) \cdot \prod_{j=1}^{n} W(Y, \underset{\sim}{B}; y_j, \underset{\sim}{q}_j)\ , \tag{10}$$

with

$$W(Y, \underset{\sim}{B}; y_j, \underset{\sim}{q}_j) \equiv \int d^2b_j\ e^{i\underset{\sim}{q}_j\cdot\underset{\sim}{b}_j}\ W(Y, \underset{\sim}{B}; y_j, \underset{\sim}{b}_j)\ . \tag{11}$$

If parity and time reversal are good symmetries then the S-matrix must be symmetric. As a result, we must associate a factor of W with the destruction of each incident pion. The diagram associated with the connected amplitude M_{nm} is shown in figure 3b. In impact parameter space we have

$$M_{nm}(Y, \underset{\sim}{B}; y_1', \underset{\sim}{b}_1', \ldots y_m', \underset{\sim}{b}_m';\ y_1, \underset{\sim}{b}_1, \ldots y_n, \underset{\sim}{b}_n)$$

$$= S(Y, \underset{\sim}{B})\,(i/2s)^{n+m-1} \prod_{j=1}^{m} W(Y, \underset{\sim}{B}; y_j', \underset{\sim}{b}_j') \cdot \prod_{k=1}^{n} W(Y, \underset{\sim}{B}; y_k, \underset{\sim}{b}_k)\ . \tag{12}$$

In momentum space the obvious generalization of Eq. (10) holds.

The elastic amplitude $M_{oo} \equiv M$, is given by Eq. (4). In addition to the connected amplitudes defined by Eqs. (4), (10) and (12) there are also disconnected ones. However, since all interactions in our model involve the participation of both nucleons, only pions can be disconnected. As a result, the general disconnected amplitude is just M_{nm} multiplied by momentum conservation delta functions for the non-interacting pions.

III. UNITARITY

Having written down the most general S-matrix element in our model, we shall now show that S is unitary. We write the differential phase space volume element for n identical pions and two nucleons as

$$d\Phi_n = \frac{1}{n!} \prod_{i=1}^{n} \frac{d^2q_i}{(2\pi)^2} \frac{dy_i}{4\pi} \frac{d^2p_a'}{(2\pi)^2} \frac{dy_a'}{4\pi} \frac{d^2p_b'}{(2\pi)^2} \frac{dy_b'}{4\pi}$$

$$\cdot \frac{1}{2} (2\pi)^4 \delta^4(P-p_a'-p_b' - \sum_{i=1}^{n} q_i) . \tag{13}$$

After making the change of variables

$$\begin{aligned} \underset{\sim}{P}' &= \underset{\sim}{p}_a' + \underset{\sim}{p}_b' \\ \underset{\sim}{p}' &= \frac{1}{2}(\underset{\sim}{p}_a' - \underset{\sim}{p}_b') , \end{aligned} \tag{14}$$

we can perform the y_a', y_b', and $\underset{\sim}{P}'$ integrations by making use of the momentum conservation delta function. We then have

$$d\Phi_n = \frac{1}{n!} \frac{1}{4s} \prod_{i=1}^{n} \frac{d^2q_i}{(2\pi)^2} \frac{dy_i}{4\pi} \cdot \frac{d^2p}{(2\pi)^2} . \tag{15}$$

Notice that the restriction imposed on the y_i by Eq. (8) allows us to drop the variables q_{iz} and $E_i = q_{io}$ from the argument of the delta functions provided $n << (s/m^2)^{\epsilon}$. For all forms of the function

W that we have investigate, the average multiplicity grows only like a power of ℓns so this approximation is valid. As a result, the y_i may be taken to be independent, and Eq. (8) remains the only restriction on their range of integration.

Let us start by considering the elastic amplitude. It is convenient to work in $\underset{\sim}{B}$ space so we write

$$\begin{aligned} \mathrm{ImM}(Y,\underset{\sim}{B}) &\equiv \int \frac{d^2p}{(2\pi)^2}\, e^{i\underset{\sim}{p}\cdot\underset{\sim}{B}}\ \mathrm{ImM}(Y,\underset{\sim}{p}) \\ &= \frac{1}{4s}\,|M(Y,\underset{\sim}{B})|^2 + \sum_{n=1}^{\infty} |M_{no}(Y,\underset{\sim}{B};y_1,\underset{\sim}{b}_1,\ldots y_n,\underset{\sim}{b}_n)|^2 d\varphi_n, \end{aligned} \tag{16}$$

where

$$d\varphi_n = \frac{1}{n!}\ \frac{1}{4s} \prod_{i=1}^{n} d^2b_i\ dy_i/4\pi\ . \tag{17}$$

Making use of Eq. (10) we see that

$$\begin{aligned} \mathrm{ImM}(Y,\underset{\sim}{B}) &= \frac{1}{4s}\ |M(Y,B)|^2 + s\sum_{n=1}^{\infty} \frac{1}{n!}\ |S(Y,B)|^2\ C(Y,B)^n \\ &= \frac{1}{4s}\ |M(Y,B)|^2 + s|S(Y,B)|^2[e^C-1] \end{aligned} \tag{18}$$

with

$$C(Y,\underset{\sim}{B}) \equiv \frac{1}{16\pi s^2}\int d^2b dy|W(Y,\underset{\sim}{B};y,\underset{\sim}{b})|^2. \tag{19}$$

Since we are interested in diffraction scattering we expect the elastic amplitude to be pure imaginary. It is therefore convenient to write

$$\frac{i}{2s}\ \delta(Y,\underset{\sim}{B}) \equiv -\ A(Y,\underset{\sim}{B}) \tag{20}$$

so that

$$M(Y,\underset{\sim}{B}) = 2is[1-e^{-A}] \ . \tag{21}$$

Notice that A is actually a function of B^2 rather than $\underset{\sim}{B}$ from rotational invariance. Substituting Eq. (20) into both sides of Eq. (18) and assuming that A is real gives

$$2s[1-e^{-A}] = s[1-e^{-A}]^2 + s\ e^{-2A}[e^{C}-1] \ . \tag{22}$$

Thus, the elastic amplitude satisfies the multiparticle unitarity condition exactly provided

$$A(Y,\underset{\sim}{B}) = \tfrac{1}{2}C(Y,\underset{\sim}{B}) = \frac{1}{32\pi s^2}\int d^2b dy\,|W(Y,\underset{\sim}{B};y,\underset{\sim}{b})|^2 \ . \tag{23}$$

This verifies our earlier assertion that all amplitudes in the model are completely determined once the function W is given.[8]

Let us now turn to the unitarity equation for the production amplitude M_{no}. Here the unitarity sum has contributions from disconnected graphs, and one must count them carefully.

We denote the full m to n amplitude including disconnected graphs by $\overline{M}_{nm}$. The contribution to $\overline{M}_{nm}$ in which k of the pions are disconnected is

$$\Sigma\, M_{n-k,m-k}(Y,\underset{\sim}{p};y'_{i_{k+1}},\underset{\sim}{g}'_{i_{k+1}},\ldots y'_{i_m},\underset{\sim}{g}'_{i_m};y_{j_{k+1}},\underset{\sim}{g}_{j_{k+1}},\ldots y_{j_n}\underset{\sim}{g}_{j_n})$$

$$\cdot \langle q'_{i_1}\ldots q'_{i_k}|q_{j_1}\ldots q_{j_k}\rangle \ . \tag{25}$$

In Eq. (25) $i_1\ldots i_m$ take on the values $1,2,\ldots$ m and $j_1,\ldots j_n$ the values 1,2,...n. The sum is over all distinct partitions of 1,...m into two groups of size k and m-k and of 1,...n into two groups of size k and n-k. Clearly there are a total of $\frac{n!}{k!(n-k)!} \cdot \frac{m!}{k!(m-k)!}$ terms. $|q_1 \ldots q_k\rangle$ is a state of k identical pions. Our normalization is

$$\langle q_1' \ldots q_k' | q_1 \ldots q_k \rangle$$

$$= \sum_{P(i_1 \ldots i_k)} 2(2\pi)^3 \delta(y_{i_1}' - y_1) \delta(\underset{\sim}{g}_{i_1}' - \underset{\sim}{g}_1) 2(2\pi)^3 \delta(y_{i_2} - y_2) \delta(\underset{\sim}{g}_{i_2}' - \underset{\sim}{g}_2)$$

$$\ldots 2(2\pi)^3 \delta(y_{i_k}' - y_k) \delta(\underset{\sim}{g}_{i_k}' - \underset{\sim}{g}_k) \ , \tag{26}$$

where $\sum_{P(i_1 \ldots i_k)}$ indicates that $i_1 \ldots i_k$ are to be summed over all permutations of 1,2,...k.

The unitarity equation for the amplitude M_{no} is

$$\text{ImM}_{no} = \sum_{N=0}^{\infty} \int d\Phi_N \, \overline{M}_{nN}^{\,*} \, M_{No}$$

$$= \sum_{N=0}^{\infty} \sum_{k=0}^{[n,N]} \int d\Phi_{N-k} \, M_{n,N-k}^{\,*} \, M_{No} \frac{n!}{(n-k)!k!} \tag{27}$$

Here [n,N] means the lesser of n and N. In the last step of Eq.(27) we have made use of the fact that the sum of Eq. (25) gives rise to $\frac{n!}{k!(n-k)!} \cdot \frac{m!}{k!(n-k)!}$ terms in Eq. (27) which are numerically equal.

It is again most convenient to work in impact parameter space. Denoting by ImM_{no}^{N} the contribution to ImM_{no} coming from the intermediate state containing N pions we find for $N \neq 0$

$$\mathrm{Im}M^N_{no}(Y,B;y_1,\underset{\sim}{b}_1,\ldots) = \left(\frac{-i}{2s}\right)^n W(1)\ldots W(n)e^{-2A}$$

$$\cdot\ s\sum_{k=0}^{[n,N]} \frac{C^{N-k}}{(N-k)!} \quad (-)^k \frac{n!}{k!(n-k)!}$$

$$+\ \delta_{n,N}\frac{1}{2i} M_{no} \quad , \tag{28}$$

and for N=0

$$\mathrm{Im}M^o_{no} = \left(\frac{-i}{2s}\right)^n W(1)\ldots W(n)se^{-2A} - \frac{i}{2s}M^*_{no} \ . \tag{29}$$

Here we have written

$$W(i) \equiv W(Y,\underset{\sim}{B};y_i,\underset{\sim}{b}_i) \ . \tag{30}$$

As a result,

$$\begin{aligned}\mathrm{Im}M_{no} &= \frac{1}{2i}[M_{no} - M^*_{no}] \\ &+ \left(\frac{-i}{2s}\right)^n W(1)\ldots W(n)e^{-2A}s\left[\sum_{N=0}^{\infty}\sum_{k=0}^{[n,N]} \frac{C^{N-k}}{(N-k)!}\frac{(-)^k\, n!}{k!(n-k)!} + 1\right] \\ &= \mathrm{Im}M_{no} + \left(\frac{-i}{2s}\right)^n W(1)\ldots W(n)e^{-2A}\ s\sum_{k=0}^{n}\frac{(-)^k\, n!}{k!(n-k)!}\sum_{N'=0}^{\infty}\frac{C^{N'}}{N'!} \\ &= \mathrm{Im}M_{no} \quad ,\end{aligned} \tag{31}$$

since $\displaystyle\sum_{k=0}^{n}\frac{(-)^k\, n!}{k!(n-k)!} = 0$.

Using precisely the same method one finds after slightly more algebra, that the general connected amplitude M_{nm} also satisfies unitarity identically. So, our S-matrix is exactly unitary. The model is definitely non-trivial in the sense that there is a finite contribution to $\mathrm{Im}M_{nm}$ from intermediate states with arbitrary numbers of pions. There is to be sure a rather amazing cancellation between contributions from intermediate states with different numbers of pions. This is illustrated by the fact that in Eq. (30) the coefficient of each power of C vanishes. However, one knows that in any model which satisfies unitarity exactly there must be strong correlations between states containing different numbers of particles.[9]

We have written out the verification of unitarity in detail in order to emphasize the strong correlation between states with different numbers of particles. However, the discussion of unitarity and of other properties of the model can be made more concise by writing the S-matrix in operator form. This formulation also provides a natural way to generalize the model to allow more pions to be emitted from each chain.

The operator S-matrix will be written in the explicitly unitary form

$$S = e^{i(\chi^{+}+\chi)} , \tag{32}$$

where

$$\chi = \frac{1}{8\pi s} \int d^2b dy \quad W(Y,\underset{\sim}{B};y,\underset{\sim}{b}) \quad a(y,\underset{\sim}{b}) \tag{33}$$

and χ^+ has a replaced by a^+. As always, W is real. The operator $a^+(y,\underset{\sim}{b})$ creates a pion with impact parameter $\underset{\sim}{b}$ and rapidity y, while $a(y,\underset{\sim}{b})$ destroys such a pion. The commutation relation in our normalization is

$$[a(y,\underset{\sim}{b}),\ a^+(y',\underset{\sim}{b}')] = 4\pi\delta(y-y')\delta^2(\underset{\sim}{b}-\underset{\sim}{b}') \ . \tag{34}$$

Matrix elements of the scattering operator, M, defined by

$$M = 2is(1-S) \ , \tag{35}$$

are to be taken between states labeled by the pion coordinates and normalized according to

$$|y_1,\underset{\sim}{b}_1,\ldots y_n \underset{\sim}{b}_n\rangle = a^+(y_1,\underset{\sim}{b}_1)\ldots a^+(y_n,\underset{\sim}{b}_n)|0\rangle \ . \tag{36}$$

In evaluating matrix elements of M it is convenient to write S in normal ordered form:

$$S = e^{i\chi^+} e^{-A(Y,\underset{\sim}{B})} e^{i\chi} \ , \tag{37}$$

where A is again given by Eq. (23). The connected part of the $m \to n$ amplitude is easily seen to be

$$M_{nm} = e^{-A(Y,B)}(i/2s)^{n+m-1} \prod_{j=1}^{m} W(j) \prod_{k=1}^{n} W(k) \tag{38}$$

which is identical to Eq. (12) provided $S(Y,\underset{\sim}{B})$ is given by Eqs. (5) and (23). The impact parameter space analogue of Eq. (25) is also easily read off. One now sees without any calculation that the scattering amplitudes all satisfy unitarity since the S-matrix is defined in terms of a unitary operator.

IV. PROPERTIES OF THE MODEL

Our model is obviously too crude to withstand a close comparison with experiment; nevertheless, it is amusing to extract some of its simpler properties. As we have stressed, unitarity is satisfied for any choice of W; however, to obtain specific results, one must commit himself to a definite form for this function. In this section we shall consider two very simple forms for W. One leads to a bootstrap equation for the elastic amplitude; the other to a model which is in agreement with present trends in the high energy data.

Let us start by considering the bootstrap problem. Guided by the multiperipheral model we write W as a product of two-body amplitudes as in Eq. (7). For simplicity we take the vertex function, g, to be a constant. Combining Eqs. (7), (21) and (23) we obtain a non-linear bootstrap equation for the elastic amplitude:

$$1 + \frac{i}{2s} M(Y,\underset{\sim}{B}) = e^{-A}$$

$$= \exp\left[\frac{-g^2}{32\pi s^2} \int d^2b\,dy \left| M(\tfrac{1}{2}Y-y;\tfrac{1}{2}\underset{\sim}{B}-\underset{\sim}{b}) \right|^2 \cdot \left| M(\tfrac{1}{2}Y+y;\tfrac{1}{2}\underset{\sim}{B}+\underset{\sim}{b}) \right|^2 \right] . \tag{39}$$

In the center of mass one has $y_a = - y_b = \frac{1}{2} Y$. At high energies an approximate solution to the bootstrap equation can be obtained by iteration. Using the input

$$M_{in}(Y,\underset{\sim}{B}) = 2im^2 e^Y \theta(R_o Y-B), \tag{40}$$

where $B = |\underset{\sim}{B}|$, then A is given by

$$A(Y,\mathbf{B}) = \frac{g^2m^4}{2\pi} \int d^2b\,dy\,\theta[R_o(\tfrac{1}{2}Y-y) - |\tfrac{1}{2}\mathbf{B}-\mathbf{b}|]$$

$$\cdot\,\theta[R_o(\tfrac{1}{2}Y+y)-|\tfrac{1}{2}\mathbf{B}+\mathbf{b}|]$$

$$= \frac{g^2m^4}{2\pi R_o} \int d^2b_1 d^2b_2 \delta^2(\mathbf{B}-\mathbf{b}_1-\mathbf{b}_2)[R_oY-b_1-b_2]$$

$$\cdot\,\theta[R_oY-b_1-b_2]$$

$$= c(R_o{}^2Y^2-B^2)^{3/2} \;\theta(R_oY-B) \;, \tag{41}$$

with $c = g^2m^4/24R_o$.[10] Then, as $s \to \infty$,

$$M_{out}(Y,\mathbf{B}) = M_{in}(Y,B)[1-e^{-A}] \tag{42}$$

As $s \to \infty$, e^{-A} goes to zero faster than any power of $1/s$ unless $B \approx R_oY$.[11] Thus, the bootstrap is complete for all values of B except those lying in a ring of radius R_oY whose width is of order $c^{-2/3}(R_oY)^{-1}$. The area of this ring is independent of energy in the asymptotic region. It is therefore a negligible fraction of the total interaction area which is $\pi(R_oY)^2$. Let us imagine performing another iteration of Eq. (39) using the amplitude of Eq. (42) as input. Notice that A will again vanish for $B > R_oY$. For $B < R_oY$ the fractional change in A will clearly be at most of order $1/Y^2$ except in the troublesome "grey ring." Thus, for most applications, Eqs. (41) and (42) can be regarded as a satisfactory solution of the boostrap problem.[12]

The amplitude of equation (42) corresponds to scattering from a black disc with a narrow grey edge. In the asymptotic region the radius of the disc grows like ℓns. This type of behavior for the elastic amplitude has been arrived at recently in several different models.[2-5] The results that can be read off directly from the form of the elastic amplitude are well known. For the total and elastic cross-sections one has

$$\sigma_{tot} = 2\sigma_{e\ell} = 2\pi (R_o Y)^2 , \tag{43}$$

so the Froissart bound is saturated. Writing M in terms of the invariant momentum transfer, t, we have

$$M(Y,t) = 2\pi is(R_o Y)^2 \cdot \left[\frac{J_1(R_o Y \sqrt{-t})}{\frac{1}{2} R_o Y \sqrt{-t}} \right] , \tag{44}$$

which corresponds to a leading ℓ-plane singularity of the form $[(\ell-1)^2 - R_o{}^2 t]^{-3/2}$. Logarithmic growth of total cross-sections is by no means ruled out by present data. However, Eq. (44) indicated that the width of the diffraction peak of the elastic cross-section shrinks like $(1/\ell ns)^2$. This is in even greater disagreement with the most recent p-p data than the simple Regge pole model. Of course one can avoid this difficulty by taking R_o to be small.[13]

We now turn to the single particle inclusive distributions. The distribution function, ρ, is given by

$$\rho(Y;y,\underset{\sim}{q}) \equiv \frac{d\sigma}{d^3q/E_q} = \frac{1}{2(2\pi)^3}\frac{1}{4s^2}\int\frac{d^2p}{(2\pi)^2}$$

$$\cdot\sum_{n=0}^{\infty}\frac{1}{n!}\,|\langle 0|M|y,\underset{\sim}{q},y_1,\underset{\sim}{q}_1,\ldots y_n,\underset{\sim}{q}_n\rangle|^2$$

$$\cdot\prod_{i=1}^{n}\frac{dy_i d^2q_i}{2(2\pi)^3}\ . \tag{45}$$

Writing

$$a(y,\underset{\sim}{q}) = \int d^2b e^{i\underset{\sim}{q}\cdot\underset{\sim}{b}}\ a(y,\underset{\sim}{b})\ , \tag{46}$$

we have

$$\rho(Y;y,\underset{\sim}{q}) = \frac{1}{2(2\pi)^3}\int\frac{d^2p}{(2\pi)^2}\sum_{n=0}^{\infty}\frac{1}{n!}\ \langle 0|S\ a^+(y,\underset{\sim}{q})|n\rangle$$

$$\cdot\langle n|a(y,\underset{\sim}{q})S^+|0\rangle\ \prod_{i=1}^{n}\frac{dy_i d^2q_i}{2(2\pi)^3}$$

$$= \frac{1}{2(2\pi)^3}\int\frac{d^2p}{(2\pi)^2}\ \langle 0|[a^+(y,\underset{\sim}{q}) + \frac{i}{2s}W(Y,\underset{\sim}{p};y,\underset{\sim}{q})]$$

$$\cdot SS^+[a(y,\underset{\sim}{q}) - \frac{i}{2s}W(Y,\underset{\sim}{p};y,\underset{\sim}{q})]|0\rangle$$

$$= \frac{1}{8s^2(2\pi)^3}\int d^2B|W(Y,\underset{\sim}{B};y,\underset{\sim}{q})|^2. \tag{47}$$

This result, should be contrasted with the exclusive cross-section for the production of a single pion.

$$\rho_{ex}(Y;y,\underset{\sim}{q}) = \frac{1}{2(2\pi)^3}\int\frac{d^2p}{(2\pi)^2}\ |\langle 0|S|y,\underset{\sim}{q}\rangle|^2$$

$$= \frac{1}{2(2\pi)^3}\int\frac{d^2p}{(2\pi)^2}\ |\langle 0|[a^+(y,\underset{\sim}{q}) + \frac{i}{2s}W(Y,\underset{\sim}{p};y,\underset{\sim}{q})]S|0\rangle|^2$$

$$= \frac{1}{8s^2(2\pi)^3}\int d^2B|W(Y,\underset{\sim}{B};y,\underset{\sim}{q})|^2 e^{-2A(Y,\underset{\sim}{B})}\ . \tag{48}$$

Eqs. (47) and (48) of course hold for any choice of W. Adopting the bootstrap solution defined by Eqs. (41) and (42) we see that in the exclusive experiment virtually all of the pions are produced peripherally, since the absorption factor, e^{-2A}, vanishes faster than any power of 1/s except in the "grey ring." On the other hand the inclusive cross-section is free of the absorption factor because of the summation over the unobserved particles. Here the pions are produced uniformly from the entire area of the interaction disc, $B \leq R_o Y$.

The inclusive cross-section takes on a particularly simple form if one integrates over the transverse momentum of the detected pion. Then

$$\frac{d\sigma}{dy} = \frac{1}{16\pi s^2} \int d^2B d^2b \left| W(Y,B;y,\underset{\sim}{b}) \right|^2$$

$$= \pi g^2 (mR_o)^4 \left(\tfrac{1}{4}Y^2 - y^2\right)^2 \quad . \tag{49}$$

Clearly the distribution is peaked at y=0; the width and height of the peak increase with energy like Y and Y^4 respectively. This should be contrasted with the flat distribution in rapidity which arises if the Pomeron is a simple Regge pole. The average multiplicity can be read off from Eq. (49).

$$\bar{n} = \frac{1}{\sigma_{tot}} \int_{-\frac{1}{2}Y}^{\frac{1}{2}Y} \frac{d\sigma}{dy} dy = \frac{2}{5} c(R_o Y)^3 \tag{50}$$

The unintegrated inclusive cross-section is a bit more complicated. We have

$$\rho(Y;y,\underset{\sim}{q}) = \frac{g^2 m^4}{16\pi} [R_o^{\,2}(\tfrac{1}{4}Y^2-y^2)]^4 \tag{51}$$

$$\cdot \int d^2p \left[\frac{J_1[R_o(\frac{1}{2}Y-y)|\underset{\sim}{p}-\frac{1}{2}\underset{\sim}{q}|]}{\frac{1}{2}R_o(\frac{1}{2}Y-y)|\underset{\sim}{p}-\frac{1}{2}\underset{\sim}{q}|} \cdot \frac{J_1[R_o(\frac{1}{2}Y+y)|\underset{\sim}{p}+\frac{1}{2}\underset{\sim}{q}|]}{\frac{1}{2}R_o(\frac{1}{2}Y+y)|\underset{\sim}{p}+\frac{1}{2}\underset{\sim}{q}|} \right]^2 .$$

For small q (of order $q \gtrsim \frac{3}{2}/R_oY$), one can make use of the fact that

$$J_1(x)/\tfrac{1}{2}x \simeq \exp[-\tfrac{1}{8}x^2] \qquad 0 \leq x \lesssim 3 \tag{52}$$

to write

$$\rho(Y;y,\underset{\sim}{q}) \simeq \frac{1}{4} g^2 m^4 \cdot \frac{[R_o^{\,2}(\frac{1}{4}Y^2-y^2)]^4}{2R_o^{\,2}(\frac{1}{4}Y^2+y^2)}$$

$$\cdot \exp\left[-\frac{1}{4}\underset{\sim}{q}^2 \cdot \frac{[R_o^{\,2}(\frac{1}{4}Y^2-y^2)]^2}{2R_o^{\,2}(\frac{1}{4}Y^2+y^2)}\right] . \tag{53}$$

Clearly ρ is very sharply peaked about $\underset{\sim}{q} = 0$.

Finally, we consider the exclusive cross-section for the production of n pions.

$$\sigma_n = \frac{1}{n!}\int d^2B |\langle 0|S|y_1,\underset{\sim}{b}_1,\ldots y_n,\underset{\sim}{b}_n\rangle|^2 \prod_{i=1}^{n} \frac{dy_i}{4\pi} d^2b_i$$

$$= \frac{1}{n!}\int d^2B e^{-2A(Y,\underset{\sim}{B})} [\frac{1}{16\pi s^2}\int dy d^2b |W(Y,\underset{\sim}{B};y,\underset{\sim}{b})|^2]^n$$

$$= \frac{1}{n!}\int d^2B e^{-2A(Y,\underset{\sim}{B})} [2A(Y,\underset{\sim}{B})]^n$$

$$\equiv \int d^2B\ \sigma_n(B) . \tag{54}$$

For fixed n, $\sigma_n(B)$ is peaked at a radius B_M given by

$$B_M^2 = (R_oY)^2 \; [1 - (\frac{2}{5}\frac{n}{\bar{n}})^{2/3}] \; . \tag{55}$$

So, for small n virtually all of the pions are produced peripherally. However, as n increases more and more pions are produced in the central region. For $n > \frac{5}{2}\bar{n}$ there is no maximum of $\sigma_n(B)$ inside the interaction disc. The partial cross-sections are negligible for values of n this large. For $n \leqslant \frac{5}{2}\bar{n}$ we have

$$\sigma_n = \frac{\pi}{n!} \int_0^{R_o^2Y^2} dB^2 e^{-2c(R_o^2Y^2-B^2)^{3/2}} [2c(R_o^2Y^2-B^2)^{3/2}]^n$$

$$\simeq \frac{\pi}{n!}\frac{2}{3} \cdot (2c)^{-2/3} \; \Gamma(n + 2/3) \; . \tag{56}$$

One is certainly not forced to adopt the bootstrap model. Interestingly enough, there is a very simple choice for W which leads to a model that is in substantial agreement with present trends in the high energy data. We write

$$W(Y,\underset{\sim}{B};y,\underset{\sim}{b}) = sg\theta[\tfrac{1}{2}R_o-|\tfrac{1}{2}\underset{\sim}{B}-\underset{\sim}{b}|]\theta[\tfrac{1}{2}R_o-|\tfrac{1}{2}\underset{\sim}{B}+\underset{\sim}{b}|] \; . \tag{57}$$

Then

$$A(Y,\underset{\sim}{B}) = \frac{g^2}{32\pi} Y \; f(B)\theta(R_o-B) \; , \tag{58}$$

where

$$f(B) = \tfrac{1}{2}[R_o^2 \cos^{-1}(B/R) - B(R_o^2-B^2)^{\frac{1}{2}}] \; . \tag{59}$$

Notice that for $B \leqslant R_o$, f(B) is monotonically decreasing, and that

$\lim_{B\to R} f(B) = \frac{1}{2} R_o^{-1}(R_o^2-B^2)^{3/2}$. As a result, e^{-A} goes to zero for large Y everywhere inside the disc $B \leq R_o$, except for a ring of radius R_o and width of order $Y^{-2/3}$. Since the area of this grey ring goes to zero for large Y, the two body amplitude becomes

$$M(Y,\underset{\sim}{B}) \simeq 2is\theta(R_o-B) \ , \tag{60}$$

which corresponds to scattering from a black disc with a constant radius. The total and elastic cross-sections are now given by

$$\sigma_{tot} = 2\sigma_{e\ell} = 2\pi R_o^2 \ . \tag{61}$$

Clearly at very high energies the width of the diffraction peak for elastic scattering is independent of energy.

The single particle inclusive distribution function $\rho(Y;y,\underset{\sim}{q})$ can be read off from Eq. (47). There are two interesting features to notice. First, ρ is independent of Y and y. As a result, there is a central plateau in the rapidity plot, i.e. $\frac{d\sigma}{dy}$ is a constant except at the edges. The average multiplicity increases like $Y = \ell n(s/m^2)$. To be explicit

$$\bar{n} = \frac{1}{2} Y \left(\frac{gR_o}{16}\right)^2 \ . \tag{62}$$

Second, ρ is sharply peaked in the transverse momentum about $\underset{\sim}{q} = 0$. In the present case, the width of the peak is independent of energy.

The exclusive cross-sections, σ_n, are again given by Eq. (53).

For $n \ll \bar{n}$, the pions are produced peripherally, i.e. at a radius $B \approx R_o$, and $\sigma_n \sim Y^{-2/3}$. For $n \approx \bar{n}$ the pions are produced in the central region of the disc $0 \leq B < R_o$. In this case $\sigma_n = c/Y$, with c independent of n. It is the contribution from this range of n that gives rise to the constant total cross-section. Finally for $n \gg \bar{n}$, the σ_n decreases exponentially with Y.

It has recently been suggested that a study of relative transverse momentum variables can yield information on the transverse spatial structure of reactions and hence of the production mechanism.[14] In this model, the distribution in the relative transverse momentum between the nucleons is easily seen to be

$$\frac{d\sigma}{dp^2} = \frac{1}{(2\pi)^2} \int d^2B d^2B' e^{i\underset{\sim}{p}\cdot(\underset{\sim}{B}-\underset{\sim}{B}')} e^{-A(Y,\underset{\sim}{B})-A(Y,\underset{\sim}{B}')} G(Y,\underset{\sim}{B},\underset{\sim}{B}') \ . \tag{63}$$

The p^2 distribution is directly seen to explore absorption and its effect on the spatial structure of the total imput parameter B. For the exclusive reaction in which only one extra pion is produced, one finds that $G = G_1$, where

$$G_1(Y,B,B') = \frac{1}{16\pi s^2} \int d^2b dy W(Y,\underset{\sim}{B};y,\underset{\sim}{b}) W(Y,B';\underset{\sim}{y},\underset{\sim}{b}) \tag{64}$$

and

$$G_1(Y,B,B) = 2A(Y,B) \ . \tag{65}$$

For the inclusive reaction, in which all final states are detected <u>except</u> elastic scattering, one finds

$$G = e^{G_1} - 1 \ . \tag{66}$$

It is easy to see that the integral over p^2 yields the inelastic cross-section.

V. A PARTON MODEL OF FRAGMENTATION

One of the most serious shortcomings of the previous model is that it does not include any effects associated with the fragmentation of the incident nucleons. They are treated like point particles and it is assumed that they always retain a large fraction of their incident momenta. This is common feature of the eikonal model and it is obviously a very undesirable one. One way to remedy this difficulty is to think of our nucleons as point-like partons. The physical nucleon, or any other hadron state, can then fragment into an arbitrary number of partons. Consider a general hadron state, A, with finite invariant mass m_A. In general A contains more than one physical particle. We denote the probability amplitude for the state A to dissociate into N partons by $\psi_A^N(x_1,\underset{\sim}{r}_1,\ldots x_N,\underset{\sim}{r}_N)$, where x_i is the rapidity of the nth parton, and $\underset{\sim}{r}_i$ is its perpendicular distance from the center of mass of the hadron system. We then write the amplitude for the reaction $A+B \to A'+B'$ in the form

$$M_{A'B';AB}(Y,\underset{\sim}{\Delta}) = \int d^2B e^{i\underset{\sim}{\Delta}\cdot\underset{\sim}{B}} \sum_{N,M} \prod_{i=1}^{N} dx_i d^2r_i \prod_{j=1}^{M} dx_j' d^2r_j'$$

$$\cdot\psi_A^N(x_i,\underset{\sim}{r}_i)\psi_{A'}^{N^*}(x_i,\underset{\sim}{r}_i)\psi_B^M(x_j',\underset{\sim}{r}_j')\psi_{B'}^{M^*}(x_j',\underset{\sim}{r}_j')$$

$$\cdot 2is[1 - e^{-A^{NM}(Y,\underset{\sim}{B};x_i,\underset{\sim}{r}_i;x_j',\underset{\sim}{r}_j')}] , \qquad (67)$$

with analogous expressions for amplitudes in which pions are emitted or absorbed from chains. Here

$$A^{NM} = \sum_{i=1}^{N} \sum_{j=1}^{M} A(x_i - x_j', \underset{\sim}{B} + \underset{\sim}{r}_i - \underset{\sim}{r}_j'). \tag{68}$$

A is again given by Eq. (23), and $\underset{\sim}{B}$ is the perpendicular distance between the centers of mass of the hadron systems.

The S-matrix for this model will be unitary provided 1) the only interaction between left moving and right moving partons are exchanges of the type we have been discussing; and 2) the states $\psi_A{}^N(x_i, \underset{\sim}{r}_i) \otimes |y_i, \underset{\sim}{b}_1, \ldots y_n, \underset{\sim}{b}_n\rangle \otimes \psi_B{}^M(x_j', \underset{\sim}{r}_j')$ form a complete set. This result follows at once from the fact that one can still write the S-matrix as in Eq. (32) only χ is now given by

$$\chi = \int d^2r d^2r' d^2b \int_{(\frac{1}{2}-\epsilon)Y}^{\frac{1}{2}Y} dx \int_{-(\frac{1}{2}-\epsilon)Y}^{(\frac{1}{2}-\epsilon)Y} dy \int_{-\frac{1}{2}Y}^{-(\frac{1}{2}-\epsilon)Y} dx' \; b^+(x, \underset{\sim}{r}) b(x, \underset{\sim}{r})$$

$$\cdot a(y, \underset{\sim}{b}) b^+(x', \underset{\sim}{r}') b(x', r') W(Y, \underset{\sim}{B}; x, \underset{\sim}{r}; x', r'; y, \underset{\sim}{b}) \tag{69}$$

Here b^+ and b are creation and annihilation operators for the partons.

It is clear that if W has the form discussed in Section IV, the total cross-section and the elastic cross-section will both grow like Y^2. On the other hand, the cross-section for diffractive dissociation will at most be a constant in the asymptotic region since it comes only from the "grey ring." It is difficult to make any more quantitative statements without adopting a definite form for the parton wave functions.

VI. DISCUSSION

Clearly the most appealing features of our models are that they are solvable and lead to a unitary S-matrix. It appears to be possible to generalize the approach considerably to yield more sophisticated models which possess these properties.

The fact that the model is solvable rests on two important features. First, the chains from which particles are produced are uncorrelated in impact parameter space. This is the property which makes it possible to perform the phase space integrals over the transverse momenta. Second, the nucleons or partons retain a large fraction of their incident momenta. As a result, the longitudinal momenta of the pions are not correlated by energy-momentum conservation, so we can do the phase space integrations over their rapidities. Both of these features come directly from the eikonal model.

It is clearly unrealistic to treat the physical hadrons as point like particles which neither fragment nor lose an appreciable fraction of their incident momenta. The parton model introduced in Section V allows one to introduce fragmentation effects while retaining exact unitarity. It should lead to an interesting class of models.

The most straightforward generalization of the model is to allow more than one particle to be emitted or absorbed from each chain. The structure of the model becomes much more complicated, but there does not seem to be any essential difficulty in solving it. We shall discuss this subject and the parton model elsewhere.

REFERENCES

1. This philosophy should be contrasted with that of the dual resonance model where one starts by imposing crossing symmetry and hopes to introduce unitarity at a later stage. The present approach would seem more appropriate for a discussion of diffraction scattering.

2. H. Cheng and T. T. Wu, Phys. Rev. Lett. 24, 1456 (1970).

3. S. J. Cheng and T. M. Yan, Phys. Rev. Lett. 25, 1586 (1970).

4. J. Finkelstein and F. Zachariasen, Phys. Lett. 34B, 631 (1971).

5. J. R. Fulco and R. L. Sugar, Phys. Rev. D15, 1919 (1971).

6. R. L. Sugar, 1971 Boulder Summer School Lectures (to be published).

7. S. J. Chang and S. Ma, Phys. Rev. Lett. 22, 1334 (1969); M. Levy and J. Sucher, Phys. Rev. 186, 1656 (1969); H.D.I. Abarbanel and C. I. Itzykson, Phys. Rev. Lett. 23, 53 (1969).

8. In the argument leading to Eqs. (9) and (12) we make use of the fact that at high energies $e^{-A(Y,\underset{\sim}{B})}$ is expected to be small for all values of $\underset{\sim}{B}$ for which there is appreciable scattering. However, if Eqs. (9) and (12) are taken as given, unitarity is satisfied whether e^{-A} is small or not.

9. In our unitarity calculations we have assumed that W is a real function. It is not difficult to check that the entire unitarity calculation goes through for complex W's provided we associate a factor of W with the creation of each pion and a factor of W^* with the annihilation, or vice versa. However,

in this case the S-matrix will not be symmetric, so we shall only consider real W.

10. One easily verifies that the restriction on the range of the y integration implied by Eq. (8) can be neglected provided $\varepsilon << 1$.

11. This justifies our statement in Section II that the two body S-matrix factor strongly damps chains from which pions are produced with large relative rapidities.

12. Notice that the contribution from the "grey ring" can be made arbitrarily small by taking g to be large.

13. Notice that the bootstrap solution given by Eqs. (41) and (42) holds for all values of R_o.

14. T. L. Neff, R. Savit and R. Blankenbecler, SLAC Pub. 988 (Nov. 1971).

FIGURE CAPTIONS

Figure 1: A general multi-chain production diagram.

Figure 2: The single particle production diagram associated with the function W.

Figure 3:

(a) The production amplitude M_{no}.

(b) The connected amplitude M_{nm}.

Figure 1

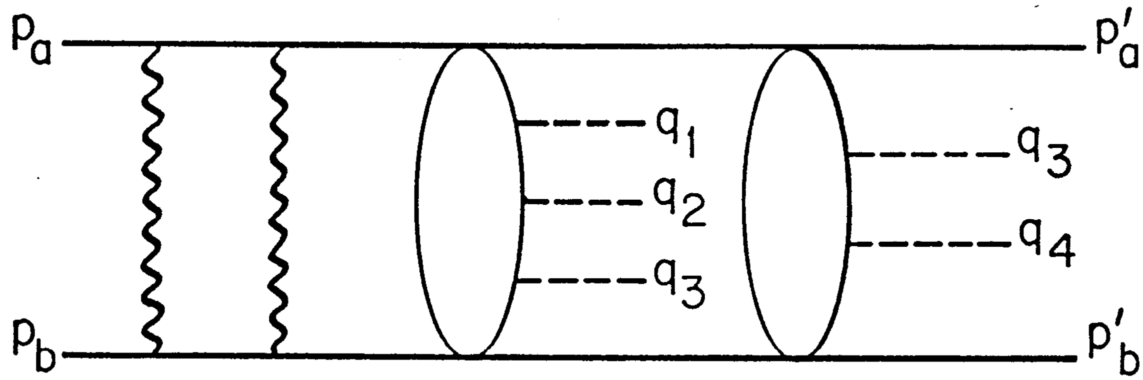


Figure 2

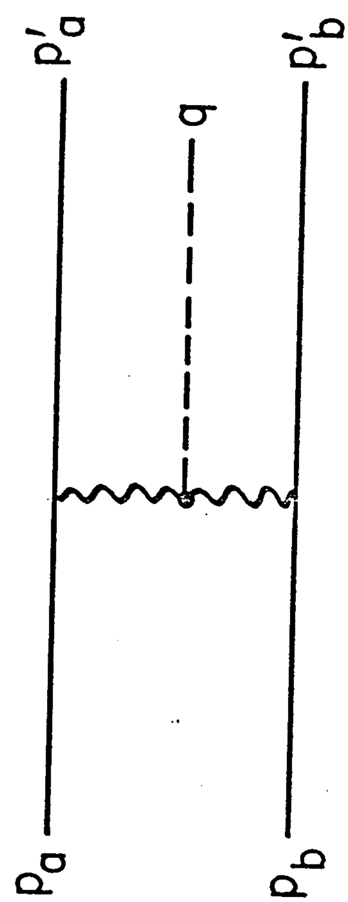
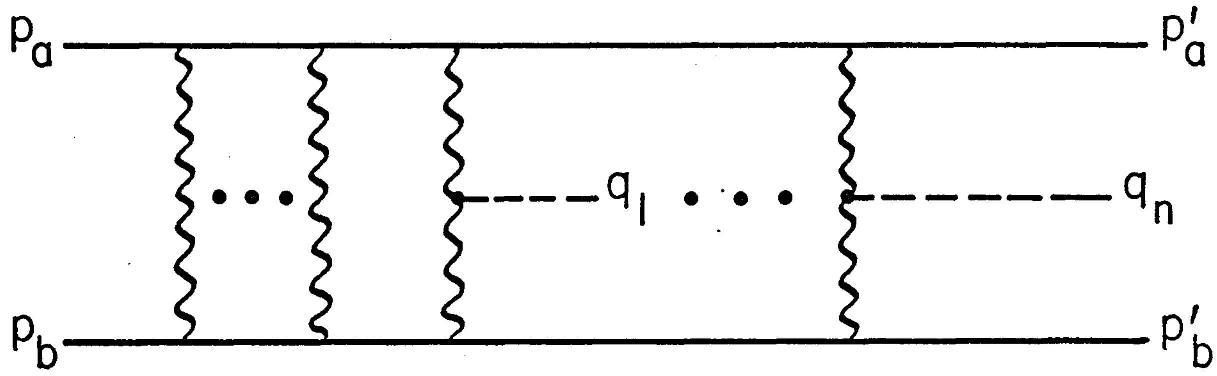


Figure 3

a)



b)