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LIGHT-CONE DOMINANCE IN THE PRESENCE OF SINGULARITIES ELSEWHERE IN COORDINATE SPACE*

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Abstract

To exemplify the special role played by the light-cone ($y^2 = 0$) in highly inelastic electroproduction, we study the contribution to electroproduction of singularities on other hyperboloids, $y^2 = a^2 > 0$. A light-cone singularity is shown to dominate by at least the power $\nu^{3/4}$ over an equivalent singularity on the hyperboloid $y^2 = a^2$ in the Bjorken limit. Moreover, the leading contribution from a singularity at $y^2 = a^2$ is shown not to scale as a power of ν multiplying a function of $x = Q^2/2M\nu$, but instead to oscillate with the (asymptotically infinite) frequency $\sqrt{a^2 M\nu\,(1-x)}$ in the Bjorken limit.

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Light-cone dominance and the analysis of light-cone singularities have provided a useful framework for the study of inelastic electroproduction in the Bjorken limit ($Q^2, \nu \rightarrow \infty$, $x \equiv Q^2/2M\nu$ fixed).[1] Derivations of light-cone dominance usually include the assumption that the matrix element whose Fourier transform yields the observed inelastic structure function, $W_{\mu\nu}$, has no singularities except on the light-cone. Our object is to study this assumption and, in particular, whether singularities elsewhere in coordinate space produce additional (perhaps scaling) contributions to the structure functions in the Bjorken limit.[2]

To explore the effects of singularities not on the light-cone we compare contributions to the inelastic structure function from a given singularity on the light-cone ($y^2 = 0$) and from an identical singularity on a mass-like hyperboloid, $y^2 = a^2$. If the singularity is on the light-cone, it is well known that the resulting structure function scales in the Bjorken limit with a power of ν determined by the strength of the singularity. If the same singularity is on the hyperboloid $y^2 = a^2$, we find that it contributes a term to the structure function which vanishes at least as fast as $\nu^{3/4}$ relative to the light-cone singularity. In addition, the asymptotically leading term does not reduce in the Bjorken limit to a power of ν multiplying a function of x, but rather oscillates with a frequency of the order of $\sqrt{a^2 M\nu}$. While this result does not prove that scaling could not arise from behavior that has nothing to do with the light-cone, it illustrates clearly how the light-cone is singled out relative to other hyperboloids in the Bjorken limit.

For simplicity, we consider scalar currents. Extension to the realistic case (for inelastic electroproduction) of vector currents is straightforward and will be discussed at the end of the paper. The structure function is the imaginary part of the forward scattering amplitude for a scalar current of spacelike mass

$q^2 \equiv -Q^2 < 0$, and laboratory energy ν, off of a target nucleon:

$$V(Q^2, \nu) \equiv \int d^4y\, e^{iq\cdot y} \langle P | [J(y), J(0)] | P \rangle \tag{1}$$

We define the current correlation function:

$$C(y^2, y\cdot P) \equiv \langle P | [J(y), J(0)] | P \rangle$$

and note that C vanishes for $y^2 < 0$ and is odd in $y \cdot P$.

Suppose $C(y^2, y\cdot p)$ had a δ-function light-cone singularity of the form:

$$C(y^2, y\cdot P) = i\,\delta(y^2)\,\epsilon(y\cdot P)\,F(y\cdot P) \tag{2}$$

where $F(y\cdot P)$ is even in $y\cdot P$. As is well known,[1] such a singularity contributes to $V(Q^2, \nu)$ a leading term of the following form in the Bjorken limit:

$$\mathrm{Lim}_{bj}\, V(Q^2, \nu) = \frac{4\pi^2}{2M\nu}\, f(x) \tag{3}$$

where $f(x)$ is the Fourier transform of $F(y\cdot P)$. The requirement that V vanish below threshold ($2M\nu < Q^2$) demands that $f(x)$ vanish for $|x| > 1$.

Let us now compare with Eq. (3) the contribution to $V(Q^2, \nu)$ from a δ-function singularity on the hyperboloid $y^2 = a^2$. We choose to make the comparison for a δ-function singularity because it is the light-cone singularity which arises if the currents are built up of free fields, as seems to be the case in electroproduction.[3] Extension to both weaker and stronger singularities is straightforward and is discussed at the end of the paper. Suppose, then, that $C(y^2, y\cdot P)$ has the following form:

$$\tilde{C}(y^2, y\cdot P) = i\,\delta(y^2 - a^2)\,\epsilon(y\cdot P)\,F(y\cdot P)$$

where causality restricts a^2 to be positive (or zero as already discussed). Writing $F(y \cdot P)$ in terms of its Fourier transform:

$$F(y \cdot P) = \int_{-\infty}^{\infty} d\alpha \, e^{i\alpha y \cdot P} f(\alpha)$$

we find for $V(Q^2, \nu)$:

$$\tilde{V}_a(Q^2, \nu) = (2\pi)^3 \int_{-\infty}^{\infty} d\alpha \, f(\alpha) \, \Delta(q + \alpha p, a^2) \tag{4}$$

where

$$\Delta(R, a^2) \equiv \frac{i}{(2\pi)^3} \int d^4k \, e^{ik\cdot R} \, \delta(k^2 - a^2) \, \epsilon(k_0)$$

is the usual causal propagator function. $\Delta(R, a^2)$ may also be written as[4]:

$$\Delta(R, a^2) \equiv \frac{1}{2\pi} \left[\delta(R^2) \, \epsilon(R_0) - \frac{a^2}{2} \, \theta(R^2) \epsilon(R_0) \, \frac{J_1\left(a\sqrt{R^2}\right)}{a\sqrt{R^2}} \right] \tag{5}$$

If $a = 0$, the second term in Eq. (5) vanishes and the integral of Eq. (4) may be performed to obtain the result given in Eq. (3).

We have yet to enforce the vanishing of $\tilde{V}_a(Q^2, \nu)$ below threshold. Using $\Delta(R, a^2) = 0$ for $R^2 < 0$, it is easily seen that $\tilde{V}_a$ vanishes at and below threshold in the Bjorken limit only if $f(\alpha)$ vanishes for $|\alpha| \geq 1$. Thus the result, usually derived when $a^2 = 0$, is actually independent of a^2. We shall use the experimental observation that $V(Q^2, \nu)$ vanishes at threshold and therefore require that $f(\alpha)$ vanish for $|\alpha| \geq 1$. In addition, we assume that $f(\alpha)$ and its derivatives are continuous for $-1 \leq \alpha \leq 1$ except perhaps at $\alpha = 0$.[5] We shall point out what pathologies may arise if $f(\alpha)$ or one of its derivatives is singular in the interval $0 < \alpha \leq 1$.

To proceed, we divide the integral of Eq. (4) into two pieces corresponding to the singular and regular parts of Eq. (5). First consider the contribution of the singular part:

$$\tilde{V}_a^S(Q^2, \nu) = (2\pi)^2 \int_{-1}^{1} d\alpha\, f(\alpha)\, \delta(R^2)\, \epsilon(R_0)$$

The roots of the δ-function occur at $R^2 \equiv (q + \alpha p)^2 = 0$:

$$\alpha_{\pm} = 1/M \left(-\nu \pm \sqrt{\nu^2 + Q^2} \right)$$

The root α_- approaches $-2\nu/M$ in the Bjorken limit and may be ignored since it falls outside the range of the α integral. The root α_+ approaches x in the Bjorken limit and is between zero and one for physical electroproduction. Some elementary algebra yields:

$$\tilde{V}_a^S(Q^2, \nu) = \frac{4\pi^2}{2M^2\lambda} f(\alpha_+) \tag{6}$$

where $2\lambda \equiv \alpha_+ - \alpha_-$. In the Bjorken limit, this becomes exactly the a^2-independent scaling result obtained in Eq. (3).

Turning to the regular part:

$$\tilde{V}_a^R(Q^2, \nu) = -(2\pi)^2 \frac{a^2}{2} \int_{-1}^{1} d\alpha\, \theta(R^2)\, \epsilon(R_0) \frac{J_1\left(a\sqrt{R^2}\right)}{a\sqrt{R^2}}$$

The θ-function restricts the integral to the regions $\alpha \geq \alpha_+$ and $\alpha \leq \alpha_-$.

Since $\alpha_- < -1$, we obtain:

$$\tilde{V}_a^R(Q^2,\nu) = -2\pi^2 a^2 \int_0^{1-\alpha_+} d\alpha\, f(\alpha+\alpha_+)\, \frac{J_1\left(aM\sqrt{\alpha(\alpha+2\lambda)}\right)}{aM\sqrt{\alpha(\alpha+2\lambda)}} \qquad (7)$$

By means of the following identity between Bessel functions:

$$\frac{J_1(\beta)}{\beta} = -\frac{1}{a^2M^2(\alpha+\lambda)} \frac{d}{d\alpha} J_0(\beta)$$

where $\beta = aM\sqrt{\alpha(\alpha+2\lambda)}$, Eq. (7) may be partially integrated:

$$\tilde{V}_a^R(Q^2,\nu) = 2\pi^2 a^2 \left\{ \frac{f(\alpha+\alpha_+)\, J_0(\beta)}{a^2M^2(\alpha+\lambda)} \Bigg|_0^{1-\alpha_+} - \int_0^{1-\alpha_+} d\alpha\, J_0(\beta) \frac{d}{d\alpha}\left(\frac{f(\alpha+\alpha_+)}{a^2M^2(\alpha+\lambda)}\right) \right\} \qquad (8)$$

Since f(1) = 0, the surface term vanishes at the upper limit. At the lower limit, we obtain:

$$-\frac{2\pi^2 f(\alpha_+)}{M^2\lambda}$$

This term exactly cancels the scaling term, $\tilde{V}_a^S$, of Eq. (6).

It remains to be shown that the integral of Eq. (8) is lower order in the Bjorken limit. To continue, we repeatedly partially integrate Eq. (8) by means of another relation among Bessel functions:

$$\beta^k J_k(\beta) = \frac{1}{a^2M^2(\alpha+\lambda)} \frac{d}{d\alpha} \beta^{k+1} J_{k+1}(\beta)$$

and obtain

$$\tilde{V}_a(Q^2,\nu) = \left\{ \frac{2\pi^2}{M^2(\alpha+\lambda)} \sum_{k=1}^{n} (-1)^k \beta^k J_k(\beta) \left[\left(\frac{d}{d\alpha} \frac{1}{a^2M^2(\alpha+\lambda)} \right)^k f(\alpha+\alpha_+) \right] \Bigg|_{\alpha=1-\alpha_+} \right.$$

$$\left. -2\pi^2 a^2 (-1)^n \int_0^{1-\alpha_+} d\alpha\, \beta^n J_n(\beta) \left[\left(\frac{d}{d\alpha} \frac{1}{a^2M^2(\alpha+\lambda)} \right)^{n+1} f(\alpha+\alpha_+) \right] \right\} \qquad (9)$$

The surface terms at $\alpha = 0$ always vanish for $k \geq 1$ because $\beta^k J_k(\beta) = 0$ at $\alpha = 0$. Note that we do not assume that $f'(1) = 0$ and therefore $k = 1$ already gives a non-vanishing contribution in Eq. (9). We emphasize that the possibility of performing all these partial integrations rests heavily on the assumption that $f(\alpha)$ and its derivatives are smooth on the interval $\alpha_+ \leq \alpha \leq 1$. (It is now evident why singularities are allowed at $\alpha = 0$, since that point never falls within the range of integration.)

So far, Eq. (9) is exact. We now turn to the Bjorken limit and bound the contributions of each term. Note that a^2 <u>is to be held fixed</u> as ν and Q^2 are taken to infinity. Otherwise (as we shall see below) quite a different result is obtained. As $\nu \rightarrow \infty$ λ approaches ν/M so that the largest term for any k is that in which all the derivatives act on the function f, rather than on $\frac{1}{a^2M^2(\alpha+\lambda)}$. For any k we obtain in the Bjorken limit[6]:

$$\tilde{V}_a^k(Q^2,\nu) \sim \frac{a^2 f^{(k)}(1)}{(a^2 M\nu)^{k/2+5/4}} \qquad (10)$$

$\tilde{V}_a^{k=1}$ gives the leading term asymptotically and <u>vanishes like $1/\nu^{3/4}$ relative to the contribution of an equivalent singularity on the light-cone</u> (<u>cf</u> Eq. (3)). Finally, note that the integral remaining after n - 1 partial integration is of order $\frac{|\text{Max}\, f^{(n)}|}{\nu^{n/2+5/4}}$ where $\text{Max}\, f^{(n)}$ is the bound on $f^{(n)}(\alpha)$ on the interval $\alpha_+ \leq \alpha \leq 1$.

In bounding the terms in Eq. (9), we have ignored the oscillations of the Bessel function $J_k\left(aM\sqrt{(1-\alpha_+)(1-\alpha_-)}\right)$. In the Bjorken limit, the argument reduces to $a\sqrt{M\nu\,(1-x)}$. Therefore, the terms in Eq. (9) not only vanish rapidly as $Q^2, \nu \to \infty$ but also oscillate with infinite frequency. This is reminiscent of the one dimensional case: If F(x) has a δ-function singularity at x = 0, then its Fourier transform is F(0). If the δ-function is at x = a, the Fourier transform is $F(a)e^{i\alpha a}$ which oscillates with infinite frequency as $\alpha \to \infty$.

To summarize: $\tilde{V}_a(Q^2, \nu)$ vanishes faster than $V(Q^2, \nu)$ by at least a factor of $\frac{1}{\nu^{3/4}}$ in the Bjorken limit. If $f'(\alpha)$ and some number of its derivatives vanish at $\alpha = 1$ (as is, in fact, the case in electroproduction), then $\tilde{V}_a(Q^2, \nu)$ vanishes even faster as $\nu, Q^2 \to \infty$. In addition, $\tilde{V}_a(Q^2, \nu)$ oscillates with infinite frequency in the Bjorken limit.

We now return to discuss some problems raised in this analysis and to state some generalizations. First we ask what happens to Eq. (9) if we take the Bjorken limit but keep the product $a^2M\nu$ fixed and less than 1. Since β is small in this limit, $J_k(\beta)$ may be replaced by $(\beta/2)^k \frac{1}{k!}$ and the leading term proportional to each derivative of $f(\alpha)$ may be summed. The result is:

$$\lim_{\substack{M\nu, Q^2 \to \infty \\ a^2M\nu \text{ fixed } < 1}} \tilde{V}_a(Q^2, \nu) = \frac{2\pi^2}{M\nu} \sum_{k=1}^{\infty} (x-1)^k \frac{f^k(1)}{k!} + \text{lower order terms}$$

$$= \frac{2\pi^2}{M\nu} f(x) + \text{lower order terms}$$

which is exactly the Bjorken limit of Eq. (3). As expected, if the singularity is within $1/M\nu$ of the light-cone, it is indistinguishable from a light-cone singularity.

Second, we ask what happens if $f(\alpha)$ or its derivatives become singular somewhere on the interval $\alpha_+ \leq \alpha \leq 1$. Clearly the partial integrations leading to Eq. (9) are not allowed and $\tilde{V}_a$ might not vanish as rapidly as we have asserted. A singularity in $f(\alpha)$ or some derivative at $\alpha = \alpha_0$ would seem to contribute a term to $\tilde{V}_a(Q^2, \nu)$ proportional to $J_k\left(aM\sqrt{(\alpha_0 - \alpha_+)(\alpha_0 - \alpha_-)}\right)$. While it might be large in the scaling limit, such a highly oscillatory term offers little hope of reproducing the experimentally observed smooth scaling behavior of $V(Q^2, \nu)$. Such behavior is consistent with our conclusion in any event: If equivalent singularities are located on the light-cone and at $y^2 = a^2$, the light-cone dominates; it is necessary to make the singularity at $y^2 = a^2$ stronger (e.g., by letting $f(\alpha)$ be singular) in order to make its contribution comparable in the Bjorken limit.

Third, we state the extensions of our results to the case of singularities other than δ-functions. Singularities of the form:

$$C(y^2, y \cdot P) = i\,\delta^{(n)}(y^2 - a^2)\,\epsilon(y \cdot P)\,F(y \cdot P)$$

or

$$\tilde{C}(y^2, y \cdot P) = i\,\theta(y^2 - a^2)\,\epsilon(y \cdot P)\,F(y \cdot P)$$

contribute to $\tilde{V}_a(Q^2, \nu)$ terms which vanish at least as fast as $1/\nu^{3/4}$ relative to the same singularity on the light-cone.[7]

Since the structure functions for electroproduction $\left(W_1(Q^2, \nu)\text{ and } W_2(Q^2, \nu)\right)$ are the Fourier transforms of δ-function and θ-function singularities on the light-cone,[3] these generalizations include the physically interesting cases. Lastly we note that weaker singularities $\left((\text{e.g.}, (y^2 - a^2)\,\theta(y^2 - a^2)\,\epsilon(y \cdot P)\,F(y \cdot P)\right)$ can be shown to be negligible in the Bjorken limit by more elementary means.[8]

To conclude: It would be of mathematical, and perhaps also of physical, interest to find forms for $C_2(y^2, y \cdot P)$ which generate structure functions which scale in the Bjorken limit and have nothing to do with the light-cone. We have shown that the most obvious candidates, singularities on hyperboloids, do not provide such examples, but rather that the light cone, $y^2 = 0$, is a favored location relative to other hyperboloids, $y^2 = a^2$, in the Bjorken limit.

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Footnotes and References

1. R. A. Brandt and G. Preparata, Nucl. Phys. B27, 541 (1971).

   Y. Frishman, Phys. Rev. Letters 25, 966 (1970).

2. This possibility has recently been raised by C. Ryan, Dublin preprint (1971).

3. R. Jackiw, R. van Royen and G. West, Phys. Rev. D2, 2473 (1970).

4. See, for example, N. N. Bogoliubov and D. V. Shirkov, Introduction to the Theory of Quantized Fields (Interscience, London, 1959).

5. In the case of singularities on the light-cone, we must allow for singularities in $f(\alpha)$ or its derivatives to at $\alpha = 0$ account for the leading Regge behavior ($\alpha(0) \gtrsim -1$) in the electroproduction structure functions. See, for example, R. Jaffe, SLAC-PUB-1022.

6. As $\nu, Q^2 \to \infty$, β becomes $\sqrt{a^2 M\nu (1-x)}$ and approaches infinity except at threshold ($x=1$). Asymptotically we replace $J_k(\beta)$ by $\sqrt{\frac{2}{\pi\beta}}$, hence the factor of $(a^2 M\nu)^{-1/4}$ in Eq. (10). Therefore, Eq. (10) is only valid for $x \neq 1$. However, if $x = 1$, we can still bound $J_k(\beta)$ by unity, lose the factor of $(a^2 M\nu)^{-1/4}$ and find that $\widetilde{V}_a$ vanishes like $1/\nu^{1/2}$ relative to an equivalent light-cone singularity.

7. The proof for a θ-function singularity relies on the observation that in this case both $f(1)$ and $f'(1)$ must be zero for $\widetilde{V}_a(Q^2, \nu)$ to vanish at threshold.

8. See, for example, R. Jaffe, SLAC-PUB-999 (Revised).