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RANGE, LIGHT-CONE DOMINANCE AND SCALING AT LOW-Q^2 IN ELECTROPRODUCTION*

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ERRATUM

In the second equation on page 13, change

$A_k =$ to read $D_{2k} =$.

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ABSTRACT

The analysis of light cone singularities is used to connect the $y \cdot P$, or range, dependence of the current correlation function with the Q^2 dependence of the inelastic electroproduction structure functions. We study for what regions of the Q^2, ν plane and for what $y \cdot P$ dependence the leading light-cone singularity dominates contributions from less singular terms with the same $y \cdot P$ dependence. When the leading singularity can be shown to dominate for a particular region of Q^2 and ν, we study whether this implies scaling for νW_2 in that kinematic region. It is shown that a division of the current-correlation function into short and long range contributions is fundamentally ambiguous and not related to scaling at low Q^2. Short range terms which are shown to be light-cone dominated for all Q^2 so long as $\nu \to \infty$, are found but are shown not to scale at low Q^2 and to be indistinguishable from corrections to long range terms which produce the leading Regge behavior. We show that leading Regge terms may receive contributions far away from the light cone for small virtual photon mass, but that light-cone dominance and scaling are recovered when the photon mass is taken very large.

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I. INTRODUCTION

The current correlation function which describes inelastic electroproduction depends both on the invariant coordinate space separation, y^2, between the absorption and emission of the virtual photon and on the invariant $y \cdot P$ which measures the time delay between absorption and emission in the rest frame of the target nucleon. The latter has come to be known as the range dependence[1] of the interaction.

The y^2 dependence of the current correlation function has been the subject of much recent work.[2] In particular, it was observed that in the deep inelastic limit the region $y^2 \approx 0$ yields the dominant contribution to the structure functions $W_1(Q^2, \nu)$ and $\nu W_2(Q^2, \nu)$. This allows a simple parameterization of the experimentally observed scaling in terms of the strength of the leading light-cone singularity and also simplifies the application of models to highly inelastic electroproduction.

Our object is to study the range, or $y \cdot P$, dependence of the current correlation function and, in particular, to explore whether the observed onset of scaling at low virtual photon mass (Q^2) can be related to the range dependence of the current correlation function. The $y \cdot P$ dependence of the leading light-cone singularity has been extracted from the scaling data by Pestieau, Roy and Terazawa.[3] More recently suri and Yennie[4] have studied the role of range when Q^2 is not assumed to be infinite. The difference between our results and theirs are discussed at the end of Section V.

We use the light-cone singularity algebra, especially as developed by Frishman,[5] to connect the $y \cdot P$ dependence of the current correlation function with the Q^2 dependence of the inelastic structure functions. Since the light-cone need not dominate at low Q^2 we apply Frishman's techniques not only to the leading y^2 singularity but to lower order singularities as well. In order to apply light-cone techniques we must assume that the current correlation function factors into separate functions of y^2 and $y \cdot P$ (aside from the derivative operators required by Lorentz and gauge invariance). This is not in conflict with any general principle and is consistent with the assumptions made by suri and Yennie.[4] In a later section we show that many of our results do not depend critically on this assumption.

Assuming factorization we proceed as follows. For a given $y \cdot P$ dependence of the current correlation function we calculate the contribution to $\nu W_2(Q^2, \nu)$ from successively lower order light-cone singularities in y^2. As is well known[2] a leading singularity proportional to $\theta(y^2)$ produces the observed scaling behavior in the Bjorken limit. We study in what regions of the Q^2, ν plane and for what $y \cdot P$ dependence the leading singularity dominates contributions from less singular terms. When the leading singularity can be shown to dominate for a particular region of Q^2 and ν, we study whether this in fact implies scaling for νW_2 in that kinematic region.

Our primary conclusion is that a division of the current correlation function into long and short range contributions is fundamentally ambiguous and not related to scaling at low Q^2 and for these reasons not very useful in confronting the data. We find that it is possible for a piece of the current correlation function to decrease so rapidly in $y \cdot P$ that its contribution to electroproduction is light-cone dominated for all Q^2 so long as $\nu \longrightarrow \infty$. These are essentially the short range terms found by suri and Yennie.[4]

Only terms in νW_2 which decrease faster than $1/\nu^2$ in the Regge region fall into this category. In contrast to suri and Yennie we find that such short range terms do not scale for small Q^2; rather their character changes completely in passing from the region of high energy photoproduction to the Bjorken region. Moreover, these short range terms cannot be distinguished, even in principle, from corrections to the long range terms which produce the leading Regge behavior. We show that leading Regge pieces in νW_2 may receive contributions far away from the light-cone in the Regge limit, i.e., for finite photon mass, but that light-cone dominance (and scaling) is recovered when the photon's mass is taken very large. Finally, we find that there is some reason to think that a term in νW_2 proportional to $1/\nu$ in the Regge limit is light-cone dominated for all Q^2. This would explain the observation of suri and Yennie that such a term can be extracted from the data with the same coefficient in both the Bjorken and photoproduction limits.

In Section II we review the necessary kinematics. In Sections III and IV we parameterize the y^2 and $y \cdot P$ dependence of νW_2, and in Section V combine the techniques developed in III and IV to obtain our results and discuss the relation of our conclusions to the work of suri and Yennie. A reader who wishes to avoid the mathematics may skip Sections III and IV provided he is willing to accept Eq. III. 3 and Table I which summarize those sections. Finally, in Section VI we discuss the generality of our results in light of the original ansatz of factorization.

II. KINEMATICS

The usual structure functions for inelastic electroproduction are defined by the Fourier transform of the current correlation function.[6]

$$W_{\mu\nu} \equiv \frac{4\pi^2 E_P}{M}\int d^4y\, e^{iq\cdot y} < P|[J_\mu(y), J_\nu(0)]|P>$$

$$= -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) W_1(q^2, \nu) + \frac{1}{M^2}\left(P_\mu - \frac{P\cdot q}{q^2} q_\mu\right)\left(P_\nu - \frac{P\cdot q}{q^2} q_\nu\right) W_2(q^2, \nu) \tag{II. 1}$$

A similar decomposition may be made in coordinate space:

$$4\pi^2 E_P < P|[J_\mu(y), J_\nu(0)]|P> = (g_{\mu\nu}\Box - \partial_\mu\partial_\nu)\, C_1(y^2, y\cdot P)$$

$$+ (P_\mu P_\nu \Box - P\cdot\partial(P_\mu\partial_\nu + P_\nu\partial_\mu) + g_{\mu\nu}(P\cdot\partial)^2) C_2(y^2, y\cdot P)$$

The coordinate and momentum space structure functions are related by:

$$\nu W_2(Q^2, \nu) = Q^2 M\nu \int d^4y\, e^{iq\cdot y}\, C_2(y^2, y\cdot P) \tag{II. 2}$$

$$MW_1(Q^2, \nu) = - Q^2\int d^4y\, e^{iq\cdot y}\, C_1(y^2, y\cdot P)$$

$$+ M^2\nu^2\int d^4y\, e^{iq\cdot y}\, C_2(y^2, y\cdot P) \tag{II. 3}$$

The functions $C_i(y^2, y\cdot P)$ are odd in $y\cdot P$ and must vanish for $y^2 < 0$. Bjorken scaling obtains if the functions $C_i(y^2, y\cdot P)$ are smooth away from the light-cone and behave like

$$C_1(y^2, y\cdot P) \propto \delta(y^2)\,\epsilon(y\cdot P)\, G(y\cdot P)$$

$$C_2(y^2, y\cdot P) \propto \theta(y^2)\,\epsilon(y\cdot P)\, F(y\cdot P)$$

to leading order near the light-cone.[2] Notice that G and F are even under $y\cdot P \leftrightarrow -y\cdot P$. With $W_{\mu\nu}$ defined as above the combinations MW_1 and νW_2 scale:

$$\lim_{Q^2,\nu\to\infty} MW_1(Q^2,\nu) = F_1(x)$$

$$x \equiv Q^2/2M\nu$$

$$\lim_{Q^2,\nu\to\infty} \nu W_2(Q^2,\nu) = F_2(x)$$

III. CONTRIBUTION TO $\nu W_2(Q^2, \nu)$ FROM VARIOUS LIGHT-CONE SINGULARITIES

We calculate the contribution to $\nu W_2(Q^2, \nu)$ from a term in $C_2^{(n)}(y^2, y \cdot P)$ of the form:

$$C_2^{(n)}(y^2, y \cdot P) = \frac{i}{8\pi^2} (\mu^2 y^2)^n \theta(y^2) \epsilon(y_0) F(y \cdot P) \qquad \text{(III. 1)}$$

using generalized function theory. The factor μ^{2n} is necessary to keep $C_2^{(n)}$ dimensionless. More singular terms are ruled out by the observed scaling of νW_2 in the Bjorken limit. Only integer values of n are considered. Extension to fractional values of n is straightforward but unnecessary for our analysis since we wish only to determine when $C_2^{(0)}$ dominates lower order terms and not exhaustively parameterize possible behavior.

Our analysis is valid for all values of Q^2 and ν. For large n and for $F(y \cdot P)$ which do not fall sufficiently fast as $y \cdot P \longrightarrow \infty$ some of the integrals which follow may be formally divergent. The finite results we obtain must be interpreted as the result of multiplying Eq. (III. 1) by a convergence factor such as $e^{-\epsilon y^2}$ and passing to the $\epsilon \longrightarrow 0$ limit.

We assume that the Fourier transform of $F(y \cdot P)$ exists:[7]

$$F(y \cdot P) \equiv \int_{-\infty}^{\infty} d\alpha \quad e^{i\alpha y \cdot P} f(\alpha) \qquad \text{(III. 2)}$$

The contribution of $C_2^{(n)}$ to $\nu W_2(Q^2, \nu)$ is given by (see Eq. II. 2):

$$\nu W_2^{(n)}(Q^2, \nu) = \frac{Q^2 M \nu i}{8\pi^2} \int_{-\infty}^{\infty} d\alpha \, f(\alpha) \int d^4y \, e^{i(q+\alpha P) \cdot y} (\mu^2 y^2)^n \theta(y^2) \epsilon(y_0)$$

The spatial integral may be performed following Frishman:[5]

$$\int d^4y\, e^{ik\cdot y}\, (y^2)^n\, \theta(y^2)\, \epsilon(y_0) = \pi^2 i\, 2^{(2n+4)}\, \Gamma(n+1)\, \epsilon(k_0) \left(\frac{d}{dk^2}\right)^{n+1} \delta(k^2)$$

where $k^2 \equiv \alpha^2 M^2 + 2\alpha M\nu - Q^2$ and $\epsilon(k_0) = \epsilon(\nu + \alpha M)$. Using this, $\nu W_2^{(n)}$ may be rewritten:

$$\nu W_2^{(n)}(Q^2, \nu) = \frac{1}{2} Q^2 M\nu (-2\mu^2)^n\, \Gamma(n+1)\, \frac{1}{|\alpha M^2 + M\nu|} \left(\frac{d}{d\alpha}\, \frac{1}{\alpha M^2 + M\nu}\right)^{n+1} f(\alpha)\Bigg|_{\alpha_-}^{\alpha_+} \quad \text{(III. 3)}$$

where $\alpha_\pm$ are the roots of $k^2 = 0$:

$$\alpha_\pm \equiv \frac{1}{M}\left(-\nu \pm \sqrt{\nu^2 + Q^2}\right) \quad \text{(III. 4)}$$

In the limit $\nu \to \infty$, $\alpha_\pm$ reduce to x and $-\frac{2\nu}{M}$ respectively:

$$\alpha_+ = x\left(1 - \frac{Mx}{2\nu}\right) + O\left(\frac{M^2 x^3}{\nu^2}\right)$$

$$\alpha_- = -\frac{2\nu}{M}\left(1 + \frac{Mx}{2\nu}\right) + O\left(\frac{Mx^2}{\nu}\right)$$

The factor $\epsilon(k_0)$ dictates that Eq. (III. 3) be evaluated with a minus sign at α_- as indicated by $\Big|_{\alpha_-}^{\alpha_+}$. In the Bjorken limit Eq. (III. 3) reduces to the familiar light-cone result[5] except for the root at $\alpha_- \approx -\frac{2\nu}{M}$. Usually one argues from the spectral restrictions on Eq. (II. 1) ($W_{\mu\nu}(Q^2, \nu) = 0$ for $2M\nu < Q^2$) that $f(\alpha) = 0$ unless $-1 \leq \alpha \leq 1$, making $f(\alpha_-) \equiv 0$. Simple models for F(y· P) may not satisfy this restriction. The various choices of $f(\alpha)$ which enter our analysis are not identically zero for $|\alpha| > 1$ but always vanish fast enough as $|\alpha| \to \infty$ so that the root at α_- may be ignored.[8]

Equation (III. 3) allows us to study the contribution of a specific light-cone singularity with a specific $y \cdot P$ dependence in various regions of the Q^2, ν plane. In order to apply it we must study the behavior of $f(\alpha)$ for various choices of $F(y \cdot P)$.

IV. THE FORM OF $f(\alpha)$

Since we are particulatly interested in the Regge region ($\nu \longrightarrow \infty$, $x \longrightarrow 0$) where α_+ and α_- tend to zero and negative infinity respectively we study $f(\alpha)$ in these limits. We assume that $F(y\cdot P)$ has the following properties:

I. $F(y\cdot P)$ has no singularities.

II. $F(y\cdot P)$ vanishes as a power of $\frac{1}{y\cdot P}$ or faster as $y\cdot P \longrightarrow \infty$.

Assumption I is standard in light-cone analyses and is borne out in simple models.[9] Assumption II is made for calculational convenience and because all of the observed properties of νW_2 in the Regge and Bjorken limits are accounted for with such forms. Modifications which arise when more complicated $y\cdot P$ dependence is allowed are discussed in Section VI.

$f(\alpha)$ is defined by the inverse of Eq. (III. 2):

$$f(\alpha) = \frac{1}{\pi} \int_0^\infty d(y\cdot P) \cos(\alpha y\cdot P)\, F(y\cdot P) \qquad \text{(IV. 1)}$$

and is an even function of α. We categorize the various terms in $F(y\cdot P)$ according to their behavior as $y\cdot P \longrightarrow \infty$. First consider a term in $F(y\cdot P)$, denoted $F^\infty(y\cdot P)$, which falls faster than any power. $f^\infty(\alpha)$ may be expanded in a power series in α^2 for $\alpha \approx 0$:

$$f^\infty(\alpha) = \sum_{k=0}^{\infty} C_{2k}\alpha^{2k} \qquad \text{(IV. 2)}$$

where

$$C_{2k} \equiv \frac{1}{\pi} \frac{(-1)^k}{(2k)!} \int_0^\infty \lambda^{2k} F(\lambda) d\lambda$$

For large α, we obtain an expansion of $f^\infty(\alpha)$ by partial integration of Eq. (IV. 1):

$$\lim_{\alpha \longrightarrow \infty} f^\infty(\alpha) = \sum_{k=1}^{\infty} (-1)^k \alpha^{-2k} F^{\infty(2k-1)}(0) \qquad \text{(IV. 3)}$$

If $F^{\infty}(y \cdot P)$ is regular at $y \cdot P = 0$ all of its odd derivatives, $F^{\infty(i)}(0)$, must vanish and $f(\alpha)$ goes to zero faster than any power as $\alpha \longrightarrow \infty$. If not, then $f^{\infty}(\alpha)$ vanishes at least as fast as $\frac{1}{\alpha^2}$.

Now consider $F(y \cdot P)$ which vanishes as $\frac{1}{(y \cdot P)^b}$ as $y \cdot P \longrightarrow \infty$. As we shall see in the next section only $b \geq 1$ need be considered. This behavior cannot be extended back to $y \cdot P = 0$ without contradicting Assumption I above. This problem has complicated the manipulation of such terms in the past.[4,10] An advantage of our approach is the relatively straight forward way such terms can be accommodated. We define $F^b(y \cdot P)$ with a damping function so that the singularity at $y \cdot P = 0$ is removed:

$$F^b(y \cdot P) \equiv N_b \frac{D_b(\delta, y \cdot P)}{(y \cdot P)^b} \qquad \text{(IV.4)}$$

where N_b is an overall normalization and $D_b(\delta, y \cdot P)$ must satisfy:

$$\lim_{y \cdot P \longrightarrow 0} D_b(\delta, y \cdot P) \propto (y \cdot P)^b$$

$$\lim_{y \cdot P \longrightarrow \infty} D_b(\delta, y \cdot P) = 1 + 0(e^{-y \cdot P/\delta}) \qquad \text{(IV.5)}$$

The parameter δ measures the values of $y \cdot P$ over which the damping is effective. Note that any function which vanishes as a power of $y \cdot P$ at infinity may be written as a series of terms like Eq. (IV.4) plus a term which vanishes faster than any power at infinity.

We choose a particular form for $D_b(\delta, y\cdot P)$ which allows us to perform the Fourier transform of Eq. (IV.1):

$$D_b(\delta, y\cdot P) = b \int_0^{y\cdot P} d\xi\, \xi^{b-1}\, e^{-\xi/\delta}$$

$$\equiv b\delta^b\, \gamma(b, y\cdot P/\delta)$$

where $\gamma(a,x)$ is the incomplete gamma function. $\gamma(b, y\cdot P/\delta)$ satisfies Eq.'s (IV.5) (for example $D_2(\delta, y\cdot P) = 2\delta^2(1 - e^{-y\cdot P/\delta} - \frac{y\cdot P}{\delta}\, e^{-y\cdot P/\delta})$)

Performing the integral of Eq. (IV.1):

$$f^b(\alpha) = \frac{N_b b\delta^b}{\pi} \int_0^{1/\delta} \frac{x^b dx}{x^2 + \alpha^2} \qquad \text{(IV. 6)}$$

$$\equiv N_b b\delta^b\, I_b(\alpha, \delta)$$

The integral $I_b(\delta, \alpha)$ may be expanded about $\alpha = 0$ (for the moment we exclude $b = 1,3,5...$ since for these cases I_b may be evaluated using elementary techniques) yielding:

$$f^b(\alpha) = N_b b\delta^b \left\{ \sum_{k=0}^{\infty} (-1)^k\, \alpha^{2k}\, I_{b-2k}(0, \delta) + C^b\, |\alpha|^{b-1} \right\}$$

$I_j(0, \delta)$ may be read off Eq. (IV.6):

$$I_j(0, \delta) = \frac{\delta^{1-j}}{\pi(j-1)} \qquad \text{(IV. 7)}$$

This expanison is derived in the Appendix where it is shown that:

$$C^b = (-1)\ \frac{m}{2\pi}\ B\left(\frac{1+b-2m}{2}\ ,\ 1 - \frac{1+b-2m}{2}\right).$$

$B(x,y)$ is the Riemann beta function and m is the smallest integer such that $b - 2m < 1$.

The essential point is that $f^b(\alpha)$ is a power series in α^2 plus a term proportional to $|\alpha|^{b-1}$. For large α the behavior of $f^b(\alpha)$ may be read off Eq. (IV.6):

$$\lim_{b \to \infty} f^b(\alpha) = \sum_{k=1}^{\infty} D_{2k}\alpha^{-2k}$$

where

$$A_k = \frac{N_b b(-1)^{k-1}}{\pi(2k-1+b)\delta^{2k-1}}$$

For b=1,3,5... the integral of Eq. (III.10) may be performed by elementary means. For b=1 one obtains

$$f^{b=1}(\alpha) = \frac{N_1 \delta}{2\pi} \log \frac{1/\delta^2 + \alpha^2}{\alpha^2}$$

which for small α can be written as a sum over even powers of α plus a logarithm. As $\alpha \to \infty$ $f^{b=1}(\alpha)$ vanishes as α^{-2}. For b = 3,5,... $f^b(\alpha)$ is simply a power series in α^2 for small α and vanishes as α^{-2} as $\alpha \to \infty$. The various forms for $f(\alpha)$ are summarized in Table I.

It is important to determine the relative size of the coefficients d_{2k} which occur in the expansion of $f(\alpha)$ about $\alpha = 0$. Referring to Eq. (IV.7) we see that $\frac{d_{2k+2}}{d_{2k}} \propto \delta^2$ for $F^b(y \cdot P)$ given by Eq. (IV.4). Similarly for $F^{\infty}(y \cdot P)$, $\frac{d_{2k+2}}{d_{2k}} \propto \delta^2$ where δ is a measure of the value of $y \cdot P$ beyond which $F^{\infty}(y \cdot P)$ is small (e.g., $F^{\infty}(y \cdot P) = e^{-y \cdot P/\delta}$). In the next section we see that δ plays a role in determining the extent to which various choices of $F(y \cdot P)$ are light-cone dominated. Terms in $F(y \cdot P)$ which are to be strictly light-cone dominated for all Q^2 must not only fall faster than $\left(\frac{1}{y \cdot P}\right)^3$ as $y \cdot P \to \infty$ but also are required to have δ small.

V. RESULTS

Our initial assumption that $C_2(y^2, y\cdot P)$ factors, together with Eq. (III. 3) and Table I allow us to choose a particular $F(y\cdot P)$ and study in what kinematic regions its contribution to $\nu W_2(Q^2, \nu)$ is light-cone dominated. It is useful to keep in mind the following simple picture of how light-cone dominance might arise at low Q^2. If $F(y\cdot P)$ falls rapidly for $y\cdot P > \delta$ for a small number δ, then major contributions to Eq. (II. 2) should come from $y_0 \leq \delta/M$, or using locality, $0 \leq y^2 \lesssim \delta^2/M^2$. This limit on y^2 is independent of Q^2. Such a situation is illustrated in Figure 1. The remainder of this section places this simple picture on a firmer footing.

First consider the Bjorken limit for finite x ($0 < x \leq 1$). For $0 < \alpha \leq 1$ the function $f(\alpha)$, studied in the previous section, and all its derivatives are finite. Therefore the leading term in $\nu W_2^{(n)}(Q^2, \nu)$ in Eq. (III. 3) is of order $(1/M\nu)^n$. The contribution of the root at α_- is always negligible.

$$\nu W_2^{(n)} = x \left(\frac{-2\mu^2}{M\nu}\right)^n \Gamma(n+1) \left.\frac{d^{n+1}}{d\alpha^{n+1}} f(\alpha)\right|_{\alpha = x} + O\left(\frac{M}{\nu}\right)^{n+1}$$

The leading light-cone singularity (n=0) dominates as expected.

If x is allowed to approach zero complications arise. Derivatives of $f(\alpha)$ (cf Table I, Rows 1 and 2) may become singular near $\alpha = 0$ enhancing the contributions of lower order singularities. To study the region $x \approx 0$ we consider the Regge limit, $\nu \longrightarrow \infty$ Q^2 fixed, but distinguish between the low mass limit (R) where $Q^2 \approx M^2, \mu^2$,[11] and the deep Regge limit (D) where $Q^2 \gg M^2, \mu^2$. It is sufficient to study only the first few singularities given by Eq. (III. 3), specifically n=0,1,2:

$$\nu W_2^{(0)} = \frac{1}{2}\frac{Q^2 M\nu}{|D|}\left[\frac{f'(\alpha)}{D} - \frac{M^2 f(\alpha)}{D^2}\right]_{\alpha_-}^{\alpha_+} \qquad \text{(V. 1a)}$$

$$\nu W_2^{(1)} = \frac{1}{2}\frac{Q^2 M\nu}{|D|}(-2\mu^2)\left[\frac{f''(\alpha)}{D^2} - \frac{3M^2 f'(\alpha)}{D^3} + \frac{3M^4 f(\alpha)}{D^4}\right]_{\alpha_-}^{\alpha_+} \qquad \text{(V. 1b)}$$

$$\nu W_2^{(2)} = \frac{1}{2}\frac{Q^2 M\nu}{|D|}(8\mu^4)\left[\frac{f'''(\alpha)}{D^3} - \frac{6M^2 f''(\alpha)}{D^4} + \frac{15M^4 f''(\alpha)}{D^5} - \frac{15M^6 f(\alpha)}{D^6}\right]_{\alpha_-}^{\alpha_+}$$

(V. 1c)

where $D \equiv \alpha M^2 + M\nu$.

a. $\underline{\lim_{y\cdot p \to \infty} F(y\cdot P) \sim \frac{1}{y\cdot P}}$

In this case Eq. (V.1a) reduces to:

$$\nu W_{2\,b=1}^{(0)} = x\left(\frac{C_1}{x} + C_1 \frac{M}{\nu} \log x\right) + O\left(x^2, \frac{Mx}{\nu}\right)$$

The $1/x$ singularity in $f'^{\,b=1}(x)$ removes the overall factor of x so that $\nu W_{2\,b=1}^{(0)}$ goes to a constant as $\nu \to \infty$. It is well known that $1/y\cdot P$ behavior in $F(y\cdot P)$ produces Pomeron-like behavior in νW_2.[1,3] Had we permitted falloff proportional to $1/(y\cdot P)^b$ for $b < 1$, νW_2 would diverge like $1/x^{1-b}$ as $x \to 0$.[12] Notice that corrections to $\nu W_{2\,b=1}^{(0)}$ vanish at least as fast as $\log \nu/\nu^2$ in the R and D limits.

The contributions of lower order light-cone singularities for b=1 may be determined from Eq. (V.1b and c):

$$\nu W^{(1)}_{2\,b=1} = x\left(\frac{-2\mu^2}{M\nu}\right)\left(\frac{-C_1}{x^2}\right) + O\left(\frac{\mu^2}{\nu^2}\right)$$

$$= \left(\frac{4\mu^2}{Q^2}\right) C_1 + O\left(\frac{\mu^2}{\nu^2}\right)$$

$$\nu W^{(2)}_{2\,b=1} = 2x\left(\frac{-2\,\mu^2}{M\nu}\right)^2\left(\frac{2C_1}{x^2}\right) + O\left(\frac{\mu^4}{Q^2\nu^2}\right)$$

$$= \left(\frac{4\mu^2}{Q^2}\right)^2 4\;C_1 + O\left(\frac{\mu^2}{Q^2\nu^2}\right)$$

The increasing powers of $\frac{1}{\nu}$ in lower order singularities are compensated by the inverse powers of x in successive derivatives of $f^{b=1}(x)$. In the R limit $\left(\frac{\mu^2}{Q^2}\text{ finite}\right)$ all singularities contribute and the Pomeron is not light-cone dominated. In the D limit $\left(\frac{\mu^2}{Q^2} \ll 1\right)$, lower singularities are negligible and light-cone dominance is restored.[13] As $Q^2 \longrightarrow 0$ the contribution of all singularities must add up in such a way that $\nu W_{2\,b=1}$ vanishes like Q^2, as required by gauge invariance.

b. $\lim\limits_{y\cdot P\to\infty} F(y\cdot P) \propto \frac{1}{(y\cdot P)^b}$ $\quad b\neq 2$

Terms with b<3 behave just like the Pomeron. Successive differentiation of $C_b\,|x|^{b-1}$ (cf Table I) yields inverse powers of x which compensate inverse powers of ν and yield contribution to νW_2 which go like

$$\nu W^{(n)}_{2b} \propto \left(\frac{\mu^2}{Q^2}\right)^n x^{b-1} + O\left(\frac{x^{b-1}}{\nu^2}\right)$$

In the R limit all light-cone singularities contribute, while in the D limit light-cone dominance is recovered.

If $b>3$, the power series in α^2 rather than the $|\alpha|^{b-1}$ term determines the leading Regge behavior. To see this consider $\nu W_{2b}^{(0)}$:

$$\nu W_{2b}^{(0)} = x\left[\sum_{k=0}^{\infty} 2kd_{2k}\, x^{2k-1} + (b-1)\, C_{b-1}\, |x|^{b-2} - \frac{M}{\nu}\left(\sum_{k=0}^{\infty} d_{2k}\, x^{2k} + C_{b-1}|x|^{b-1}\right)\right]$$

$$+ \text{ terms lower order in } \nu$$

$$= x\left[2d_2 x + (b-1)\, C_{b-1} \cdot x^{b-2} - \frac{M}{\nu}\left(d_0 + C'_{b-1}|x|^{b-1}\right) + \ldots\right]$$

When $b>3$ the terms proportional to d_0 and d_2 dominate as $x \longrightarrow 0$. $\nu W_{2b}^{(n)}$ behaves the same. $C_b\,|\alpha|^{b-1}$ may be ignored so $F^b(y\cdot P)$ for $b>3$ behaves as if it fell faster than any power as $y\cdot P — \infty$ and is subsumed into the discussion of $F^{\infty}(y\cdot P)$ in the second section following. This result has also been obtained by suri and Yennie.[4]

c. $\displaystyle \lim_{y\cdot P \to \infty} F(y\cdot P) \propto \frac{1}{(y\cdot P)^2}$

In this case the leading singularity contributes:

$$\nu W_{2\,b=2}^{(0)} = C_2 x + O\left(\frac{Mx}{\nu}, x^2\right)$$

Subsequent differentiation of $f^{b=2}(x)$ <u>does not</u> yield inverse powers of x, so that lower order light-cone singularities vanish at least as fast as $\frac{xM}{\nu}$.

No other $F(y \cdot P)$ (see esp. $F^{b=1}(y \cdot P)$) contributes like $\frac{1}{\nu}$ to νW_2. Within the framework of our assumptions a piece of νW_2 which is proportional to $\frac{1}{\nu}$ for large ν is light-cone dominated and proportional to $\frac{Q^2}{\nu}$ in both the D and R regions (including $Q^2 = 0$). suri and Yennie[4] find a term in the data proportional to $\frac{Q^2}{\nu}$ with approximately the same coefficient at $Q^2 = 0$ and in the scaling region (they find $G = 106.5 \pm 13.5\ \mu b$ GeV in real photoproduction and $G = 117.5 \pm 7\ \mu b$ GeV in the scaling region, where $\sigma_{TOT}(``\gamma" P) = \sigma_{TOT}(\infty) + \frac{G}{\nu}$ and $\sigma_{TOT} \propto \frac{1}{Q^2}\ \nu W_2$ as $x \longrightarrow 0$).[14,15] Unfortunately, as discussed in Section VI, the presence of logarithms in $F(y \cdot P)$ would invalidate this result.

d. <u>$F(y \cdot P) \longrightarrow 0$ faster than any power</u>

As explained above $F(y \cdot P) \xrightarrow{y \cdot P \to \infty} \frac{1}{(y \cdot P)^b}$ with $b>3$ may also be included in this section. In this case Eq's. (V.1) reduce to:

$$\nu W_{2\infty}^{(0)} = x \left[2d_2 x - d_0 \frac{M}{\nu}\right] \qquad \text{(V.2a)}$$

$$\nu W_{2\infty}^{(1)} = -\left(\frac{2\mu^2}{M\nu}\right) x \ (2d_2) \qquad \text{(V.2b)}$$

$$\nu W_{2\infty}^{(2)} = 2\left(\frac{2\mu^2}{M\nu}\right)^2 x \left(24 d_4 x - 12\, d_2 \frac{M}{\nu}\right) \qquad \text{(V.2c)}$$

to leading order in x and $\frac{M}{\nu}$. $\nu W_{2\infty}^{(2)}$ and all lower order singularities are negligible as $\nu \longrightarrow \infty$. However $\nu W_{2\infty}^{(0)}$ and $\nu W_{2\infty}^{(1)}$ both appear to go as $\frac{1}{\nu^2}$ in the R region. In Section IV we noted that $\frac{d_{2k+2}}{d_{2k}} \propto \delta^2$, in particular $d_2 \propto \delta^2 d_0$

If δ is small $\nu W_{2\infty}^{(0)}$ dominates in the R region: $\nu W_{2\infty}^{(0)} \propto x\, d_0 \frac{M}{\nu}$. For large Q^2, $\nu W_{2\infty}^{(0)}$ also dominates, however $-2d_2 x^2$ is the leading term. If δ is not small then $\nu W_{2\infty}^{(0)}$ and $\nu W_{2\infty}^{(1)}$ are comparable in the R region, while only $\nu W_{2\infty}^{(0)}$ contributes in the D region. Terms falling faster than $\frac{1}{(y\cdot P)^3}$ with δ small are unambigrously short range (i.e., light-cone dominated for all Q^2). If δ is not small $F(y\cdot P)$ may extend far away from the light-cone despite its rapid falloff as $y\cdot P \longrightarrow \infty$. It is not surprising then that lower order singularities contribute to νW_2 when δ is not small.

Short range pieces of $F(y\cdot P)$ (falling faster than $\frac{1}{(y\cdot P)^3}$ as $y\cdot P \longrightarrow \infty$ with δ small) do not scale early. As shown above, their character changes completely in going from the R region to the D region. The utility of dividing $F(y\cdot P)$ into short and long range pieces is further diminished by the following observations. First, corrections to power behavior ($F(y\cdot P) \sim \frac{1}{(y\cdot P)^b}$ for $1 \leq b < 3$) are of the same order as these short range pieces. This follows because $f^b(\alpha)$ always contains a power series in α^2 like the series which produced Eq.'s (V.2). Second, the short range terms vanish as $\frac{1}{\nu^2}$ in the R region and as x^2 in the D region making extraction from the data exceedingly difficult.

The results of this section are summarized in Table II. We may now compare our conclusions with those of suri and Yennie. In the following we concur:

1. The relation of Regge behavior in νW_2 to power falloff of $F(y\cdot P)$ for leading Regge trajectories ($\alpha(0) > -1$) as summarized in Table II.
2. The identification of short range terms in νW_2 as those which decrease faster than $\frac{1}{(y\cdot P)^3}$ as $y\cdot P \to \infty$. Our criteria for short range terms (that they be light-cone dominated for all Q^2) necessitates that $\delta << 1$ which suri and Yennie do not require.

We differ with suri and Yennie regarding the properties of the short range contribution to νW_2. As we have emphasized, short range contributions do not scale at low Q^2. This conclusion is implicit in Eq. (III.9) of reference 4; however, suri and Yennie neglect the non-scaling term. This is a dynamical assumption: it does not follow from the range dependence of the term alone. The basis of their dynamical argument is that $\delta = MR_P$, where the radius of the proton is of the order $1/m_\pi$, making $\delta \sim 6$. If one further assumes that lower light cone singularities are dynamically suppressed (e.g. the constant, μ^2, in Eq. V.2b is small) then short range terms indeed scale at low Q^2.

Lastly we note that our conclusions regarding the $1/(y \cdot P)^2$ term (cf. Table I) can be obtained by the techniques of reference 4.[16]

VI. GENERALIZATIONS

In the preceding analysis it was assumed:

1. that the y^2 and $y \cdot P$ dependences of the functions $C_i(y^2, y \cdot P)$ factor, and
2. that the functions $C_i(y^2, y \cdot P)$ vanish as inverse powers of $y \cdot P$ or faster as $y \cdot P \rightarrow \infty$.

Those of our results in the form of explicit examples of allowed behavior of $\nu W_2(Q^2, \nu)$ are not invalidated by the lack of generality in Assumption 1. In this category fall the results that at low Q^2, leading Regge contributions need not be light-cone dominated nor scale and short range terms used not scale. Whether or not short range pieces remain light-cone dominated in the R limit when factorization is not assumed depends on how they are defined as simultaneous functions of y^2 and $y \cdot P$, and is somewhat immaterial since light-cone dominance was only an intermediate step in relating $y \cdot P$ behavior to Q^2 dependence.

The connection between power law falloff in $y \cdot P$ with $3 > b \geq 1$ and Regge behavior in νW_2 proportional to ν^{1-b} breaks down when factorization is not assumed. Suppose, for example, $C_2(y^2, y \cdot P)$ fell as $\frac{1}{(y \cdot P)^b}$ $(3 > b \geq 1)$ when $y \cdot P$ is large and y^2 is small but fell faster than any power of $y \cdot P$ when y^2 is large. Such a model for C_2 would generate a term proportional to x^{b-1} in the D limit, but of course the power low falloff in ν would not be connected to a unique $y \cdot P$ dependence since $C_2(y^2, y \cdot P)$ has no unique $y \cdot P$ behavior. Just such a term might account for scaling a low Q^2 since it falls so rapidly in $y \cdot P$ when y^2 is not small that perhaps only $y^2 \approx 0$ contributes even at small Q^2.[17] Of course it is necessary to provide some dynamical motivation for such a choice of C_2. This example illustrates the need for a dynamical rather than kinematic understanding of the early onset of scaling.

We have studied the consequences of relaxing Assumption 2. The most important modification is that the term proportional to $\frac{1}{\nu}$ in $\nu W_2(Q^2, \nu)$ need not scale for all Q^2. This comes with the inclusion of logarithms of $y \cdot P$ in $F(y \cdot P)$. Recall that $F^{b=2}(y \cdot P) \propto \frac{D_2(\delta, y \cdot P)}{(y \cdot P)^b}$ generated a term in $f(\alpha)$ proportional to $|\alpha|$. When combined with a θ-function singularity or the light-cone this contributed to νW_2 a term proportional to $\frac{Q^2}{\nu}$. Lower order light-cone singularities do not modify this result even at low Q^2 because derivatives of $f^{b=2}(\alpha)$ vanish after the first. If a term of the form

$$\tilde{F}^{b=2}(y \cdot P) \propto \frac{D_2(\delta, y \cdot P)}{(y \cdot P)^2} \log (y \cdot P)$$

is allowed, $f(\alpha)$ receives a contribution of the form $\tilde{f}^{b=2}(\alpha) \propto |\alpha| \log |\alpha|$. Derivatives of this do not vanish but instead provide the successively stronger singularities for $\alpha \approx 0$ necessary to enhance lower light-cone singularities in the R limit. Modifying other power law behavior by including logarithms or powers of logarithms does not change the results of Table II.

Finally we note that the inclusion of a simple oscillation in $F(y \cdot P)$, e.g.:

$$F(y \cdot P) \propto \frac{\cos \gamma y \cdot P \, D_b(\delta, y \cdot P)}{(y \cdot P)^b}$$

generates only an even power series in α for any $b \geq 1$. Instead of a term proportional to $|\alpha|^{b-1}$ one obtains

$$|\alpha - \gamma|^{b-1} + |\alpha + \gamma|^{b-1}$$

Such terms behave like terms which fall faster than any power and, in particular, are light-cone dominated in both the R and D limits so long as γ (analogous to $\frac{1}{\delta}$) is chosen large.

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APPENDIX

Here we derive the expansion of $I_b(\alpha, \delta)$ used in Section IV.

$$I_b(\alpha, \delta) = \frac{1}{\pi} \int_0^{1/\delta} \frac{x^b dx}{x^2 + \alpha^2} \qquad \begin{matrix} b>1 \\ b \neq 3,5,7\ldots \end{matrix} \tag{A1}$$

Since $b>1$, $I_b(0, \delta)$ is defined and equals

$$I_b(0, \delta) = \frac{\delta^{1-b}}{\pi(b-1)} \tag{A2}$$

then

$$I_b(\alpha, \delta) - I_b(0, \delta) = -\frac{\alpha^2}{\pi} \int_0^{1/\delta} \frac{dx\, x^{b-2}}{(x^2 + \alpha^2)}$$

If $b>3$ this integral also exists at $\alpha=0$ and equals $-\alpha^2 I_{b-2}(0, \delta)$. Pursuing this we obtain an even power series in α^2 until the $\alpha \longrightarrow 0$ limit of the remaining integral does not exist:

$$I_b(\alpha, \delta) = I_b(0, \delta) - \alpha^2 I_{b-2}(0, \delta) + \alpha^4 I_{b-4}(0, \delta) + \ldots \pm \frac{\alpha^{2n}}{\pi} \int_0^{1/\delta} \frac{dx\, x^{b-2n}}{x^2 + \alpha^2}$$

where n is the smallest integer such that $b-2n<1$. Consider the remaining integral:

$$G_b(\alpha, \delta) \equiv \frac{(-1)^n}{\pi} \alpha^{2n} \int_0^{1/\delta} \frac{dx\, x^{b-2n}}{x^2 + \alpha^2}$$

and scale $y = x/\alpha$

$$G_b(\alpha, \delta) = (-1)^n \frac{\alpha^{b-1}}{\pi} \int_0^{\frac{1}{\alpha\delta}} \frac{dy\, y^{b-2n}}{y^2 + 1}$$

$$= (-1)^n \frac{\alpha^{b-1}}{\pi} \left[\int_0^{\infty} \frac{dy\, y^{b-2n}}{y^2 + 1} - \int_{\frac{1}{\alpha\delta}}^{\infty} \frac{dy\, y^{b-2n}}{y^2 + 1} \right]$$

The integral from zero to infinity is tabulated:

$$\int_0^{\infty} \frac{dy\, y^{b-2n}}{y^2+1} = \frac{1}{2}\, B\left(\frac{1+b-2n}{2},\, 1-\frac{1+b-2n}{2}\right)$$

and the second integral may be rescaled back to the variable x:

$$G_b(\alpha,\delta) = (-1)^n \frac{\alpha^{b-1}}{2\pi}\, B\left(\frac{1+b-2n}{2},\, 1-\frac{1+b-2n}{2}\right)$$

$$-\frac{(-1)^n}{\pi}\,\alpha^{2n}\int_{\frac{1}{\delta}}^{\infty}\frac{dx\, x^{b-2n}}{x^2+\alpha^2}$$

We may now proceed to expand the remaining integral in powers of α^2:

$$\int_{\frac{1}{\delta}}^{\infty}\frac{dx\, x^{b-2n}}{x^2+\alpha^2} = -\, I_{b-2n}(0,\delta) + \alpha^2 I_{b-2n-2}(0,\delta) - \dots$$

and obtain finally:

$$I_b(\alpha,\delta) = \sum_{k=0}^{\infty} (-1)^k \alpha^{2k} I_{b-2k}(0,\delta) + \frac{(-1)^n \alpha^{b-1}}{2\pi}\, B\left(\frac{1+b-2n}{2},\, 1-\frac{1+b-2n}{2}\right)$$

as quoted in Section IV.

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3. J. Pestieau, P. Roy and H. Terazawa, Phys. Rev. Letters 25, 402 (1970).

4. A. suri and D. Yennie, SLAC-PUB-954, to be published in Annals of Physics (N.Y.).

5. Y. Frishman, Annals of Physics (N.Y.) 66, 373 (1971).

6. We have normalized stated to $\delta^3(P-P')$ and use the metric $(a \cdot b = a^0 b^0 - \vec{a} \cdot \vec{b})$ and other conventions of J. D. Bjorken and S. D. Drell, Relativistic Quantum Mechanics (McGraw-Hill, New York, 1964).

7. This assumption is discussed in Section IV.

8. If $f(\alpha)$ did not vanish sufficiently fast to allow the root at $\alpha = \alpha_-$ to be ignored, one also finds that νW_2 would diverge as $\nu \to \infty$ and would not vanish at threshold in the Bjorken limit. Since only forms of $f(\alpha)$ which are compatible with the gross features of the data are of interest, $f(\alpha)$ will always be such to allow us to ignore root at α_-.

9. For example, in the parton model (cf. R. Jaffe, SLAC-PUB-999) or equivalently in the light-cone analysis of Jackiw et al. (Ref. 2). The

smoothness of F(y·P) is displayed most graphically in Ref. 3. In these papers only the form of the F(y·P) multiplying the leading y^2 singularity is discussed. Having assumed factorization we carry over their results for all y^2.

10. S. P. deAlwis, Cambridge preprint DAMTP 71/39.

11. Photoproduction, $Q^2 = 0$, is included in the R limit.

12. If one is willing to contemplate νW_2 diverging as $x \to 0$, terms with $b < 1$ are allowed. In our analysis they behave like the Pomeron ($b = 1$).

13. This is not a proof that the D-limit (Q^2 fixed and large, $\nu \to \infty$) and the $x \to 0$ limit of the Bjorken limit ($Q^2, \nu \to \infty$, x fixed; then $x \to 0$) are identical. For any fixed Q^2 the coefficient, K_n, of some lower order light-cone singularity might be so large that $(\mu^2/Q^2)^n K_n$ is not negligible and the light cone does not dominate. It is only when Q^2 becomes infinite (i. e., in the Bjorken limit) that this is excluded.

14. In suri and Yennie's analysis $\sigma_{TOT}(\infty)$ changes considerably in passing from high energy photoproduction to highly inelastic electroproduction. In photoproduction $\sigma_{TOT}(\infty) = 108.0 \pm 1.8\ \mu b$; in the scaling data $\sigma_{TOT}(\infty)$ depends on how the transverse and longitudinal parts are separated. Using a combined transverse and longitudinal vector dominance model to parametrize the data they obtain $\sigma_{TOT}(\infty) = 153.8 \pm 8\ \mu b$ in the scaling data.

15. More recent photoproduction data may change the $Q^2 = 0$ figure substantially in the wrong direction (A. suri, private communication).

16. D. Yennie, private communication.

17. Note that if a Regge term were light-cone dominated for Q^2 small it would also scale for Q^2 small. This is evident in the analysis of Section Vb: there is no analog to the d_0, d_2 phenomenon which prevented the short range pieces from scaling.

TABLE I

Behavior of $F(y\cdot P)$ for large $y\cdot P$	Behavior of $f(\alpha)$	
	Near $\alpha = 0$*	as $\alpha \longrightarrow \infty$*
1. $\lim_{y\cdot P \to \infty} F^{b=1}(y\cdot P) \propto \frac{1}{y\cdot P}$	$f^{b=1}(\alpha) = \sum_{k=0}^{\infty} d_{2k}\alpha^{2k} + C^1 \log \|\alpha\|$	$f^{b=1}(\alpha) = \sum_{k=1}^{\infty} D_{2k}\alpha^{-2k}$
2. $\lim_{y\cdot P \to \infty} F^{b}(y\cdot P) \propto \frac{1}{(y\cdot P)^b}$ $b > 1$	$f^{b}(\alpha) = \sum_{k=0}^{\infty} d_{2k}\alpha^{2k} + C^b \|\alpha\|^{b-1}$	$f^{b}(\alpha) = \sum_{k=1}^{\infty} D_{2k}\alpha^{-2k}$
3. $F^{\infty}(y\cdot P)$ vanishes faster than any power	$f^{\infty}(\alpha) = \sum_{k=0}^{\infty} d_{2k}\alpha^{2k}$	$f^{\infty}(\alpha) = \sum_{k=1}^{\infty} D_{2k}\alpha^{-2k}$

* The coefficients D_{2k} and d_{2k} are, of course, different for each choice of $F(y\cdot P)$.

TABLE II

Asymptotic y· P dependence of F(y· P)	Leading contribution to νW_2		Light-Cone Dominance		Scaling	
	R limit	D limit	R limit	D limit	R limit	D limit
$\propto \frac{1}{(y\cdot P)^b}$ $1 \leq b < 3$, $b \neq 2$	$\frac{F(Q^2)}{\nu^{b-1}}$	x^{b-1}	no	yes	no	yes
$\propto \frac{1}{(y\cdot P)^2}$	$\frac{Q^2}{\nu}$	x	yes in the absence of loga-rithms	yes	yes in the absence of loga-rithms	yes
$\propto \frac{1}{(y\cdot P)^b}$ $b \geq 3$ or $\longrightarrow 0$ faster than any power	$\frac{F(Q^2)}{\nu^2}$	x^2	yes if $\delta \ll 1$; no if $\delta \gtrsim 1$	yes	no	yes

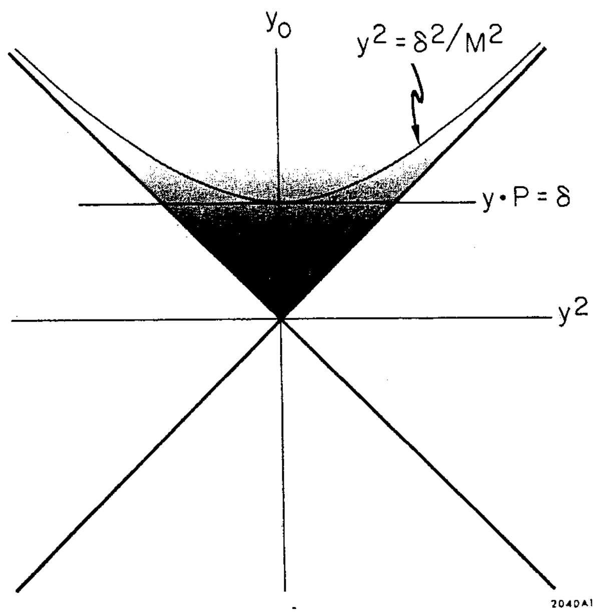

Fig. 1

The contour lines of a function $F(y \cdot P)$ which vanishes rapidly for $y \cdot P \gtrsim \delta$ are shown. Note that the $y \cdot P$ dependence ensures that the function is negligible unless $y^2 \lesssim \delta^2/M^2$.