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ON CORRELATIONS IN INCLUSIVE REACTIONS†

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ABSTRACT

We consider the nature of correlations in inclusive hadronic reactions. Assuming the existence of hadronic scaling, we study the constraints imposed by energy-momentum conservation, and deduce that they seem not to lead to correlations of long range in rapidity. We show that short-range correlations must obey various restrictions in order to guarantee consistency with conservation laws. Following Le Bellac and Wilson, we investigate the nature of correlations when diffractive processes are present, and show that these processes lead to long-range correlations. In Mueller's Regge analysis of inclusive reactions these would correspond to failures of the Pomeranchuk singularity to factorize. Lower bounds are established for the average correlation between two particles of similar type when there is nonshrinking diffraction. The existence of correlations between particles of different types cannot be established unless specific additional assumptions are made. The implications of our results for experiment are discussed.

1. Introduction

One of the most interesting questions in the field of many-particle hadronic interactions is the nature of the correlations among produced particles. It is still an open question whether particle production processes are basically weakly correlated phenomena or whether strong correlations are present. The matter is complicated by the fact that conservation laws impose certain kinematical correlations which cannot always be trivially separated from correlations of a more dynamical nature.

A very natural way to study correlations has emerged in the last few years through the study of inclusive reactions. It is possible to define a hierarchy of correlation functions for inclusive processes such that these functions together with the single particle number functions determine the theory completely[1). We examine here various simple features that these correlation functions must possess due to kinematical and dynamical reasons. For the most part we shall concentrate on two-particle correlation functions since they are the ones which are most amenable to experimental investigation. We shall study these correlation functions at high energy where it is expected that inclusive processes exhibit scaling properties[2). Thus we assume asymptotically constant total cross sections, as well as the existence of limiting fragmentation and pionization, so that multiplicities increase logarithmically with energy. These assumptions, together with some definitions, are discussed in section 2.

The kinematical constraints imposed on two-particle correlation functions by energy momentum conservation have been much discussed recently[3). We reexamine these constraints in section 3 and find, as might be expected, that when both particles are going fast in the same direction in the centre-of-mass (c.m.) frame nonzero correlations must exist because of conservation laws.

On the other hand, when one particle is show and the other is fast, or when both particles are fast but moving in opposite directions we find no kinematical constraints for the correlation functions and they may therefore vanish.

We examine in detail in section 4 the nature of two-particle correlations in weakly correlated models — models for which the two-particle correlation functions vanish for sufficiently large separation in the rapidities of the two particles in question. We find that it is physically necessary for these correlation functions to obey one of the following three conditions if the multiplicities of hadronic reactions are not to increase more than logarithmically with increasing energy:

(1) The correlation functions must change sign as the rapidity of one of the particles approaches its kinematical limit.

(2) The correlation length $\xi < 1$.

(3) The correlation functions must change sign as a function of the transverse components of momenta, in such a way that, when integrated over transverse momenta, they obey conditions (1) or (2). The Regge approach to inclusive reactions, pioneered by Mueller[4)], chooses in general condition (1). This gives rise to various sum rules relating Regge couplings which are displayed in Appendix A.

In section 5 we examine the nature of the correlation functions when there is a nonvanishing contribution to the total cross section at infinite energy which comes from diffractive processes. We consider two converses of a result of Le Bellac[5)], who showed that if the average n-particle correlation functions, $< C_n >$, increased only logarithmically with energy then the cross sections for producing n particles fall faster than any power of $(\ln s)^{-1}$. We show that if the elastic cross section vanishes at infinite energy as $(\ln s)^{-1}$ then it is necessary either that all $\langle C_{2n} \rangle$, $n > 1$, increase with energy as $\langle C_{2n} \rangle \sim (\ln s)^{2n-1}$ or that some $\langle C_n \rangle$ increases faster than $(\ln s)^{n-1}$. If exclusive diffractive cross sections are nonvanishing, we find that the lowest average correlation function, $\langle C_2 \rangle$, must increase as $(\ln s)^2$. In this case $\langle C_2 \rangle$ is bounded by the square of the average multiplicity:

$$\langle C_2 \rangle \geq \frac{\sigma^{\text{elastic}}}{\sigma^{\text{total}}} \langle n \rangle^2 \quad . \tag{1}$$

A more stringent bound on $\langle C_2 \rangle$ can be obtained, which includes other diffractive cross sections on the right-hand side of the relation (1). We have no proof that higher average n-particle correlation functions increase with energy as $(\ln s)^n$, but this seems quite likely.

We study in detail the case of nonvanishing diffraction and we show that it implies the existence of long-range two-particle correlations among particles of the same species when they are well separated in rapidity, both from each other and from the incoming particles. Because experimentally $\sigma^{\text{elastic}}/\sigma^{\text{total}}$ is at least 1/5 at accelerator energies, this correlation will also be at least 1/5 of the square of the average multiplicity if diffraction cross sections do not vanish. The presence of these long-range correlations implies a certain amount of nonfactorizability of the Pomeranchuk singularity, which, however, may not necessarily show up in total cross section or single particle distribution measurements. If the Pomeranchuk singularity has I=0 and is even under charge conjugation then particles will have long-range correlations with all other particles in the same isomultiplet and their antiparticles. For example there would be correlations between different pions and between different kaons, but the correlation functions among members of different isomultiplets need not have a long-range component.

We discuss in section 6 the feasibility of detecting these long-range correlations at the ISR and make some comments on theoretical models in the light of our results.

2. Definitions and Scaling Properties

We shall be mainly interested in one- and two-particle inclusive reactions which we denote generically as $a+b \rightarrow c+X$ and $a+b \rightarrow c_1+c_2+X$. We shall adopt as a convenient notation the particle type as a subscript for its momentum (e.g., p_a is the momentum of a). The differential cross section for the process $a+b \rightarrow c+X$ will be written as

$$E_c \frac{d\sigma^c_{ab}}{d^3p_c} = \sigma^{total}_{ab}(s)\ N^c_{ab}(p_a, p_b; p_c) \quad , \tag{2}$$

so that

$$\int \frac{d^3p_c}{E_c} N^c_{ab}(p_a, p_b; p_c) = \langle n^c \rangle_{ab} \tag{3}$$

is the average multiplicity of particle c in the ab process.

We write, similarly, for the process $a+b \rightarrow c_1+c_2+X$

$$E_{c_1} E_{c_2} \frac{d\sigma^{c_1c_2}_{ab}}{d^3p_{c_1}\, d^3p_{c_2}} = \sigma^{total}_{ab}(s)\ N^{c_1c_2}_{ab}(p_a, p_b; p_{c_1}, p_{c_2}) \quad , \tag{4}$$

and we have

$$\int \frac{d^3p_{c_1}}{E_{c_1}} \frac{d^3p_{c_2}}{E_{c_2}} N^{c_1c_2}_{ab}(p_a, p_b; p_{c_1} p_{c_2}) = \left\langle n^{c_1} n^{c_2} - \delta^{c_1c_2} n^{c_1} \right\rangle_{ab} \quad . \tag{5}$$

For ease in writing we will drop the ab subscripts whenever they are unnecessary.

We will define a two-particle correlation function $C_{ab}^{c_1c_2}$ as

$$C_{ab}^{c_1c_2}(p_a,p_b;p_{c_1},p_{c_2}) = N_{ab}^{c_1c_2}(p_a,p_b;p_{c_1},p_{c_2}) - N_{ab}^{c_1}(p_a,p_b;p_{c_1})\, N_{ab}^{c_2}(p_a,p_b;p_{c_2}) \quad . \tag{6}$$

We will return in section 5 to alternative definitions of correlation functions. It is clear that if we had totally uncorrelated production of c_1 and c_2 that $C_{ab}^{c_1c_2}=0$. This cannot be the case because overall energy momentum conservation imposes the constraints[3)]

$$\sum_c \int \frac{d^3p_c}{E_c}\, N_{ab}^{c}(p_a,p_b;p_c)\, p_c^{\mu} = (p_a+p_b)^{\mu} \, , \tag{7}$$

$$\sum_{c_2} \int \frac{d^3p_{c_2}}{E_{c_2}}\, N_{ab}^{c_1c_2}(p_a,p_b;p_{c_1},p_{c_2})\, p_{c_2}^{\mu} = N_{ab}^{c_1}(p_a,p_b;p_{c_1})(p_a+p_b-p_{c_1})^{\mu} \, , \tag{8}$$

which yield in particular

$$-\sum_{c_2} \int \frac{d^3p_{c_2}}{E_{c_2}}\, C_{ab}^{c_1c_2}(p_a,p_b;p_{c_1},p_{c_2})\, p_{c_2}^{\mu} = N_{ab}^{c_1}(p_a,p_b;p_{c_1})\, p_{c_1}^{\mu} \quad . \tag{9}$$

Thus eq. (9) informs us that at least some of the correlation functions $C_{ab}^{c_1c_2}$ are nonvanishing for purely kinematical reasons. We should remark that there are further constraints than those written above that follow from energy momentum conservation[3)]. These constraints involve, however, number functions for more than two particles.

We will be using in what follows, as a convenient kinematical variable, the particle's rapidity. It is defined in terms of the particle's momentum by

$$p_c = (m_{c_\perp} \cosh y_c, \quad p_{c_x}, \quad p_{c_y}, \quad m_{c_\perp} \sinh y_c) \ , \tag{10}$$

where $m_{c_\perp} = \sqrt{m_c^2 + p_{c_x}^2 + p_{c_y}^2}$ is the transverse mass. For large initial energies the rapidities of the produced particles in the c.m. system have kinematical regions that extend to $\pm Y$ with $Y \sim \frac{1}{2} \ln s$. Another useful variable is Feynman's[2)]

$$x_c = \frac{2p_{c_\parallel}}{\sqrt{s}} \ , \tag{11}$$

where $p_{c_\parallel}$ is the parallel momentum of particle c in the c.m. system. For large energies the kinematical limits on x are $-1 \leq x \leq 1$.

From recent studies of inclusive reactions by a variety of approaches[6)] has emerged the expectation that N_{ab}^c, $N_{ab}^{c_1 c_2}$ and higher number functions should exhibit scaling properties at high energy. By scaling we mean here that at high energy in the c.m. system as $s \to \infty$:

$$N_{ab}^c(p_a, p_b; p_c) \longrightarrow N_{ab}^c(x_c, \vec{p}_{c_\perp}) \quad , \tag{12}$$

$$N_{ab}^{c_1 c_2}(p_a, p_b; p_{c_1}, p_{c_2}) \longrightarrow N_{ab}^{c_1 c_2}(x_{c_1}, x_{c_2}, \vec{p}_{c_{1\perp}}, \vec{p}_{c_{2\perp}}) \quad . \tag{13}$$

We should note that in expression (13) the point $x_{c_1} = x_{c_2} = 0$ is somewhat special: in that case it is simpler to express $N_{ab}^{c_1 c_2}$ in terms of rapidities. The property of scaling is defined to be, for $x_{c_1} = x_{c_2} = 0$,

$$\lim_{Y \to \infty} N_{ab}^{c_1 c_2}(Y, y_{c_1}, y_{c_2}, \vec{p}_{c_{1\perp}}, \vec{p}_{c_{2\perp}}) \longrightarrow N_{ab}^{c_1 c_2}(y_{c_1} - y_{c_2}, \vec{p}_{c_{1\perp}}, \vec{p}_{c_{2\perp}}) \ . \tag{14}$$

Some limited confirmation of scaling of the single-particle number functions has been obtained recently at the ISR[7, 8)]. We shall assume in what follows that all these scaling properties obtain.

A simple consequence of the scaling behaviour of inclusive reactions is the expectation that multiplicities grow logarithmically with energy

$$\langle n^c \rangle_{ab} \sim \ln s \quad .$$

Strictly speaking, the above does not follow only from scaling. It requires also that $N^c_{ab}(x_c=0, \vec{p}_{c_\perp})$ be nonvanishing at least for some values of $\vec{p}_{c_\perp}$, and that the number function be cut off in the transverse momentum. For pions, which constitute the bulk of produced particles, both these properties have experimental backing[7]. We shall assume in this paper that this logarithmic growth obtains.

3. Energy Momentum Constraints

Here we would like to examine the constraints imposed on the two-particle correlation functions by four-momentum conservation. Much of the discussion that follows is based on the paper of Brown[3] and represents a modest extension of his work.

We begin by examining the constraint (9) for the energy component. Then we have

$$E_{c_1} N^{c_1} = - \sum_{c_2} \int d^3p_{c_2} \, C^{c_1 c_2} \quad . \tag{15}$$

Since N^{c_1} is a positive function it follows that the integral on the RHS of (15) must be negative. Thus energy momentum conservation tells us that the correlation functions $C^{c_1 c_2}$ cannot be positive definite. This is a reasonably intuitive result since it says that the inclusive distribution function for two particles cannot always exceed the product of the two one-particle distribution functions, something we expect especially when the two particles observed have large parallel momentum.

We can be more specific about the nature of the correlations demanded by energy momentum conservation by examining also the longitudinal constraint in eq. (9). We will assume we are in the scaling regime and work in the c.m. system. Then we can write in the limit as $s \to \infty$,

$$(|x_{c_1}| \pm x_{c_1}) N^{c_1}(x_{c_1}, \vec{p}_{c_{1\perp}}) = -\sum_{c_2} \int d^2 p_{c_{2\perp}} dx_{c_2} C^{c_1 c_2}(x_{c_1}, x_{c_2}, \vec{p}_{c_{1\perp}}, \vec{p}_{c_{2\perp}}) \times$$

$$\times \left(\frac{|x_{c_2}| \pm x_{c_2}}{|x_{c_2}|} \right) . \tag{16}$$

If $x_{c_1} > 0$ we have the two equations

$$N^{c_1}(x_{c_1}, \vec{p}_{c_{1\perp}}) = - \sum_{c_2} \int_{x_{c_2} > 0} d^2 p_{c_{2\perp}} dx_{c_2} C^{c_1 c_2}(x_{c_1}, x_{c_2}, \vec{p}_{c_{1\perp}}, \vec{p}_{c_{2\perp}}) , \tag{17}$$

$$0 = - \sum_{c_2} \int_{x_{c_2} < 0} d^2 p_{c_{2\perp}} dx_{c_2} C^{c_1 c_2}(x_{c_1}, x_{c_2}, \vec{p}_{c_{1\perp}}, \vec{p}_{c_{2\perp}}) . \tag{18}$$

Thus we learn that we need a negative integrated correlation for $x_{c_1} > 0$, $x_{c_2} > 0$ but need no integrated correlation for $x_{c_1} > 0$, $x_{c_2} < 0$. Similarly we find that for $x_{c_1} = 0$ we have that the integrated correlation must vanish for all x_{c_2}. Finally if $x_{c_1} < 0$ and $x_{c_2} < 0$ we find again a negative integrated correlation, but no such correlation need arise when $x_{c_1} < 0$ but $x_{c_2} > 0$. When both x_{c_1} and x_{c_2} are zero we cannot say anything from eq. (16). The pattern of integrated correlations demanded by energy momentum conservation in the scaling regime is shown in fig. 1. Its correspondence to an intuitive picture of the structure of correlations from energy momentum is manifest.

When $|x_{c_1}+x_{c_2}|>1$ the two-particle number function $N^{c_1c_2}$ vanishes because it is outside its kinematically allowed region. Thus we have

$$C^{c_1c_2}(x_{c_1},x_{c_2},\vec{p}_{c_{1\perp}},\vec{p}_{c_{2\perp}}) = -N^{c_1}(x_{c_1},\vec{p}_{c_{1\perp}})\,N^{c_2}(x_{c_2},\vec{p}_{c_{2\perp}}), \qquad |x_{c_1}+x_{c_2}|>1 \ . \tag{19}$$

In particular when $x_{c_1}\to\pm1$ the above holds for all $x_{c_2}\gtrless 0$. In this case the constraints in (16) are satisfied trivially because of eq. (7).

4. Constraints on Weakly Correlated Models

Although it is not possible for the correlation functions $C^{c_1c_2}$ to vanish identically it is fruitful to consider models in which these correlation functions are in some sense small. If this were indeed the case then it would be sufficient to determine the single-particle number functions N^{c_1} and N^{c_2} to obtain a good picture of what $N^{c_1c_2}$ was.

A useful concept, introduced by Wilson[9], is to suppose that the correlation function $C^{c_1c_2}$ vanishes for sufficiently large separation between the rapidities of c_1 and c_2, and to parameterize this by introducing a correlation length ξ_{12} so that

$$C^{c_1c_2} \sim \exp\left[-\frac{|y_{c_1}-y_{c_2}|}{\xi_{12}}\right], \qquad |y_{c_1}-y_{c_2}| \text{ large} \ . \tag{20}$$

Such a behaviour of the correlation function is characteristic of multiperipheral models[6], dual models[10], and the Feynman gas model[9], and follows in general from an analysis of inclusive processes along the lines of Mueller[4], provided the Pomeranchuk singularity is factorizable[11]. We shall refer to (20) as the weak correlation condition and call models for which (20) obtains weakly correlated models.

A decrease in the correlation function as indicated in (20), along with an assumed limitation in the transverse momentum, implies that the average correlation

$$\langle C^{c_1c_2}\rangle = \int \frac{d^3p_{c_1}}{E_{c_1}} \frac{d^3p_{c_2}}{E_{c_2}} C^{c_1c_2} \quad , \tag{21}$$

increases at most logarithmically with energy,

$$\langle C^{c_1c_2}\rangle \lesssim \ln s \quad . \tag{22}$$

Relation (22) is a general property of weakly correlated models.

We remark that the behaviour of weakly correlated models for large rapidity separation indicated in (20) cannot hold when one of the rapidities is near the kinematical boundary $y_{c_i} \sim \pm Y$ without some additional restrictions. That this is the case can be seen by considering the following sum rule, trivially derivable from eq. (9):

$$\langle n^{c_1}\rangle = \frac{-1}{m^2_{c_1}} \sum_{c_2} \int \frac{d^3p_{c_1}}{E_{c_1}} \frac{d^3p_{c_2}}{E_{c_2}} (p_{c_1} \cdot p_{c_2})\, C^{c_1c_2} \; . \tag{23}$$

Since $(p_{c_1} \cdot p_{c_2})$ is just a function of the relative rapidity, $y_{c_1} - y_{c_2}$,

$$(p_{c_1} \cdot p_{c_2}) = m_{c_{1\perp}} m_{c_{2\perp}} \cosh(y_{c_1} - y_{c_2}) - \vec{p}_{c_{1\perp}} \cdot \vec{p}_{c_{2\perp}} \quad , \tag{24}$$

we can, using (20), perform the integrand over $(y_{c_1} + y_{c_2})$ to obtain a contribution proportional to $\ln s$ on the RHS of eq. (23). If the multiplicity is not to grow more than logarithmically with energy then the integral over the relative rapidity $y_{c_1} - y_{c_2}$ must just give a constant factor. This will not be the case unless one of the following three conditions are satisfied:

(1) The correlation function changes sign for large values of $(y_{c_1} - y_{c_2})$, so that (20) ceases to be valid near the kinematic limits.

(2) The correlation length $\xi_{12} < 1$, in which case the integral over $y_{c_1} - y_{c_2}$ in (23) is clearly constant.

(3) There are cancellations in the integrals over transverse momenta in (23), so that the correlation functions integrated over transverse momenta obeys either (1) or (2).

Conditions (2) and (3) are not met generally in models. So in general the way to avoid contradictory behaviour is that condition (1) must be satisfied. In the usual Regge picture[11], with a factorizable Pomeranchuk singularity, one has $\xi_{12} \approx 2$ and there is no dependence on $(\vec{p}_{c_{1\perp}} \cdot \vec{p}_{c_{2\perp}})$. (The behaviour discussed here is the leading behaviour. There are nonleading terms which are proportional to $\vec{p}_{c_{1\perp}} \cdot \vec{p}_{c_{2\perp}}$ (Ref. 11). In fact these terms must be there if one is to satisfy the transverse momentum constraints in eq. (9) as emphasized by L. S. Brown[3].) No contradiction arises since in this model (20) is only valid when y_{c_1} and y_{c_2} are both not near $\pm Y$. There are additional contributions to (20) when y_{c_1}, y_{c_2} are near $\pm Y$ and a cancellation occurs between these terms and the terms coming from the behaviour indicated in (20), when one performs the integral indicated in (23). The net result is that $\langle n^c \rangle$ as calculated from (23) indeed only grows like ln s. What we learn then is that, in weakly correlated models, the two-particle correlation functions must change sign as we let one of the rapidities approach the kinematical limit.

In a particular model it is possible to make use of the constraints implied by (20), or more generally by (9), to relate various parameters of the model. For the Regge model most of the constraints implied by (9) follow already from (7). These types of constraints have been analyzed recently by Tye and Veneziano[12)]. The constraints that follow only from (9) involve functions in which both c_1 and c_2 are in the same fragmentation region. The relations that ensue do not appear to be directly amenable to experimental investigation. They are discussed in detail in Appendix A.

It is amusing to point out that experimentally $\langle n^{\pi}\rangle / \langle n^{K}\rangle$ is not too dissimilar from the mass ratio m_K^2/m_π^2 so that it appears that

$$\sum_{c_2} \int \frac{d^3p_{c_1}}{E_{c_1}} \frac{d^3p_{c_2}}{E_{c_2}} (p_{c_1} \cdot p_{c_2}) \, C^{c_1 c_2} \tag{25}$$

is roughly independent of whether c_1 is a pion or a kaon. We do not have a satisfactory explanation for this observation, but remark that the factor $(p_{c_1} \cdot p_{c_2})$ weights the integral in (25) toward higher invariant masses of the pair of particles in question. Thus it appears reasonable to suppose that some of the strongly SU(3) breaking effects that arise for small invariant masses such as from the presence of low mass resonances are damped out by the weighting in (25).

5. The Role of Diffractive Processes

Some recent work of Le Bellac[5)] has questioned the validity of weakly correlated models. A characteristic of weakly correlated models is that the average of the n-particle correlation function $\langle C_n \rangle$, which is defined and discussed in Appendix B, grows only logarithmically with energy

$$\langle C_n \rangle \sim \ln s \quad . \tag{26}$$

What Le Bellac showed is that the behaviour (26) is incompatible with the presence of exclusive diffractive processes. This realization is contained in the work of Wilson[9], although in a less general form.

We shall here discuss two converses of Le Bellac's result. We consider, with Le Bellac, the obviously positive quantity

$$<(n^c - <n^c>)^2> = \frac{1}{\sigma^{\text{total}}} \sum_{n=0}^{\infty} (n^c - <n^c>^2)\, \sigma_{nc} \quad . \tag{27}$$

Here σ_{nc} is the partial cross section for producing precisely n particles of type c. By "type c" we could mean particles of a given set of quantum numbers (e.g., π^+) or we could mean, for example, all charged particles, or all particles. In this last case, σ_{nc} is the usual n-particle production cross section. In general it is a partially inclusive cross section. We can write the left-hand side of (27) as

$$\begin{aligned} <(n^c - <n^c>)^2> &= <n^c(n^c-1)> + <n^c> - <n^c>^2 \\ &= <n^c> + <C^{cc}> \quad . \end{aligned} \tag{28}$$

The RHS of (27) can be bounded by the term which contains the elastic contribution so that at large energies we may write the inequality

$$<C^{cc}> + <n^c> \gtrsim <n^c>^2 \frac{\sigma^{\text{elastic}}}{\sigma^{\text{total}}} \quad . \tag{29}$$

If $\sigma^{\text{elastic}}/\sigma^{\text{total}} \to$ constant as $s \to \infty$ we see that we obtain a contradiction with weakly correlated models. For if $<n^c> \sim \ln s$ we must have also that

$$<C^{cc}> \sim (\ln s)^2 \quad . \tag{30}$$

If $\sigma^{\text{elastic}}/\sigma^{\text{total}} \to 1/\ln s$ as $s \to \infty$ (shrinking Pomeranchuk singularity) then there are no inconsistencies at this level, but a consideration of higher correlation functions again restores the inconsistency[5]. These matters are considered in detail in Appendix B where we show that if $\sigma^{\text{elastic}}/\sigma^{\text{total}} \to 1/\ln s$ as $s \to \infty$ then, provided no individual $\langle C_n \rangle$ grows faster than $(\ln s)^{n-1}$, all $\langle C_{2n} \rangle$ must grow like

$$\langle C_{2n} \rangle \sim (\ln s)^{2n-1} \qquad (n > 1) \quad . \tag{31}$$

In Appendix B we also discuss the higher correlation functions in the case $\sigma^{\text{elastic}}/\sigma^{\text{total}} \to$ constant and find no constraints on their behaviour beyond that of eq. (30).

We would like now to consider in depth the implications of having $\sigma^{\text{elastic}}/\sigma^{\text{total}} \to$ constant as $s \to \infty$. We are motivated to do so partly by the recent ISR data[13,14] on pp elastic scattering which seem to indicate a non-shrinking diffraction peak; and partly by the fact that in this case it is possible to discuss differences, at the level of the two-particle correlation functions, from weakly correlated models. We do not need, of course, to restrict ourselves to the case where only σ^{elastic} survives as $s \to \infty$. We can more generally consider that we have

$$\sigma^{c}_{\text{DIFF}} \equiv \lim_{N \to \infty} \lim_{s \to \infty} \sum_{n=0}^{N} \sigma_{nc} \neq 0 \quad . \tag{32}$$

We note that σ^{c}_{DIFF} defined in (32) may depend on particle c; however, for any c, $\sigma^{c}_{\text{DIFF}} \geq \lim_{s\to\infty} \sigma^{\text{elastic}}_{ab}$. It is possible to have $\sigma^{c}_{\text{DIFF}} > 0$ even if particle c is never produced diffractively. It is clear that analogously to eq. (29) asymptotically for large s

$$\langle C^{cc} \rangle + \langle n^c \rangle \geq \langle n^c \rangle^2 \, \frac{\lim\limits_{s \to \infty} \sum\limits_{n=0}^{N} \sigma_{nc}(s)}{\sigma^{\text{total}}} \quad . \tag{33}$$

Equation (33) holds for all values of N so we have the asymptotic bound

$$<C^{cc}> \geq <n^c>^2 \frac{\sigma^c_{DIFF}}{\sigma^{total}} . \tag{34}$$

Experimentally diffractive contributions (e.g., from elastic scattering) are at least 1/5 of total cross sections at accelerator energies. Thus if the Pomeranchuk is flat, eq. (34) suggests that there may well be long-range two-particle correlations at very high energies of the order of 1/5 the square of the single-particle multiplicity.

Equation (34) implies that the average of the two-particle inclusive distribution at high energy is always greater than the joint average of the respective single-particle distributions:

$$<n^c(n^c-1)> \geq <n^c>^2 \left(1 + \frac{\sigma^c_{DIFF}}{\sigma^{total}}\right) . \tag{35}$$

We remark that (35) holds for all types of particles so that if it is experimentally more convenient one can sum over various types of particles (e.g., charged particles). Equation (35) is, however, an asymptotic statement and may not be verifiable even at ISR energies. At subasymptotic energies we have the inequality

$$<n^c(n^c-1)> \geq <n^c>^2 \left(1 - \frac{1}{<n^c>} + \frac{\sigma^c_{DIFF}}{\sigma^{total}}\right) , \tag{36}$$

and we see that, unless $<n^c>$ is large, there may be significant variations from (35).

The net import of Le Bellac's work is that the correlation functions cannot be as simple in structure as the ones given by weakly correlated models. If there is nonshrinking diffraction the two-particle correlation functions must have, besides a short-range piece, also a piece whose nature must be one of a long-range correlation. This latter piece is responsible for the $(\ln s)^2$ growth in these correlation functions. This separation of correlations into short-range correlations, which are a result of factorizable Regge mechanisms, and long-range correlations, which are a result of diffraction, has been extensively discussed by Wilson[9] in the context of multiperipheral models. We would like to examine it here in the light of Mueller's analysis[4].

The use of Mueller's technique to relate inclusive distributions to discontinuities of forward multiparticle amplitudes yields average n-particle correlation functions which grow like $\ln s$ if the Pomeranchuk singularity is factorizable. We show in Appendix B that if diffractive cross sections fall like $1/\ln s$, as would occur if $\alpha'_P \neq 0$ and diffraction peaks shrink, then $\langle C_n \rangle \sim (\ln s)^{n-1}$ is a possibility. This behaviour is expected in the Mueller analysis because of the presence of nonfactorizing Pomeron cut corrections to scaling falling like $1/\ln s$. Thus it may well be that the result of LeBellac and its converse reflect the incorporation of unitarity in the form of Regge cuts into the analysis of multiparticle reactions via the generalized optical theorem. As we showed earlier, long-range correlations are not required by energy-momentum conservation.

If diffractive cross sections are constant, corresponding to $\alpha'_P = 0$, then the concomitant growth of $\langle C_2 \rangle \sim (\ln s)^2$ implies that the Pomeranchuk singularity cannot exactly factorize. This lack of factorization manifests itself in the existence of long-range correlations for the central region in the rapidity space of the two particles in question. In diagramatic language this long-range correlation ensues because the diagram of fig. 2 does not factorize. We should note that in Mueller-type models one would expect from this absence of factorization at Pomeranchuk vertices that the higher average correlations $\langle C_n \rangle \sim (\ln s)^n$. Although the evidence is by no means overwhelming, there appears to be experimental support for the factorization properties of exclusive diffractive cross sections[15)] and of total cross sections and single-particle distribution functions[16)]. It is possible to envisage models in which in fact one preserves factorization for total cross sections and single-particle distribution functions but fails to obtain this property for two-particle correlation functions. For example, it may be that multi-Pomeron contributions, which presumably do not factorize, are consistently more important in Mueller-Regge analysis than they are in Regge analysis.

The conclusions that we have drawn about two-particle correlation functions from the converse of Le Bellac's result apply in the first place only to correlations among particles of the same type. That no such restrictions can follow for correlations among different kinds of particles without additional assumptions can be seen by considering the following example. Suppose there were two kinds of particles and that the partial cross section for producing n particles of type 1 and m particles of type 2 factorized:

$$\sigma_{n,m} = (\delta_{n1} + P_n)(\delta_{m1} + P_m) \quad . \tag{37}$$

Here P_n and P_m are Poisson-like distributions for particles one and two. The above model has nonvanishing diffraction but it is clear that

$$\langle n^{(1)} n^{(2)} \rangle = \langle n^{(1)} \rangle \langle n^{(2)} \rangle \quad , \tag{38}$$

so that

$$\langle C^{12} \rangle = 0 \quad . \tag{39}$$

We can, however, use physical arguments to deduce constraints among correlation functions of particles of different species. Charge conservation gives the constraint[3)]

$$Q^{c_1} N^{c_1}(p_{c_1}) = -\sum_{c_2} \int \frac{d^3 p_{c_2}}{E_{c_2}} Q^{c_2} C^{c_1 c_2}(p_{c_1}, p_{c_2}) \quad , \tag{40}$$

so that if $Q^{c_1} = \pm 1$ we have

$$\langle n^{c_1} \rangle = - \sum_{c_2} Q_{c_1} Q_{c_2} \langle C^{c_1 c_2} \rangle$$

$$= - \langle C^{c_1 c_1} \rangle - \sum_{c_2 \neq c_1} Q_{c_1} Q_{c_2} \langle C^{c_1 c_2} \rangle . \tag{41}$$

If we have a nonvanishing diffractive cross section so that $\langle C^{c_1 c_1} \rangle \sim (\ln s)^2$ then we see that the above requires that at least one of the unequal species correlation functions grows like $(\ln s)^2$. In particular if we consider that π mesons are much more likely to be produced than K mesons so that presumably $< C^{\pi\pi} > \gg < C^{\pi K} >$. We can deduce that

$$\langle C^{\pi^+ \pi^-} \rangle \approx \langle C^{\pi^+ \pi^+} \rangle \quad . \tag{42}$$

Further predictions follow if we are willing to commit ourselves some more on the nature of the Pomeranchuk singularity. As remarked above, the long-range correlations come from the nonfactorizable diagram in fig. 2, and if the Pomeranchuk singularity has I=0 and is even under charge conjugation we can deduce that

$$\langle C^{\pi_i \pi_j} \rangle = \langle C^{\pi^+ \pi^+} \rangle \ , \tag{43}$$

$$\langle C^{K_i K_j} \rangle = \langle C^{K^+ K^+} \rangle \ , \tag{44}$$

where i and j go over all charges and strangenesses. No relations among π-K correlation functions ensue. More generally, we obtain equality among all correlation functions of particles of the same I-spin multiplet or among particles which can be related by charge conjugation. No relations exist for particles belonging to different multiplets without involving some higher symmetry, and those correlation functions could be, although it is unlikely, zero.

It is possible to adopt a different type of normalization for the inclusive distributions which de-emphasizes the role of diffractive processes in the correlation functions. For example one could define a correlation function

$$\widetilde{C}^{cc} = \frac{1}{\sigma^{\text{total}}}\left[E_{c_1} E_{c_2} \frac{d\sigma^{cc}}{d^3p_{c_1}\, d^3p_{c_2}} - \left(1 + \frac{\sigma^c_{\text{DIFF}}}{\sigma^{\text{total}}}\right) \frac{1}{\sigma^{\text{total}}} E_{c_1} \frac{d\sigma^c}{d^3p_{c_1}} E_{c_2} \frac{d\sigma^c}{d^3p_{c_2}}\right] . \tag{45}$$

Then

$$<\widetilde{C}^{cc}> = <n^c(n^c-1)> - \left(1 + \frac{\sigma^c_{\text{DIFF}}}{\sigma^{\text{total}}}\right) <n^c>^2 \quad , \tag{46}$$

so that we would have

$$<(n^c - <n^c>)^2> = \frac{\sigma^c_{\text{DIFF}}}{\sigma^{\text{total}}} <n^c>^2 + <n^c> + <\widetilde{C}^{cc}> \tag{47}$$

$$\geq \frac{\sigma^c_{\text{DIFF}}}{\sigma^{\text{total}}} <n^c>^2 \quad ,$$

which yields no constraint at this level on $<\widetilde{C}^{cc}>$ except

$$<\widetilde{C}^{cc}> + <n^c> \geq 0 \quad . \tag{48}$$

The definition (45) is, however, rather unnatural and does not help to elucidate the role of diffractive processes. A more natural definition would be to normalize the inclusive distributions not with σ^{total} but with $\sigma^{\text{total}} - \sigma^{c}_{\text{DIFF}}$ (experimentally this should be at moderate energies approximately $\sigma^{\text{inelastic}}$):

$$E_c \frac{d\sigma^c}{d^3 p_c} = \left(\sigma^{\text{total}} - \sigma^{c}_{\text{DIFF}}\right) \rho^c \quad , \tag{49}$$

$$E_c E_c \frac{d\sigma^{cc}}{d^3 p_c \, d^3 p_c} = \left(\sigma^{\text{total}} - \sigma^{c}_{\text{DIFF}}\right) \rho^{cc} \quad , \tag{50}$$

and then consider the correlation functions

$$C_\rho^{cc} = \rho^{cc} - \rho^c \rho^c \quad . \tag{51}$$

However, the average of (51) still grows as $(\ln s)^2$ if there are nonvanishing diffraction processes.

Yet another definition of correlation functions has been adopted by Wilson[9]. He divides from the start events into multiperipheral, i.e., short-range, and diffractive, and weights each event by the ratio of $\sigma_{\text{MULT}} = \sigma^{\text{total}} - \sigma_{\text{DIFF}}$ or σ_{DIFF} depending on the type of event. This definition also leads to long-range correlations coming from the diffractive part and so it has the same disadvantages as the ones displayed in (49) and (50). It has the further difficulty that one must experimentally decide what events are diffractive or multiperipheral.

Finally we should remark that perhaps the whole idea of redefining correlation functions so as to de-emphasize diffractive events may not be very practical because the only correlation functions which appear affected by diffractive events are the ones that deal with particles of similar type.

Further, definitions of modified correlation functions such as those of eqs. (45) and (50) are difficult to generalize naturally to correlations between different particles unless σ^{c}_{DIFF} is independent of c. In general, by redefining these correlation functions we may get rid of diffractive effects for correlations among particles of similar type, but perhaps introduce extraneous effects in the correlation functions of unlike particles.

6. Experimental Implications and Theoretical Comments

It is worthwhile discussing whether the long-range correlations that follow from constant diffractive cross sections are observable at the ISR. The correlations we have obtained, which come from the diagrams of fig. 2, would show up when the two detected particles were well separated in rapidity from each other and from the incoming particles. The total length in rapidity space available at the ISR is about 8, so the optimal conditions for observing the correlations we discuss would be when $|Y-y_{c_1}| \approx |y_{c_1}-y_{c_2}| \approx |y_{c_2}+Y| \approx 3$. Since the correlations due to Regge exchanges have a range $\Delta y \sim 2$, it is not clear that the long-range correlations we have been discussing will be dominant in any kinematic region at the ISR. However some remarks can be made: the optimal conditions correspond to particles observed with $p_{\parallel} \sim \pm(1-2)$ GeV/c, and if little correlation were observed between particles with these momenta, that would be evidence against the picture of strong long-range correlations. If large correlations were found, they might be due to short-range effects, but by comparing correlation data at different energies and rapidity separations it might be possible to separate out any long-range component. Although we have no proof, we would expect any long-range correlations to persist into the region where the two detected particles are in different fragmentation regions. The existence of such effects would support the picture we have discussed.

We would like to comment on certain models of multiparticle production processes in the light of our results. Bjorken and Bander[17)] have considered the function I(z;s) introduced by Mueller[18)] and discussed in Appendix B. If all correlations are short-range then

$$\ln I(z,s) \approx J(z) + p(z) \ln s \quad , \tag{52}$$

because all the average correlation functions, $< C_n >$, would individually have this structure. They then consider a thermodynamic limit where

$$p(z) = \lim_{s\to\infty} \frac{1}{\ln s} \ln I(z,s) \quad . \tag{53}$$

Unfortunately the existence of diffraction processes makes this thermodynamic limit somewhat problematic. Indeed if the Pomeranchuk does have $\alpha_P' = 0$, then eq. (52) would not be a useful approximation.

Certain other authors[19-21)] have proposed models in which the predominant multiparticle production mechanisms are diffractive. The double diffraction dissociation model[19)], at least in its original form, has no pionization and so does not possess the long-range correlations we discussed in section 5. Diffractive models with pionization must either have exclusive cross sections falling as some power of $(1/\ln s)$, or have positive long-range correlations of the order of the product of single particle distributions, or violate scaling as defined in section 2. The diffractive excitation model[20)] chooses the last of these three alternatives: because its exclusive cross sections $\sigma_n \sim 1/n^2$ as $n \to \infty$, it has

$$\int \frac{d^3p_{c_1}}{E_{c_1}} \frac{d^3p_{c_2}}{E_{c_2}} N^{c_1c_2} \sim \sqrt{s} \sim \int \frac{d^3p_{c_1}}{E_{c_1}} \frac{d^3p_{c_2}}{E_{c_2}} C^{c_1c_2} \quad . \tag{54}$$

This inconsistency with scaling, as derived for example from Regge theory by Mueller[4, 18], will be shared by the version of the diffractive excitation model called the Nova model[22], as it also has $\sigma_n \sim 1/n^2$. The Nova model has in addition, because of its lack of two-fireball production processes at intermediate energies, strong negative correlations between particles in different fragmentation regions. (According to the Nova model, in most interactions the produced particles will either all be going forward or all backward in the c.m. frame.) We leave it to the reader to decide whether he or she finds palatable these strong long-range correlations or violations of scaling.

Appendix A

In this appendix we consider the constraints imposed by energy-momentum conservation on the correlation functions in the context of the Mueller analysis, where scaling distributions factorize as do the Regge corrections to scaling. For simplicity we consider the case where the leading Regge trajectory is nondegenerate. Tye and Veneziano[12)] have obtained relations between Regge residues and corrections to fragmentation distributions from the sum rule

$$\sum_c \int \frac{d^3p_c}{E_c} N^c(p_a, p_b; p_c)\, p_c^\mu = (p_a+p_b)^\mu \quad . \tag{A.1}$$

In the limit of large s, and assuming scaling in the form

$$\sigma^{\text{total}}(s) = g_P^2 + g_R^2\, s^{\alpha_R - 1} \quad ,$$

$$\sigma^{\text{total}}(s)\, N^c(p_a, p_b; p_c) = g_P N_P^c(x_c, \vec{p}_{c\perp}) + g_R N_R^c(x_c, \vec{p}_{c\perp})\, s^{\alpha_R - 1} \quad ,$$

where the quantities $N_P^c(x_c, \vec{p}_{c\perp})$ and $N_R^c(x_c, \vec{p}_{c\perp})$ are defined graphically in fig. 3, eq. (A.1) can be rewritten as

$$\sum_c \int \frac{d^2p_{c\perp}\, dx_c}{|x_c|} \left(g_P N_P^c + g_R N_R^c s^{\alpha_R - 1}\right) \cdot (|x_c|, \vec{p}_{c\perp}, x_c) = \left(g_P^2 + g_R^2 s^{\alpha_R - 1}\right) \cdot (2, \vec{0}, 0) \quad .$$

Comparing the coefficients of s^0 and $s^{\alpha_R - 1}$ we obtain:

$$g_P = \sum_c \int_0^1 dx_c\, d^2p_{c\perp}\, N_P^c(x_c, \vec{p}_{c\perp}) = \sum_c \int_{-1}^0 dx_c\, d^2p_{c\perp} N_P^c(x_c, \vec{p}_{c\perp}) \quad , \tag{A.2a}$$

and

$$g_R = \sum_c \int_0^1 dx_c\, d^2p_{c\perp}\, N_R^c(x_c, \vec{p}_{c\perp}) = \sum_c \int_{-1}^0 dx_c\, d^2p_{c\perp} N_R^c(x_c, \vec{p}_{c\perp}) \quad . \tag{A.2b}$$

We now consider the energy-momentum sum rule relating the one-particle distribution to the two-particle correlation function:

$$p^{\mu}_{c_1} N^{c_1}(p_a, p_b; p_{c_1}) = -\sum_{c_2} \int \frac{d^3 p_{c_2}}{E_{c_2}} C^{c_1 c_2}(p_a, p_b; p_{c_1}, p_{c_2})\, p^{\mu}_{c_2} \quad . \qquad (A.3)$$

(Equivalently we could have considered eq. (8), which relates the one- and two-particle distributions, and have obtained the same results.) It was observed in the main text that the sum rule (A.3) was incompatible with the scaling limit unless Regge corrections to scaling observed certain consistency relations. To see this, consider the scaling limit where eq. (A.3) becomes, in the pionization case ($x_{c_1} = 0$)

$$0 = -\sum_{c_2} \int \frac{dx_{c_2}\, d^2 p_{c_{2\perp}}}{|x_{c_2}|} C^{c_1 c_2}(p_a, p_b; p_{c_1}, p_{c_2}) \cdot (|x_{c_2}|, x_{c_2})\sqrt{s} \quad . \qquad (A.4)$$

According to Regge theory, we expect that for $x_{c_2} \neq 0$ the leading terms in $C^{c_1 c_2}$ have the form

$$C^{c_1 c_2} \simeq \frac{(\sqrt{s})^{\alpha_R - 1}}{g_P^2} \left\{ g_P g_{RP}^{c_1 c_1}(\vec{p}_{c_{1\perp}}) N_R^{c_2}(x_{c_2}, \vec{p}_{c_{2\perp}}) - N_P^{c_2}(x_{c_2}, \vec{p}_{c_{2\perp}}) g_{RP}^{c_1 c_1}(\vec{p}_{c_{1\perp}})\, g_R \right\} .$$

The quantity $g_{RP}^{c_1 c_1}(\vec{p}_{c_{1\perp}})$ is defined graphically in fig. 3. On substituting this expression into the energy-momentum conservation sum rule (A.4) we find that its satisfaction is guaranteed by the relations (A.2) between Regge and Pomeranchuk residues.

We now discuss the relations that can be obtained from eq. (A.3) when one considers the limit $x_{c_1} \neq 0$:

$$(|x_{c_1}|, \vec{p}_{c_{1\perp}}, x_{c_1}) N^{c_1}(p_a, p_b; p_{c_1}) = -\sum_{c_2} \int \frac{dx_{c_2} d^2 p_{c_{2\perp}}}{|x_{c_2}|} C^{c_1 c_2}(p_a, p_b; p_{c_1}, p_{c_2}) \cdot$$

$$\cdot (|x_{c_2}|, \vec{p}_{c_{2\perp}}, x_{c_2}) \quad . \tag{A.5}$$

In this case the leading terms of $C^{c_1 c_2}$ take the form:

$$\sigma^{total}(s)\, C^{c_1 c_2} \simeq s^{\alpha_R - 1} \left\{ N_R^{c_1}(x_{c_1}, \vec{p}_{c_{1\perp}})\, N_R^{c_2}(x_{c_2}, \vec{p}_{c_{2\perp}}) + \right.$$

$$+ \frac{g_R^2}{g_P^2} N_P^{c_1}(x_{c_1}, \vec{p}_{c_{1\perp}})\, N_P^{c_2}(x_{c_2}, \vec{p}_{c_{2\perp}}) -$$

$$- \frac{g_R}{g_P} N_R^{c_1}(x_{c_1}, \vec{p}_{c_{1\perp}})\, N_P^{c_2}(x_{c_2}, \vec{p}_{c_{2\perp}}) -$$

$$\left. - \frac{g_R}{g_P} N_P^{c_1}(x_{c_1}, \vec{p}_{c_{1\perp}})\, N_R^{c_2}(x_{c_2}, \vec{p}_{c_{2\perp}}) \right\} \quad ,$$

if sign $x_{c_1} \neq$ sign x_{c_2} ,

and

$$\sigma^{\text{total}}(s)\, C^{c_1 c_2} \simeq \left[N_P^{c_1 c_2}(x_{c_1}, x_{c_2}, \vec{p}_{c_{1\perp}}, \vec{p}_{c_{2\perp}})\, g_P - N_P^{c_1}(x_{c_1}, \vec{p}_{c_{1\perp}})\, N_P^{c_2}(x_{c_2}, \vec{p}_{c_{2\perp}}) \right]$$

$$+ s^{\alpha_R - 1} \left\{ N_R^{c_1 c_2}(x_{c_1}, x_{c_2}, \vec{p}_{c_{1\perp}}, \vec{p}_{c_{2\perp}})\, g_R + \right.$$

$$+ \frac{g_R^2}{g_P^2} N_P^{c_1}(x_{c_1}, \vec{p}_{c_{1\perp}})\, N_P^{c_2}(x_{c_2}, \vec{p}_{c_{2\perp}}) -$$

$$- \frac{g_R}{g_P} N_R^{c_1}(x_{c_1}, \vec{p}_{c_{1\perp}})\, N_P^{c_2}(x_{c_2}, \vec{p}_{c_{2\perp}}) -$$

$$\left. - \frac{g_R}{g_P} N_P^{c_1}(x_{c_1}, \vec{p}_{c_{1\perp}})\, N_R^{c_2}(x_{c_2}, \vec{p}_{c_{2\perp}}) \right\} ,$$

if $\operatorname{sign} x_{c_1} = \operatorname{sign} x_{c_2}$.

$N_{R,P}^{c_1 c_2}(x_{c_1}, x_{c_2}, \vec{p}_{c_{1\perp}}, \vec{p}_{c_{2\perp}})$ are defined in fig. 3. On substituting these expressions for $C^{c_1 c_2}$ into the sum rule (A.5) the following expressions are obtained for $N_{P,R}^{c_1 c_2}$:

for $x_{c_1} > 0$

$$(1 - x_{c_1}) N_{P,R}^{c_1}(x_{c_1}, \vec{p}_{c_{1\perp}}) = \sum_{c_2} \int_0^1 dx_{c_2}\, d^2 p_{c_{2\perp}}\, N_{P,R}^{c_1 c_2}(x_{c_1}, x_{c_2}, \vec{p}_{c_{1\perp}}, \vec{p}_{c_{2\perp}}) \quad , \tag{A.6a}$$

for $x_{c_1} < 0$

$$(1 + x_{c_1}) N_{P,R}^{c_1}(x_{c_1}, \vec{p}_{c_{1\perp}}) = \sum_{c_2} \int_{-1}^0 dx_{c_2}\, d^2 p_{c_{2\perp}}\, N_{P,R}^{c_1 c_2}(x_{c_1}, x_{c_2}, \vec{p}_{c_{1\perp}}, \vec{p}_{c_{2\perp}}) \quad . \tag{A.6b}$$

These equations (A. 6) and the original relations (A. 2) are sufficient to ensure that the energy-momentum constraints (A. 5) are obeyed. They are consistency conditions that must be imposed on any Regge model of inclusive spectra, though how useful they are is open to question. Because eqs. (A. 2b) and (A. 6) refer to the coefficients of $s^{\alpha_R - 1}$ in asymptotic expressions for the inclusive distributions, they should remain valid even in the presence of long-range correlations.

Appendix B

We define an m-particle inclusive number function for the process $a+b \to c_1+c_2+\ldots+c_m+X$ as

$$E_{c_1} E_{c_2} \ldots E_{c_m} \frac{d\sigma_{ab}^{c_1 c_2 \ldots c_m}}{d^3p_{c_1}\, d^3p_{c_2} \ldots d^3p_{c_m}} =$$

$$= \sigma_{ab}^{\text{total}}(s)\, N_{ab}^{c_1 c_2 \ldots c_m}(p_a, p_b; p_{c_1}, p_{c_2}, \ldots p_{c_m}) \quad . \qquad \text{(B.1)}$$

We shall be interested in the case in which all c_i are of the same type and for ease of notation we denote simply

$$N_{ab}^{c_1 c_2 \ldots c_m}(p_a, p_b; p_{c_1}, p_{c_2}, \ldots p_{c_m}) = N_m(1, 2, \ldots m) \quad . \qquad \text{(B.2)}$$

We define m-particle correlation functions as is done in the cluster expansion in statistical mechanics[22)] via the sequence

$$N_1(1) = C_1(1) \quad ,$$

$$N_2(1,2) = C_1(1)\, C_1(2) + C_2(1,2) \quad ,$$

$$N_3(1,2,3) = C_1(1)\, C_1(2)\, C_1(3) + C_1(1)\, C_2(2,3) + C_1(2)\, C_2(1,3)$$

$$+ C_1(3)\, C_2(1,2) + C_3(1,2,3) \quad , \qquad \text{(B.3)}$$

and in general by

$$N_m(1,2,\ldots m) = \sum_{\{\ell_i\}_m} \sum_{\text{Perm.}} \underbrace{\left[C_1(\ )\ldots C_1(\ )\right]}_{\ell_1 \text{ factors}} \underbrace{\left[C_2(,)\ldots C_2(,)\right]}_{\ell_2 \text{ factors}} \ldots \underbrace{C_m(,,\ldots,)}_{\ell_m \text{ factor}} \quad .$$

(B.4)

Here ℓ_i is either zero or a positive integer and the set of integers $\{\ell_i\}_m$ satisfy the condition

$$\sum_{i=1}^{m} i\ell_i = m \quad . \tag{B.5}$$

The arguments in the C_i functions are to be filled by the m possible momenta in any order. The sum over permutations is a sum over all distinct ways of filling these arguments. We note that for any given factor product there are precisely[22)]

$$\frac{m!}{\left[(1!)^{\ell_1}(2!)^{\ell_2}\ldots(m!)^{\ell_m}\right]\ell_1!\,\ell_2!\,\ldots\ell_m!} \tag{B.6}$$

terms.

The average of the m-particle number function for particles of type c is

$$\int \prod_{i=1}^{m} \frac{d^3p_i}{E_i} N_m(p_1,p_2,\ldots p_m) = \langle n^c(n^c-1)\ldots(n^c-m+1)\rangle \quad . \tag{B.7}$$

We define the average of the m-particle correlation function as

$$\int \prod_{i=1}^{m} \frac{d^3p_i}{E_i} C_m(p_1,p_2,\ldots p_m) = \langle C_m\rangle \quad . \tag{B.8}$$

In the text we denoted the average of the two-particle correlation function by $\langle C^{cc}\rangle$. It is clear that

$$\langle C^{cc}\rangle \equiv \langle C_2\rangle \quad . \tag{B.9}$$

We may use (B.4) to express the average of the m-particle number functions in terms of the sum of averages of correlation functions. We have

$$\langle n^c(n^c-1)\ldots(n^c-m+1)\rangle = m!\sum_{\{\ell_i\}_m}\prod_{j=1}^{m}\left(\frac{\langle C_j\rangle}{j!}\right)^{\ell_j}\frac{1}{\ell_j!} \quad . \tag{B.10}$$

In obtaining (B.10) we have made use of the counting argument (B.6).

It is convenient to consider the generating function[23)]

$$I(z;s) = \frac{1}{\sigma^{\text{total}}(s)} \sum_n \sigma_{nc}(s)\,(1+z)^n \quad . \tag{B.11}$$

Thus $I(z-1;s)$ plays the role of the partition function in statistical mechanics. Then

$$\begin{aligned} I(z;s) &= 1 + \frac{1}{\sigma^{\text{total}}} \left(\sum_n z\, n \sigma_{nc} + \sum_n z^2 \frac{n(n-1)}{2!} \sigma_{nc} + \dots \right) \\ &= 1 + \sum_{m=1}^{\infty} z^m \frac{\langle n^c (n^c - 1) \dots (n^c - m + 1) \rangle}{m!} \quad . \end{aligned} \tag{B.12}$$

Using (B.10) we can rewrite this equation as

$$I(z;s) = 1 + \sum_{m=1}^{\infty} z^m \sum_{\{\ell_i\}_m} \prod_{j=1}^{m} \left(\frac{\langle C_j \rangle}{j!} \right)^{\ell_j} \frac{1}{\ell_j!} \quad . \tag{B.13}$$

The restriction (B.5) allows us to write the above as

$$I(z;s) = 1 + \sum_{m=1}^{\infty} \sum_{\{\ell_i\}_m} \prod_{j=1}^{m} \left(\frac{\langle C_j \rangle z^j}{j!} \right)^{\ell_j} \frac{1}{\ell_j!} \quad . \tag{B.14}$$

The sums over m and $\{\ell_i\}_m$ just mean that the sum is over all finite sequences. Let $[\ell_i]_m$ be a sequence with the restriction

$$\sum_i \ell_i = m \quad .$$

Then

$$I(z;s) = 1 + \sum_{m=1}^{\infty} \sum_{[\ell_i]_m} \prod_{j=1}^{m} \left(\frac{\langle C_j \rangle z^j}{j!} \right)^{\ell_j} \frac{1}{\ell_j!} \quad . \tag{B.15}$$

It then follows easily that

$$I(z;s) = \exp\left\{ \sum_{i=1}^{\infty} \frac{<C_i> z^i}{i!} \right\} \quad . \tag{B.16}$$

We shall now make use of the generating function $I(z;s)$ to compute the expectation value of $<(n^c-<n^c>)^K>$ in terms of the average correlation functions. The method of obtaining this relation is due to Brown[24)] and we are indebted to him for the elegant derivation that follows. Consider the generating function

$$f(\lambda) = \sum_K \frac{\lambda^K}{K!} <(n^c-<n^c>)^K> \quad , \tag{B.17}$$

then

$$<(n^c-<n^c>)^K> = \frac{d^K}{d\lambda^K} f(\lambda)\Big|_{\lambda=0} \quad , \tag{B.18}$$

we may express the generating function $f(\lambda)$ in terms of $I(z;s)$

$$\begin{aligned} f(\lambda) &= \sum_K \frac{1}{K!} \lambda^K \frac{1}{\sigma_{total}} \sum_n \sigma_{nc} (n^c-<n^c>)^K \\ &= \sum_K \frac{1}{K!} \lambda^K \left\{ \left(z\frac{d}{dz} - <n^c> \right)^K I(z-1;s) \right\} \Big|_{z=1} \quad . \end{aligned} \tag{B.19}$$

Thus we can write

$$f(\lambda) = \exp\left\{ \lambda \left(z\frac{d}{dz} - <n^c> \right) \right\} I(z-1;s) \Big|_{z=1} \quad . \tag{B.20}$$

The effect of the operator $e^{\lambda z \frac{d}{dz}}$ on $I(z-1;s)$ is easily seen to be

$$e^{\lambda z \frac{d}{dz}} I(z-1;s) = I(e^{\lambda} z-1) \quad . \tag{B.21}$$

Thus we have

$$f(\lambda) = e^{-\lambda<n_c>} \, I(e^{\lambda}-1;s) \quad , \tag{B.22}$$

which gives on using (B. 14)

$$f(\lambda) = \exp\left\{ (e^{\lambda}-1-\lambda) < n^c > + \sum_{\ell=2}^{\infty} \frac{< C_\ell >}{\ell !} (e^{\lambda}-1)^{\ell} \right\} \quad . \tag{B.23}$$

We are particularly interested in the even moments $<(n^c-<n^c>)^{2K}>$, since for these moments we can establish bounds. It is now a straightforward matter to generate any of these moments using (B.18) and (B.23). We write down the first few even moments

$$<(n^c-<n^c>)^2> = <n^c> + <C_2> \quad ,$$

$$<(n^c-<n^c>)^4> = <n^c> + 7<C_2> + 6<C_3> + <C_4> +$$

$$+ 3\,(<n^c> + <C_2>)\,(<n^c> + <C_2>) \quad . \tag{B.24}$$

We can obtain in general the 2Kth moment by Faà di Bruno's relation[25)]

$$<(n^c-<n^c>)^{2K}> = \left. \frac{d^{2K}}{d\lambda^{2K}} f(\lambda) \right|_{\lambda=0}$$

$$= \sum_{\{\ell_i\}_{2K}} d_{\ell_2 \ell_3 \ldots \ell_{2K}} \underbrace{\left(h^{(2)} \ldots h^{(2)}\right)}_{\ell_2 \text{ factors}} \underbrace{\left(h^{(3)} \ldots h^{(3)}\right)}_{\ell_3 \text{ factors}} \ldots \underbrace{h^{(2K)}}_{\ell_{2K} \text{ factor}} \quad . \tag{B.25}$$

Here $d_{\ell_2 \ell_3 \ldots \ell_{2K}}$ is given by (B.6) with m=2K and

$$h^{(i)} = \frac{d^i}{d\lambda^i} h(\lambda) \Big|_{\lambda=0} \quad , \tag{B.26}$$

where

$$h(\lambda) = (e^\lambda - 1 - \lambda) < n^c > + \sum_{\ell=2}^{\infty} \frac{< C_\ell >}{\ell !} (e^\lambda - 1)^\ell \quad . \tag{B.27}$$

Further the set $\{\ell_i\}_{2K}$ obey the restriction

$$\sum_{i=2}^{2K} i \, \ell_i = 2K \quad . \tag{B.28}$$

As can be easily seen the $h^{(i)}$ are given in general by

$$h^{(i)} = \sum_{j=1}^{i} d_j < C_j > \quad , \tag{B.29}$$

where the d_j are positive coefficients.

We are now in a position to prove the observations made in the text. Suppose that $\sigma^{\text{elastic}}/\sigma^{\text{total}} \to 1/(\ln s)^\alpha$, $\alpha > 0$, as $s \to \infty$. Then $< (n^c - < n^c >)^{2K} >$ is bounded asymptotically by

$$< (n^c - < n^c >)^{2K} > \; \geq \; < n^c >^{2K} \frac{\sigma^{\text{elastic}}}{\sigma^{\text{total}}} \sim (\ln s)^{2K-\alpha} \quad . \tag{B.30}$$

It is clear that this bound can be satisfied if $< C_{2K} > \sim (\ln s)^{2K-\alpha}$ for, according to (B.25), $< (n^c - < n^c >)^{2K} >$ contains a term proportional to $< C_{2K} > \sim (\ln s)^{2K-\alpha}$. If no $< C_K >$ increases faster than $(\ln s)^{K-\alpha+1}$ then it is necessary that, for all $K > 1$, $< C_{2K} > \sim (\ln s)^{2K-\alpha}$. The necessity ensues because no other terms on the RHS of (B.25) except $h^{(2K)}$ can grow as fast as $(\ln s)^{2K-\alpha}$.

If $\sigma^c_{DIFF}/\sigma^{total}$ is nonzero, then we have the asymptotic bound

$$<(n^c-<n^c>)^{2K}> \geq <n^c>^{2K} \frac{\sigma^c_{DIFF}}{\sigma^{total}} \sim (\ln s)^{2K} \quad . \qquad \text{(B.31)}$$

This bound necessitates, for K=1, that $<C_2> \sim (\ln s)^2$. This condition is also sufficient to guarantee that $<(n^c-<n^c>)^{2K}> \sim (\ln s)^{2K}$, for the RHS of (B.25) contains the term $\left(h^{(2)}\right)^K \sim (\ln s)^{2K}$.

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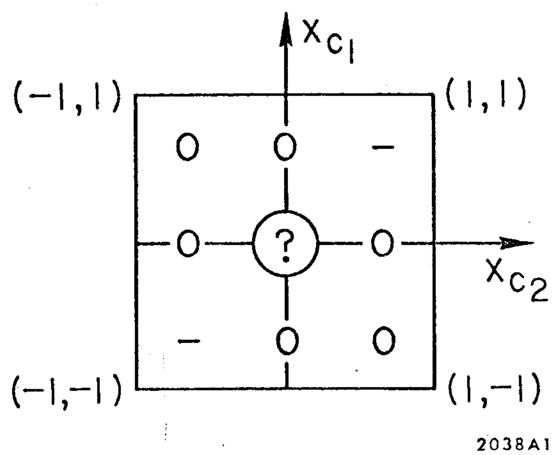
Figure Captions

1) Patterns of integrated correlations due to energy-momentum conservation.

2) Mueller diagram that does not factorize when there is diffraction.

3) Graphical definitions of functions used in Appendix A.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0
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0
x_{c_2}
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(−1,−1)
(1,−1)
2038A1

Fig. 1

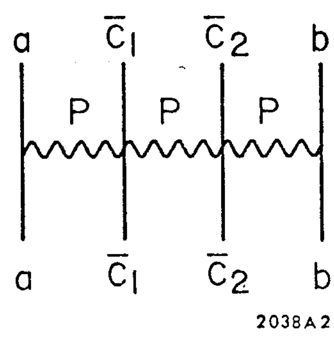
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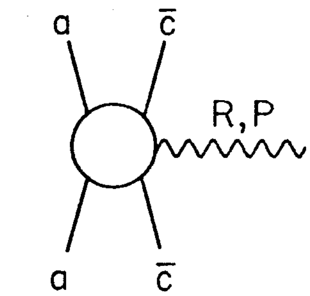
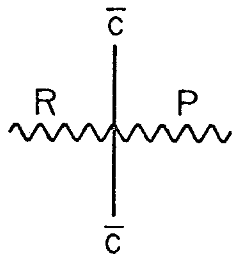
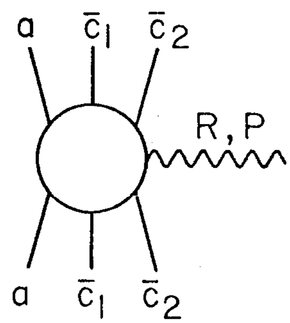
Fig. 2



$$\equiv N^{c}_{R,P}\,(x_c\,,\vec{p}_{c\perp})$$



$$\equiv g^{cc}_{RP}\;(\vec{p}_{c\perp})$$



$$\equiv N^{c_1,c_2}_{R,P}\left(x_{c_1},x_{c_2},\vec{p}_{c_1\perp},\vec{p}_{c_2\perp}\right)$$

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Fig. 3