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HIGH ENERGY BOUNDS FROM LOW ENERGY DATA*

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ABSTRACT

A method for deriving rigorous upper bounds on asymptotic values of total cross sections is discussed. The size of the bound is determined by low energy data. A functional form for the total cross section at high energies is assumed and bounds on the parameters introduced are obtained. The method is applied to $\pi - \pi$ scattering, and numerical estimates, using experimental data, are given for different parametrizations. For an asymptotically constant $\sigma_T(s)$, we find $\sigma_T(\infty) \lesssim 40$ mb. Better data would improve the accuracy of this bound and could lower it significantly.

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One problem often faced in physics is to estimate, or at least to limit, an experimental quantity in regions where it has not been measured. In this note the Froissart-Gribov representation and a knowledge of finite energy scattering will be used to limit the value of the total cross section at asymptotic energies, assuming a functional form for σ_T in this region.

Suppose that the experimental situation is as follows: the total cross section is known for x, the square of the center-of-mass momentum, between zero and c. Further, a partial wave analysis of elastic scattering has been performed for $0 \leq x \leq b \leq c$, so that a finite number of partial wave elastic cross sections are known. For values of x above c, the functional form of σ_T will be assumed. For example, one could take the cross section to be given by

$$\sigma_T(x) = \sigma_T(\infty) + \left(\sigma_T(c) - \sigma_T(\infty)\right)\sqrt{c/x}\ ;\ x \geq c \tag{1}$$

as suggested by Regge theory, and derive an upper bound on $\sigma_T(\infty)$.[1]

The Froissart-Gribov formula for the D-wave scattering length, d, for spinless particles of unit mass (isospin will be neglected for the moment) is

$$d = \int_0^\infty dx\, K(x) \sum_\ell (2\ell + 1)\, a_\ell(x) P_\ell(w)\ , \tag{2}$$

where

$$K(x) = \left[30\pi\,(1 + x)^{5/2} x^{1/2}\right]^{-1}$$

$$w = 1 + 2/x\ ,$$

and d is defined by

$$d = \lim_{x \to 0} \frac{\sqrt{x+1}}{x^{5/2}} e^{i\delta_2(x)} \sin\delta_2(x) . \tag{3}$$

The a_ℓ's are the partial wave amplitudes for the absorptive part in the crossed channel.

The constraints will be written in the form

$$\Sigma_T = \sum_\ell (2\ell + 1) a_\ell(x) = x\sigma_T(x)/8\pi \tag{4}$$

for $x > b$, and

$$a_\ell \geq |f_\ell|^2$$

for $0 \leq x < b$, where $|f_\ell|^2$ is (except for kinematic factors) the partial wave elastic cross sections for those values of ℓ which are experimentally known, and zero otherwise. The inequality follows from unitarity since b may be above inelastic thresholds.

The maximization problem is written in the form

$$\begin{aligned} \mathcal{L} = & -d + \int_0^b dx K(x) \sum_\ell (2\ell + 1)\nu_\ell(x) \left[a_\ell - |f_\ell|^2\right] \\ & - \int_b^\infty dx K(x)\alpha(x)\left[\Sigma_T - \sum_\ell (2\ell + 1)a_\ell\right] \\ & + \int_0^\infty dx K(x) \sum_\ell (2\ell + 1)\lambda_\ell(x) \left[a_\ell - a_\ell^2\right] , \end{aligned} \tag{5}$$

where d is given by Eq. (2), $\nu_\ell(x)$ and $\lambda_\ell(x)$ are Lagrange inequality multipliers and $\alpha(x)$ is a Lagrange equality multiplier.

The solution for the a_ℓ's which minimizes d is easily found by standard methods[2]:

$$a_\ell = |f_\ell|^2 \qquad (0 \le x < b) ,$$

and for x above b,

$$a_\ell(x) = 1; \qquad \ell < L$$

$$a_\ell(x) = 0; \qquad \ell > L$$

where L is the integer which satisfies

$$(2L+1)\,a_L(x) = (\Sigma_T - L^2) , \tag{6}$$

with a_L between zero and one. This set of a_ℓ's when substituted into Eq. (2) will yield a lower bound for d. The existence of a solution demands that $\alpha(x)$ be positive definite; the lower bound for d is then an increasing functional of σ_T and hence will provide an upper bound for σ_T if d is known.

In order to evaluate this bound, it is convenient to define particular values of x by the integer M,

$$x_M\,\sigma_T(x_M)/8\pi = M^2 . \tag{7}$$

Then for $x > b$, and $x_L < x < x_{L+1}$, one finds

$$\sum_\ell (2\ell+1) a_\ell P_\ell(w) = \frac{x\sigma_T}{8\pi} + \sum_{n=1}^{L} (\Sigma_T - n^2)\left(P_n(w) - P_{n-1}(w)\right) .$$

The contribution to d from values of x between 0 and b is easily written in terms of the $|f_\ell|^2$. The contribution for values of x between b and ∞ is conveniently written in terms of the integer J, where $x_J < b < x_{J+1}$. This contribution is

$$\int_b^\infty dx\, K(x)\left[\Sigma_T(x) + \sum_{n=1}^{J} (\Sigma_T(x) - n^2)(P_n - P_{n-1})\right] + \sum_{n=J+1}^{\infty} \int_{x_n}^{\infty} dx\, K(x)\, (\Sigma_T(x) - n^2)(P_n - P_{n-1}). \tag{8}$$

In evaluating this expression, it is convenient to use the expansion

$$P_n(w) - P_{n-1}(w) \simeq 2n/x + \ldots$$

which is valid if $b >> 1$ and the sum over n converges rapidly enough.

One may now evaluate expression (8) by inserting for $b < x < c$ the experimentally known $\sigma_T(x)$, and for $x > c$, the assumed analytic form for $\sigma_T(x)$. The resulting inequality may then be written

$$d - \int_0^c dx\, K(x) \sum_\ell (2\ell+1)\, a_\ell P_\ell(w) \geq [\text{contribution of (8) for } x > c]\ . \tag{9}$$

Expression (9) thus provides a bound on the parameters used to characterize $\sigma_T(x)$ for $x > c$.

This bound can be improved by including a constraint to fix the value of the elastic cross section, σ_{el}. The solution is straightforward but algebrically involved. If $\sigma_{el} = \sigma_T$, then the above bound on σ_T is not changed, of course. However, if $\sigma_{el}(x) \lesssim \frac{1}{2}\sigma_T(x)$ for $x > c$, then the upper bound on $\sigma_T(x)$ in this asymptotic region will be about 50% smaller than that given by Eq. (9).

The extension of our problem to particles with isospin is straightforward. In the case of π-π scattering, the isospin zero D-wave scattering length, d^0, can be written in terms of a linear combination of isospin amplitudes of the form

$\beta_{0I} a_\ell^I$, where $\beta_{0I} = (2/3,\ 2,\ 10/3)$. The form of the cross section constraints remains the same with

$$\Sigma_T^I(x) = x\sigma_T^I(x)/8\pi \ ,$$

and the sums over ℓ are now over even or odd values depending on the isospin value.

To estimate the size of our bound for π-π scattering, it is convenient to work in units where $m_\pi^2 = 1$. We choose $b = c = 24$ and include the δ_0^0, δ_2^0, δ_1^1, and δ_0^2 partial wave phase shifts for x below 24 ($s = 100$). Inclusion of additional partial waves in this region can only improve the bound. The expressions $\sin^2\delta_\ell^I = c_\ell^I(x)^{2\ell+1}$ where used to fit phase shifts near threshold. Above threshold curves for $\sin^2\delta_\ell^I$ were broken into several regions, and a straight line fit was used in each region.[3] With this procedure, and estimating the errors, we find that

$$0.0020 \lesssim \int_0^{24} dx K(x) \left(\frac{2}{3} a_0^0 + \frac{10}{3} a_0^2 + 6P_1(w)a_1^1 + \frac{10}{3} P_2(w) a_2^0 \right) \lesssim 0.0045 \ .$$

The value of d^0 is estimated to be around 0.0030 by extrapolating a Breit-Wigner fit to the f_0 resonance.[4] Therefore, a conservative estimate for the left hand side of expression (9) is N(0.0005) where N is a number on the order of 1.

We can use this result to treat several problems. First, we evaluate the right hand side of expression (9) by assuming that for $x \geq 24$ $\sigma_T^I(x) = B\ell n^2 \frac{s}{s_0}$, independent of I. By evaluating this expression at $x = c$ we can write s_0 in terms of B. We then use our inequality (9) to derive an upper bound on the coefficient, B. It has been shown[5] that the convergence of the d-wave scattering length implies $B \leq \pi$. Rough estimates indicate that with $N \sim 2$ and $\sigma_T(c) \sim 20$ mb, $B \lesssim \frac{\pi}{2}$.

While this does not appear to be a great improvement over previous bounds on B, improved low energy data and more careful approximation techniques could considerably decrease this value, and may be used to indicate the energies at which there is hope of experimentally detecting a $\ln^2 s$ growth in the cross section. Multiplying the coefficient of $\ln^2 s$ by only $\frac{1}{4}$ means that one must square the energy for this term to achieve its previous value.

Next we wish to treat the problem outlined in footnote (1). To facilitate evaluation of the right hand side of expression (9), we first make the reasonable assumption that $\sigma_T^I(\infty)$ is independent of I, and second, as previously suggested, we assume that $\sigma_T^I(x)$ is constant for $x > 24$ and not necessarily continuous there. This second assumption is made primarily to simplify the calculation, and our numerical results do not depend strongly on it. From the inequality (9), we then find that $\sigma_T(\infty) \lesssim 7.2$ when $N = 2$, and $\sigma_T(\infty) \lesssim 4$ when $N = 1$. This latter value is an upper bound of about 80 mb for the total cross section.

If the elastic cross section is included as a constraint in the original variational problem, and if $\sigma_{el}(x) \lesssim \frac{1}{2}\sigma_T(x)$ for $x > 24$, then the upper limit on $\sigma_T(\infty)$ is about 40 mb for $N = 1$. From factorization, one expects $\sigma_T(\infty) \approx$ 15-20 mb, and it is intriguing that our bound approaches this value so closely.

A more accurate determination of the value of our bound requires a better treatment of both the asymptotic region, $x > c$, and better data to improve the left hand side of expression (9). However, the complexity induced by more realistic assumptions on the large x behavior of $\sigma_T(x)$ does not seem warranted until the low energy region is better known. Accurate data over a large x region can considerably improve the bound on $\sigma_T(\infty)$ and thus may provide a stringent test of factorization as well as other theoretical ideas.

References

1. However, to demonstrate quantitatively that this bound is non-trivial, we shall, when we treat such a problem, assume the simpler case of a constant σ_T above c, but we shall not demand that it be continuous at x = c.
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