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SCALING AND THE NUCLEON FORM FACTORS: INFORMATION ON Z_2 AND THE DRELL-YAN-WEST RELATION*

D. J. Broadhurst†
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

ABSTRACT

New rigorous bounds on the discontinuities of off-shell nucleon form factors are systematically derived and used in two applications. First, upper limits of between 0 and 0.3 are set on Z_2, the proton's wave renormalization constant, under different assumptions about R, the ratio of longitudinal to transverse virtual photoabsorption cross sections, in the Bjorken limit, and about possible subtraction constants in the sideways dispersion relations for nucleon form factors. The bounds on $Z_2^{-1}-1$ represent an improvement by factors of 8 and 32 over previous results and permit a conclusion without neglect of a subtraction constant. If the $J^\pi = \frac{1}{2}^+$ contributions to deep inelastic scattering do not scale, $Z_2 = 0$ in any case. Secondly it is suggested that the Drell-Yan-West relation is the extremum of an inequality imposed by unitarity and analyticity. This is the case if the off-shell Dirac form factor $F(s,Q^2)$, near $s=m^2$, is a smooth but nontrivial function of s, as $Q^2 \longrightarrow \infty$, whether or not the $J^\pi = \frac{1}{2}^+$ contributions scale.

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†Harkness Fellow 1971-1973.

I. INTRODUCTION

More than ten years ago Bincer[1] proved that the form factors representing the coupling of an off-shell nucleon, of mass W, to a spacelike photon and a physical nucleon are analytic in the cut W plane. That is, they satisfy sideways dispersion relations.

Recently Cooper and Pagels[2] derived rigorous bounds on the discontinuities of these off-shell form factors in terms of the nucleon spectral functions and the structure functions W_1 and νW_2, measured in inclusive electron-nucleon reactions. Subject to certain technical assumptions--essentially about the number of subtractions required in Bincer's dispersion relations--upper bounds may be set on Z_2, the proton's wave renormalization constant, in terms of the on-shell proton electromagnetic form factors at large momentum transfer and integrals over the structure functions in the Bjorken limit. Specifically, West[3] has analysed the consequences of the vanishing of $R = \sigma_L(\nu, Q^2)/\sigma_T(\nu, Q^2)$ in the Regge and Bjorken limits (by σ_L and σ_T we mean the effective cross sections for a spacelike photon of four-momentum q and energy ν, polarised longitudinally and transversely respectively, incident on a nucleon at rest; the Regge limit is $\nu \longrightarrow \infty$, with $Q^2 = -q^2$ fixed; the Bjorken limit is $Q^2 \longrightarrow \infty$, with $\omega = 2m\nu/Q^2$ fixed). West demonstrated that given asymptotically vanishing form factors and plausible, but unproven, assumptions about possible subtraction constants in the sideways dispersion relation, the vanishing of R in these limits implies $Z_2 = 0$.

This is an interesting result since the SLAC-MIT experiments[4] on deep inelastic electron scattering indicate that R is indeed small for large ν and Q^2, and the condition $Z_2 = 0$ is part of the input to Drell, Levy and Yan's parton model[5] of deep inelastic reactions and is interpreted by them as representing an entirely composite nucleon.

There are three parts to the work reported here.

First, we improve the bounds given by Cooper and Pagels[2] and by West[3] for the discontinuities of the off-shell form factors. In Section II we show how the analysis of the off-shell electromagnetic vertex is simplified considerably by working with extensions of the Sachs form factors G_E and G_M, whose imaginary parts may be bounded by σ_L and σ_T respectively. We perform covariant calculations to obtain bounds better than those given previously. Further improvements result from bounding the combination of positive and negative cut contributions, almost as efficiently as each individually, in terms of the optimal combination of σ_L and σ_T. The resultant inequalities involving Z_2 are better by factors of 8 and 32 than those given by West[3] and by Cooper and Pagels.[2] Moreover we derive limits on the subtraction constant previously ignored by West and have a new restriction on Z_2 which is independent of its value.

Secondly, in Section III, we evaluate numerical bounds on Z_2, for the proton, subject to different assumptions about the behaviour of R in the scaling limit and the possible subtraction constants in the sideways dispersion relations. The SLAC-MIT data[4] on νW_2 as a function of ω show a remarkably early onset of scaling, for $Q^2>1$ GeV^2 and $W>2$ GeV (W is the missing mass). However R is at present not well determined. The best fit in the region $\omega \leq 4$ is with $R=0.18$, but a fit with $R=0$ is almost as acceptable. Accordingly we evaluate upper limits on Z_2 subject to the following assumptions:

(A) $R=0$ for all ω, or

(B) $R\approx 0.18$ for $\omega\leq 4$ and falls off roughly as $\omega^{-1/2}$ thereafter (so that if Regge poles offer an explanation of the behavior of νW_2 at large ν and large ω, then the Pomeron does not contribute to σ_L);

(1) A possible subtraction constant in the magnetic moment dispersion relation vanishes as $Q^2 \longrightarrow \infty$, or

(2) It tends to any finite limit.

Our results are:

$$(A_1) \quad Z_2 = 0,$$
$$(A_2) \quad Z_2 \leq 0.13,$$
$$(B_1) \quad Z_2 \leq 0.13,$$
$$(B_2) \quad Z_2 \leq 0.30.$$

We find no support for West's claim[3] that $Z_2 \leq 0.1$ with assumptions roughly corresponding to case (B_1), since his input is an inequality weaker than ours by a factor of 8.

Finally, Section IV is addressed to the relation obtained by Drell and Yan[6] and by West[7] in parton models, by Bloom and Gilman,[4,8] from an extension of observed correlations between resonance electroproduction and scaling behaviour, and more recently by Drell and Lee,[9] namely that if the threshold behaviour of νW_2 in the Bjorken limit is $\nu W_2 \propto (\omega - 1)^p$, then $F_1(Q^2)$, the proton's Dirac form factor, falls as $Q^{-(p+1)}$ as $Q^2 \longrightarrow \infty$.

From the rigorous bounds of Section III we deduce the following restriction on the behaviour of the off-shell nucleon form factor $F(s, Q^2)$, which reduces to the Dirac form factor at $s = m^2$: all but the first few derivatives $d^nF(m^2, Q^2)/ds^n$ must vanish as $[\log(Q^2)]^c\, Q^{-(p+1)}$ or faster as $Q^2 \longrightarrow \infty$.

The number of derivatives we are unable to restrict depends on the number of subtractions required in the sideways dispersion relation and the asymptotic behaviour of the nucleon spectral function. For example with $p=3$ and $\rho_1(s) \longrightarrow C/s$ (which results in $Z_2=0$) our result holds for $n \geq 2$, provided, of course, a twice subtracted dispersion relation is valid.

The actual Q^2 dependence of the form factor depends on how restrictive is our application of the Schwarz inequality. We argue that in this application it is a stringent restriction.

The import of our result is that if $F(s, Q^2)$ is a smooth but sufficiently varying function of s, near $s=m^2$, for large Q^2, then the Drell-Yan-West result is the extremum of an inequality imposed by unitarity and analyticity. It is interesting to note that whilst the νW_2 data are consistent with $p=3$ there is evidence[6] that $G_M(Q^2)$ is falling faster than Q^{-4} at large Q^2. That is consistent with the direction of our inequality.

Our failure rigorously to bound the strictly on-shell form factor is perhaps not surprising when one considers the failure of an operator product expansion treatment to predict a sufficiently rapidly falling form factor.[10]

II. DERIVATION OF INEQUALITIES

Consider the process: nucleon (p) + spacelike photon (q) $\longrightarrow$ off-shell nucleon (p'), where the kinematics are those for inelastic electron scattering, namely $p^2 = m^2$, $q^2 = -Q^2 < 0$, $p'^2 = W^2 = s$ and

$$2pq = 2m\nu = s - m^2 + Q^2 = \omega Q^2. \tag{1}$$

Bincer[1] has defined off-shell electromagnetic form factors in terms of the LSZ reduced matrix element

$$\Gamma_\mu(p,\ q,\ s) = i\int d^4x\, e^{ip'x}(i\not\partial - m)\langle 0\,|\,\Theta(x_0)[\psi(x),\ j_\mu(0)]\,|\ p,\ s\rangle (p_0/m)^{1/2}$$

$$= \left(\frac{\not p' + W}{2W}\right)\left[F_1(W,\ Q^2)\,\gamma_\mu + F_2(W,\ Q^2)\, i\sigma_{\mu\nu}q^\nu + F_3(W, Q^2)q_\mu\right]u(p,s)$$

$$+\ (W \longrightarrow -W). \tag{2}$$

For $Q^2 \geq 0$ the form factors are analytic in the cut W plane[1] and their imaginary parts are given by the absorptive vertex function

$$A_\mu(p,q,s) = \frac{1}{2}\,(2\pi)^4 \sum_n \delta(p' - n)(\not p' - m)\langle 0|\psi(0)|n\rangle\langle n|j_\mu(0)|p,s\rangle (p_0/m)^{1/2}. \tag{3}$$

The form factor F_3 may be eliminated by virtue of the Ward-Takahashi identity, which we write as

$$q^\mu\Gamma_\mu(p,\ q,\ s) = \alpha\left(\frac{\not p' + W}{2W}\right)\not q\ u(p,\ s) + (W \longrightarrow -W), \tag{4}$$

where α is the charge in units of e(=1 or 0). Equation (4) then requires

$$Q^2 F_3(W,\ Q^2) = (W - m)(F_1(W,\ Q^2) - \alpha). \tag{5}$$

The projection operators for $F_{1,2}$ as given by Bincer are rather complicated. Considerable simplification results from working with the off-shell analogues of the Sachs form factors, projected out by the polarisation vectors ϵ_L and ϵ_T.

Let

$$\epsilon_L^\mu = \frac{p^\mu + (m\nu/Q^2)q^\mu}{m(1+\nu^2/Q^2)^{1/2}} \tag{6}$$

and let ϵ_T be any unit spacelike vector orthogonal to both p and q. Then $\epsilon_L q = \epsilon_T q = \epsilon_L \epsilon_T = 0$, $\epsilon_L^2 = -\epsilon_T^2 = 1$, $\epsilon_L p = m(1+\nu^2/Q^2)^{1/2}$ and $\epsilon_T p = 0$.

We define form factors $G_{E,M}$ in analogy with Eq. (4):

$$\epsilon^\mu_{L,T}\Gamma_\mu(p,q,s) = G_{E,M}(W,Q^2)\left(\frac{\not{p}' + W}{2W}\right) \not{\epsilon}_{L,T}\, u(p,s) + (W \longrightarrow -W) \tag{7}$$

and from Eqs. (2), (5) and (7) obtain

$$\begin{aligned} G_E(W,Q^2) &= F_1(W,Q^2) - (Q^2/(W+m))F_2(W,Q^2) \\ &= \frac{Q^2}{(W+m)^2} \quad \frac{m}{(W-m)^2+Q^2} \sum_s \bar{u}(p,s)\not{\epsilon}_L(\not{p}' + W)\epsilon^\mu_L \Gamma_\mu(p,q,s), \end{aligned} \tag{8}$$

$$\begin{aligned} G_M(W,Q^2) &= F_1(W,Q^2) + (W+m)\, F_2(W,Q^2) \\ &= \frac{m}{(W-m)^2+Q^2} \sum_s \bar{u}(p,s)\not{\epsilon}_T(\not{p}' + W)\epsilon^\mu_T\, \Gamma_\mu(p,q,s). \end{aligned}$$

The imaginary parts of $G_{E,M}(W,Q^2)$ can now be bounded, covariantly, in terms of the nucleon spectral functions $\rho_{1,2}(W^2)$ and the inclusive structure functions $W_{1,2}(W^2,Q^2)$, by a Schwarz inequality for the sum over the intermediate states in the absorptive vertex function <u>and</u> the sum over spin in the projection operator. From Eq. (3)

$$\left| \sum_s \bar{u}(p,s) \not{\epsilon}(\not{p}' + W) \epsilon^{\mu} A_{\mu}(p,q,s) \right|^2$$

$$= \left| \frac{1}{2}(2\pi)^4 \sum_{n,s} \delta(p' - n) \left\{ \bar{u}(p,s) \not{\epsilon}(\not{p}' + W)(\not{p}' - m) \langle 0 | \psi(0) | n \rangle \right\} \left\{ \langle n | \epsilon \, j(0) | p,s \rangle (p_0/m)^{1/2} \right\} \right|^2$$

$$\leq \frac{1}{4}(2\pi)^8 \left\{ \mathrm{Trace}\left[(\not{p} + m) \not{\epsilon} (\not{p}' + W) \not{\epsilon} \right] (W - m)^2 (W^2 \rho_1(W^2) + W \rho_2(W^2))/m \right\}$$

$$\times \left\{ 2 \epsilon_{\mu} \epsilon_{\nu} W^{\mu\nu}(W^2, Q^2)/(2\pi)^6 \right\}, \tag{9}$$

where

$$W^{\mu\nu}(W^2, Q^2) = W_2(W^2, Q^2)(1 + \nu^2/Q^2) \, \epsilon_L^{\mu} \epsilon_L^{\nu} - W_1(W^2, Q^2)(g^{\mu\nu} + q^{\mu} q^{\nu}/Q^2). \tag{10}$$

It will be convenient to work with

$$R(W^2, Q^2) = \sigma_L/\sigma_T = W_L(W^2, Q^2)/W_1(W^2, Q^2), \tag{11}$$

where

$$W_L(W^2, Q^2) = (1 + \nu^2/Q^2) W_2(W^2, Q^2) - W_1(W^2, Q^2). \tag{12}$$

From Eqs. (8) to (12) we obtain bounds on the imaginary parts of $G_{E,M}(W, Q^2)$

$$\left| \frac{\mathrm{Im}\, G_E(W, Q^2)}{W - m} \right|^2 \leq L(W, Q^2) \; \frac{(W + m)^2 + Q^2}{(W + m)^2} \; \frac{R}{R + 1}, \tag{13}$$

$$\left|\frac{\operatorname{Im} G_M(W,Q^2)}{W-m}\right|^2 \le L(W,Q^2)\,\frac{(W+m)^2+Q^2}{Q^2}\quad\frac{1}{R+1}\,, \tag{14}$$

where

$$L(W,Q^2) = \pi^2 W_2(W^2,Q^2)(W^2\rho_1(W^2) + W\rho_2(W^2))/m. \tag{15}$$

Next consider any combination of G_E and G_M

$$G(W,Q^2) = \Big[(W+m)a(W,Q^2)\,G_E(W,Q^2)$$

$$+ Qb(W,Q^2)\,G_M(W,Q^2)\Big]\Big/\Big[(W+m)^2+Q^2\Big]^{1/2}\,, \tag{16}$$

where a and b are arbitrary real functions. The orthogonality of ϵ_L and ϵ_T enables us to bound Im G in terms of any combination $(\sigma_L + c\,\sigma_T)$, with c an arbitrary positive function of W and Q^2. From inequalities (13) and (14)

$$\left|\frac{\operatorname{Im} G(W,Q^2)}{W-m}\right|^2 \le L(W,Q^2)\left(\Big[a(W,Q^2)\Big]^2 + \Big[b(W,Q^2)\Big]^2/c\right)\left(\frac{R+c}{R+1}\right), \tag{17}$$

of which the most efficient version is with $c = R^{1/2}\,|b(W,Q^2)/a(W,Q^2)|$, giving

$$\left|\frac{\operatorname{Im} G(W,Q^2)}{W-m}\right|^2 \le L(W,Q^2)\left(R^{1/2}\,\Big|a(W,Q^2)\Big| + \Big|b(W,Q^2)\Big|\right)^2/(R+1). \tag{18}$$

Inequalities (13) and (14) are then special cases of this result.

We shall be concerned with obtaining limits on Z_2, which depends only upon $\rho_1(W^2)$. The second spectral function may be eliminated by the positivity requirement $|W\rho_2(W^2)| \le W^2\rho_2(W^2)$. It is however more efficient to use this relation <u>after</u> considering the contribution of both positive and negative cuts to the dispersion relation in W. Here we use the orthogonality of $(\not{p} + W)$ and $(\not{p} - W)$ to obtain from inequality (17)

$$\left[\left|\frac{\mathrm{Im}\, G(W, Q^2)}{W - m}\right| + \left|\frac{\mathrm{Im}\, G(-W, Q^2)}{W + m}\right|\right]^2 \leq L(W, Q^2)\left(\left[a(W, Q^2)\right]^2 + \left[b(W, Q^2)\right]^2 /c\right) \frac{R + c}{R + 1} + (W \longrightarrow - W), \tag{19}$$

where c is now an arbitrary positive function of W^2 and Q^2.

The optimal value for c depends upon the sign of the 'discriminant'

$$d(W^2, Q^2) = -\left(\left[b(W, Q^2)\right]^2 - \left[b(-W, Q^2)\right]^2\right) \Big/ \left(\left[a(W, Q^2)\right]^2 - \left[a(-W, Q^2)\right]^2\right). \tag{20}$$

Our general result is

$$\left[\left|\frac{\mathrm{Im}\ G(W, Q^2)}{W - m}\right| + \left|\frac{\mathrm{Im}\ G(-W, Q^2)}{W + m}\right|\right]^2 \leq M(W^2, Q^2) N(W^2, Q^2), \tag{21}$$

where

$$M(W^2, Q^2) = \pi^2 W_2(W^2, Q^2) 2W^2 \rho_1(W^2)/m \geq L(\pm W, Q^2), \tag{22}$$

and for $d \geq 0$ we use inequality (17) with $c = d$

$$\begin{aligned} N(W^2, Q^2) &= ([a(W, Q^2)]^2 + [b(W, Q^2)]^2/d)\left(\frac{R+d}{R+1}\right), \\ &= ([a(-W, Q^2)]^2 + [b(-W, Q^2)]^2/d)\left(\frac{R+d}{R+1}\right), \end{aligned} \tag{23}$$

but for $d \leq 0$ we use inequality (18),

$$N(W^2, Q^2) = \mathrm{Max}\left\{\left(R^{1/2}\left|a(W, Q^2)\right| + \left|b(W, Q^2)\right|\right)^2, (W \longrightarrow -W)\right\} /(R+1). \tag{24}$$

From Eqs. (8), (16) and (20) we find $d = 1$ for $G = F_{1,2}$, so that by (23) our bound is independent of the value of R and will be used in our discussion of the Drell-Yan-West relation in Section IV. However, for $G = (W - m) F_{1,2}$ we find $d < 0$, giving bounds on Z_2 which will be sensitive to the behaviour of R in the Bjorken limit. The following specific cases of inequality (21) are required:

$$\left[\left|\frac{\operatorname{Im} G_E(W,Q^2)}{W-m}\right| + \left|\frac{\operatorname{Im} G_E(-W,Q^2)}{W+m}\right|\right]^2 \leq M(W^2,Q^2)\ \frac{(W-m)^2+Q^2}{(W-m)^2}\ \frac{R}{R+1}, \tag{25a}$$

$$\left[\left|\operatorname{Im} F_1(W,Q^2)\right| + \left|\operatorname{Im} F_1(-W,Q^2)\right|\right]^2 \leq \frac{M(W^2,Q^2)(W^2-m^2)^2}{(W-m)^2+Q^2}\ \frac{(R^{1/2}+Q/(W-m))^2}{R+1}, \tag{25b}$$

$$\left[\left|\operatorname{Im} F_2(W,Q^2)\right| + \left|\operatorname{Im} F_2(-W,Q^2)\right|\right]^2 \leq \frac{M(W^2,Q^2)(W+m)^2}{(W-m)^2+Q^2}\ \frac{(R^{1/2}+(W-m)/Q)^2}{R+1}, \tag{25c}$$

$$\left[\left|\frac{\operatorname{Im} F_1(W,Q^2)}{W-m}\right| + \left|\frac{\operatorname{Im} F_1(-W,Q^2)}{W+m}\right|\right]^2 \leq M(W^2,Q^2), \tag{25d}$$

with $Q^2 \geq 0$ and $W \geq (m+\mu)$, where μ is the pion mass.

Inequalities (25a, b,c) are now used to obtain bounds on Z_2, given assumptions about the asymptotic behaviour of R and about the subtraction constants required in the sideways dispersion relations.

We shall assume that Bincer's form factors $F_{1,2,3}(W,Q^2)$ are bounded in both W and Q^2 and hence satisfy once subtracted dispersion relations in W for $Q^2 \geq 0$. We are then given the subtraction constant for F_1 (or, equivalently, F_3) by the Ward identity,[5] which requires $F_1(\pm\infty, Q^2) = \alpha$ (or, equivalently, $F_3(m,Q^2) = 0$). In addition a subtraction constant for F_2 is required. It will be convenient to take this as $\Gamma(Q^2) = (Q^2/2m)\, F_2(-m,Q^2)$, since this is proportional to the residue of $G_E(W,Q^2)$ at its kinematic pole (see Eq. 8).

The dispersion relation for $G_E(m,Q^2)$ then reads

$$G_E(m,Q^2) = \alpha - \Gamma(Q^2) + \frac{1}{\pi} \int_{(m+\mu)}^{\infty} dW \left[\frac{\mathrm{Im}\, G_E(W,Q^2)}{W-m} + \frac{\mathrm{Im}\, G_E(-W,Q^2)}{W+m}\right] \tag{26}$$

and from inequality (25a) we obtain

$$\left|\alpha - \Gamma(Q^2) - G_E(m,Q^2)\right|^2 \leq \left| \int_{(m+\mu)^2}^{\infty} dW^2 \left[\rho_1(W^2) \frac{W_2(W^2,Q^2)}{2m} \left(1 + \frac{Q^2}{(W-m)^2}\right) \frac{R}{R+1}\right]^{1/2} \right|^2$$

$$\leq (Z_2^{-1} - 1) \int_{(m+\mu)^2}^{\infty} dW^2 \left[\frac{W_2(W^2,Q^2)}{2m} \left(1 + \frac{Q^2}{(W-m)^2}\right) \frac{R}{R+1}\right], \tag{27}$$

using a further Schwarz inequality for the integration. Thus, writing the integral over $\omega = 2m\nu/Q^2$,

$$\frac{Z_2}{1 - Z_2} \left|\alpha - \Gamma(Q^2) - G_E(m,Q^2)\right|^2 \leq \int_{1+\mu(2m+\mu)/Q^2}^{\infty} d\omega \; \frac{\nu W_2(W^2,Q^2)}{\omega(\omega-1)} \left(\omega + \frac{2m}{W-m}\right) \frac{R}{R+1}. \tag{28}$$

We now consider the $Q^2 \to \infty$ limit of inequality (28) and assume the Bjorken limit $\nu W_2(W^2,Q^2) \to F_2(\omega)$, consistent with the SLAC-MIT data. The form factors $G_{E,M}(m,Q^2)$ fall at least as fast as Q^{-4} experimentally. Nothing is known about the subtraction constant. This gives

$$\frac{Z_2}{1 - Z_2} \left|\alpha - \Gamma(\infty)\right|^2 \leq \int_1^{\infty} d\omega \; \frac{F_2(\omega)}{(\omega-1)} \; \frac{R}{R+1} \tag{29a}$$

and similarly from (25b,c)

$$\frac{Z_2}{1 - Z_2} \ |\alpha|^2 \leq \int_1^\infty d\omega \ \frac{F_2(\omega)}{\omega^2} \ \frac{([R(\omega-1)]^{1/2} + 1)^2}{R + 1} \ , \tag{29b}$$

$$\frac{Z_2}{1 - Z_2} \ |\Gamma(\infty)|^2 \leq \int_1^\infty d\omega \ \frac{F_2(\omega)}{\omega^2} \ \frac{([R/(\omega - 1)]^{1/2} + 1)^2}{R + 1} \ . \tag{29c}$$

From (29b,c) follow two weaker inequalities which do not depend upon R,

$$\frac{Z_2}{1 - Z_2} \ |\alpha|^2 \leq \int_1^\infty d\omega \ \frac{F_2(\omega)}{\omega} \ , \tag{29d}$$

$$\frac{Z_2}{1 - Z_2} \ |\Gamma(\infty)|^2 \leq \int_1^\infty d\omega \ \frac{F_2(\omega)}{\omega(\omega-1)} \ . \tag{29e}$$

The inequalities (29a,b) are those we shall use in Section III. The relevant integrals converge if $RF_2(\omega)$ falls as some power of ω as $\omega \to \infty$. Inequality (29a) is an improvement by a factor of 8 of a previous result of West and inequality (29d) corresponds to a result of Cooper and Pagels improved by a factor of 32 (see footnote 11).

III. EVALUATION OF NUMERICAL BOUNDS ON THE PROTON'S Z_2

We now use the data on inclusive inelastic electron-proton scattering to evaluate bounds on Z_2, subject to various assumptions about the behaviour of R in the Bjorken limit and about the subtraction constant $\Gamma(\infty)$. Inequalities (29a,b) furnish a sufficient basis for our discussion. It can be shown that the elimination of $\Gamma(\infty)$ between inequalities (29a,c) yields an inequality no more stringent than (29b), whatever the values of R.

A very good fit[4] to the νW_2 data with $Q^2 \geq 1$ GeV^2 and W≥2.0 GeV is

$$\nu W_2(W^2, Q^2) = 0.557\,(1 - 1/\omega')^3 + 2.1978\,(1 - 1/\omega')^4 - 2.5954\,(1-1/\omega')^5, \tag{30}$$

where $\omega' = \omega + m^2/Q^2$ is indistinguishable from ω in the Bjorken limit. This fit encompasses data in the region $0.8 > 1/\omega' > 0.1$ and we shall use it in evaluating the integrals. A crucial feature of this fit is that $F_2'(\omega)$ tends to a constant as $\omega \longrightarrow \infty$, corresponding to a diffractive contribution in the scaling limit. A necessary condition then for the convergence of the integrals of inequalities (29a,b) is that R vanish as $\omega \longrightarrow \infty$, i.e., that the Pomeron couple to transversely but not to longitudinally polarised photons in the scaling limit. In fact R is not well determined at present. The best fit is with $R = 0.18 \pm 0.1$ with no indication of a strong dependence upon ν and Q^2. However $R = 0$ is almost as acceptable. Accordingly we give bounds on Z_2 in the alternative situations:

(A) $R = 0$ for all ω, or

(B) $R \approx 0.18$ in the region at present well investigated, with $\omega \leq 4$, and falls roughly as $\omega^{-1/2}$ thereafter, since only nonleading trajectories with $\alpha \lesssim \frac{1}{2}$ contribute to σ_L.

It remains to consider the possible subtraction constant $\Gamma(\infty) = \lim_{Q^2 \to \infty} (Q^2/m) F_2(-m, Q^2)$. A slightly different derivation yields inequality (29a) with an alternative subtraction constant $\Gamma'(\infty) = \lim_{Q^2 \to \infty} 2m\,\partial G_E(m, Q^2)/\partial W$, which is the result of West[3] improved by a factor of 8. We know of no argument whereby the vanishing of $F_2(m, Q^2)$ or $G_E(m, Q^2)$ at large Q^2 permits one to set either subtraction constant equal to zero. Therefore we study the alternatives

(1) $\Gamma(\infty) = 0$, permitting a bound on Z_2 from (29a), or

(2) $\Gamma(\infty)$ is finite, requiring a bound on Z_2 from (29b).

The numerical results are

$$(A_1) \quad Z_2 = 0,$$

$$(A_2) \quad Z_2 \leq 0.13,$$

$$(B_1) \quad Z_2 \leq 0.13,$$

$$(B_2) \quad Z_2 \leq 0.30.$$

We remark that West[3] has claimed a limit $Z_2 \leq 0.1$ results from a fit to the deep inelastic data with $R = 0.18$. This numerical result is difficult to reconcile with his input, which is inequality (29a) weakened by a factor of 8. The relevant integral diverges unless $RF_2(\omega)$ is assumed to vanish at large ω so that no conclusion is possible without assumptions qualitatively similar to (B_1), which give $Z_2 \leq 0.54$ in his case as against our bound $Z_2 \leq 0.13$.

Finally in this section, we comment on the stringency of our inequalities. In deriving bounds on the discontinuities of the form factors a Schwarz inequality was used for the sum over intermediate states. Only those with the same quantum numbers as the nucleon contribute, so that our result may be stated in terms of the structure function for producing only $I = \frac{1}{2}$, $J^\pi = \frac{1}{2}^+$ final states, rather than the full inclusive structure functions. It is evident that the lack of experimental information or theoretical models relevant to the final states in electroproduction forces us considerably to weaken our inequalities by dealing with the inclusive process. For spacelike photons we have no bounds on the partial wave projections of the forward virtual Compton amplitudes. If, however, some t dependence is assumed for nonforward virtual Compton scattering in the Bjorken limit, greater stringency may be obtained by projecting out the $J^\pi = \frac{1}{2}^+$ component with the appropriate d-functions.[12] For example a form $\exp(Bt)$, with B a constant, would lead to the vanishing of $J^\pi = \frac{1}{2}^+$ contributions in the scaling limit and one could conclude $Z_2 = 0$, irrespective of the behaviour of R. On the other hand a t dependence of the form $\exp(t/Q^2)$ is not sufficient to permit this conclusion.

IV. THE DRELL-YAN-WEST RELATION

By the Drell-Yan-West relation we mean the prediction[6,7] that the Dirac form factor of the proton vanishes as $Q^{-(p+1)}$ as $Q^2 \longrightarrow \infty$, if $F_2(\omega) \propto (\omega-1)^p$ near $\omega = 1$. The data are consistent with this relation with $p = 3$.

Originally this result was obtained by Drell and Yan[6] from their parton model in which the nucleon is supposed to behave as a collection of instantaneously free pointlike constituents when viewed interacting with a highly virtual photon in the infinite momentum frame. In this picture the structure function is given by

$$\nu W_2(s, Q^2) = x \sum_n \sum_{a=1}^{n} f_a^n(x) \lambda_a^2 , \tag{31}$$

where $f_a^n(x)$ is the probability that a parton a, of charge λ_a, among a collection of n partons, carries a fraction $x = 1/\omega$ of the total longitudinal momentum. Drell and Yan showed that the Dirac form factor is given by

$$F_1(Q^2) = \int_0^1 dx \sum_n \sum_{a=1}^{n} g_a^n(x, Q^2) \lambda_a , \tag{32}$$

where there exists a Schwarz inequality

$$\left| g_a^n (x, Q^2) \right| < f_a^n (x). \tag{33}$$

They argue that the dominant contribution to (32) occurs when the interacting parton carries all but a fraction (m/Q) of the longitudinal momentum, with m a characteristic mass. Thus $F_1(Q^2)$ is bounded by $(m/Q)^{p+1}$ and it is assumed that the inequality reflects the actual Q dependence.

The same result was obtained by West[7] independently, in a parton model. Bloom and Gilman[4,8] have suggested that it may be viewed as an extreme case of observed correlations between resonance electroproduction and scaling behaviour. More recently Drell and Lee[9] have given a covariant model of the proton as a bound state in which the result, with p=3, follows using the Bethe-Salpeter equation in the ladder approximation, with scalar gluons.

Our concern here is to extract as much as is possible from the proven analyticity properties of the electromagnetic form factors and the rigorous unitarity bounds of Section II, without additional model-dependent input.

We define a suitable off-shell form factor in terms of Bincer's form factor $F_1(W, Q^2)$. Let

$$F(s, Q^2) = \left(\frac{W + m}{2W}\right) F_1(W, Q^2) + (W \longrightarrow -W), \tag{34}$$

where $s = W^2$. Then $F(s, Q^2)$ is analytic in the cut s plane and reduces to $F_1(Q^2)$ at $s = m^2$. Its imaginary part is restricted by inequality (25d):

$$\left| \mathrm{Im}\, F(s, Q^2) \right| \le \pi(s - m^2) \left[W_2(s, Q^2)\, \rho_1(s)/2m \right]^{1/2}. \tag{35}$$

We now show that all but the first few derivatives $d^nF(m^2, Q^2)/ds^n$ are bounded by $[\log Q^2]^c\, Q^{-(p+1)}$, where c is an arbitrary positive number. The restriction upon n depends on the unknown asymptotic behaviour in s of the nucleon spectral function $\rho_1(s)$ and the off-shell form factor $F(s, Q^2)$. Let us assume that these are bounded by s^a and s^b respectively. Then for $n > b$ we may write an n times subtracted dispersion relation and use inequality (35),

$$\left|\frac{d^nF(m^2, Q^2)}{ds^n}\right|^2 \leq \left|\int_{(m+\mu)^2}^{\infty} \frac{ds}{(s-m^2)^n}\left[W_2(s, Q^2)\,\rho_1(s)/2m\right]^{1/2}\right|^2 \tag{36}$$

$$\leq \int_{(m+\mu)^2}^{\infty} ds\,\rho_1(s)(s-m^2)^{p+1-\epsilon-2n}\int_{(m+\mu)^2}^{\infty} ds\left[W_2(s, Q^2)/2m\right](s-m^2)^{\epsilon-p-1},$$

where we have used a Schwarz inequality for the integration over s. Taking the $Q^2 \longrightarrow \infty$ limit, we obtain

$$\lim_{Q^2\to\infty}\left|Q^{p+1-\epsilon}\,\frac{d^nF(m^2, Q^2)}{ds^n}\right|^2 \leq \int_{(m+\mu)^2}^{\infty} ds\,\rho_1(s)(s-m^2)^{p+1-\epsilon-2n}\int_1^{\infty}\frac{d\omega}{\omega}\,\frac{F_2(\omega)}{(\omega-1)^{p+1-\epsilon}}\,. \tag{37}$$

The integral over the structure function is finite for arbitrarily small positive ϵ. The integral over the spectral function is finite for $n \geq (p+2+a)/2$. Thus all derivatives with $n > b$ and $n \geq (p+2+a)/2$ are bounded by $[\log Q^2]^c Q^{-(p+1)}$. To get some idea of the number of derivatives we are unable to restrict one must consider what are reasonable values for a and b. In Section II we assumed $b = 0$ corresponding to a constant term arising from the equal time commutator and given by the charge. Elementarity of the proton $(Z_2 > 0)$ would correspond to $a<-1$, but $a<0$ is a sufficient condition for the existence of the Lehmann representation. For concreteness let us assume $p=3$ and $a=-1$, which corresponds to a composite proton ($Z_2=0$) and involves only a dimensionless constant in the spectral function. Then our result holds for $n\geq 2$ provided the sideways dispersion relation requires no more than two subtractions, which is hardly a restrictive assumption. If $\lim_{Q^2\to\infty} F(s,Q^2) = f(s)g(Q^2)$, near $s=m^2$, and $f(s)$ varies at least quadratically in s then $g(Q^2)$ is bounded by $[\log Q^2]^c Q^{-(p+1)}$. Loosely stated our result is that the Drell-Yan-West relation is the extremum of an inequality imposed by analyticity and unitarity, provided the off-shell electromagnetic form factor is a smooth but sufficiently varying function of s, near $s=m^2$, as $Q^2\to\infty$.

Note our inability to exclude logarithms in Q^2. In the Drell and Lee bound state model[9] $F_1(Q^2) \propto [\log Q^2]^2 Q^{-4}$ and $F_2(\omega)\propto(\omega-1)^3$, consistent with our result.

Finally we discuss whether the bound may be made more stringent by considering only $J^\pi = \frac{1}{2}^+$ contributions to νW_2. Even if the t dependence of virtual Compton amplitudes were given by exp(Bt), so that the $J^\pi = \frac{1}{2}^+$ contributions did not scale, no improvement may be achieved. In that case the Bjorken limit of $Q^2 \nu W_2^{\left(\frac{1}{2}^+\right)}(s,Q^2)$ behaves as $(\omega-1)^{p+1}$ near $\omega=1$ and the same result is obtained from inequality (37) with $\epsilon\to -1$.

V. CONCLUSIONS

The inequalities of Section II, which set upper limits on Z_2 if R vanishes as $\omega\to\infty$, represent improvements over previous work in three respects: numerically, in their sensitivity to the behaviour of R in the Bjorken limit, and in permitting a conclusion without neglect of the subtraction constant.

The upper limits on Z_2 of between 0 and 0.3, evaluated in Section III, indicate the latitude in drawing conclusions about the compositeness of the proton from the deep inelastic data, given inadequate experimental information about R and the uncertainty of the subtraction constant. However this is a very weak application of the unitarity inequality, since $Z_2 = 0$ if the $J^\pi = \frac{1}{2}^+$ contributions to virtual photoabsorption do not scale. Our result on the Drell-Yan-West relation is a more stringent application of unitarity and analyticity. We conclude that the proton's Dirac form factor $F_1(Q^2) = F(m^2, Q^2)$ is bounded by $[\log Q^2]^c\, Q^{-(p+1)}$ if $F(s, Q^2)$ is a smooth but sufficiently varying function of s, near $s = m^2$, as $Q^2 \longrightarrow \infty$. This is true whether or not the $J^\pi = \frac{1}{2}^+$ contributions scale.

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