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FINITE MASS SUM RULES FOR INCLUSIVE REACTIONS*

Martin B. Einhorn

Lawrence Berkeley Laboratory
University of California, Berkeley, California 94720

John Ellis**

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

J. Finkelstein†

Institute of Theoretical Physics
Department of Physics
Stanford University, Stanford, California 94305

ABSTRACT

Taking account of analyticity, crossing and signature, we derive sum rules relating triple Regge vertices to integrals over low missing masses in inclusive reactions. Some implications for triple Regge couplings are discussed.

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** On leave from King's College, Cambridge, England.

† Alfred P. Sloan Foundation Fellow.

I. INTRODUCTION

Recently it has been suggested[1] that it might be possible to use analyticity to derive sum rules for single particle distributions in inclusive reactions. On the basis of model studies sum rules have been proposed relating integrals over low missing masses in inclusive spectra at high incoming energies to triple Regge vertices. In form and content these finite mass sum rules are similar to finite energy sum rules for two-body processes.

In this paper we discuss the derivation of such sum rules, paying particular attention to the correct incorporation of crossing. We emphasize which pairs of crossed reactions are related by the sum rules, what sum rules are free of wrong signature fixed poles, and how triple Regge signature should be taken into account. The sum rules we obtain take the form

$$\int_0^N d\nu\ \nu^n \left[E_c \frac{d^3\sigma}{dp_c}(a+b\to c+\text{anything}) + (-1)^{n+1} E_a \frac{d^3\sigma}{dp_a}(c+b\to a+\text{anything})\right]$$

$$= \sum_{i,j,k} \left(1+(-1)^{n+1}\tau_i\tau_j\tau_k\right)\left(\frac{\eta}{N}\right)^{\alpha_i(t)+\alpha_j(t)-1} N^{\alpha_k(0)+n}\ \xi_i(t)\,\xi_j^*(t)$$

$$\times\ \frac{\beta^i_{a\bar{c}}(t)\ \beta^{j*}_{a\bar{c}}(t)\ g^k_{ij}(t)\ \beta^k_{b\bar{b}}(0)}{\alpha_k(0)+n+1-\alpha_i(t)-\alpha_j(t)} \tag{1}$$

where $\nu = p_b\cdot(p_a-p_c) = \frac{1}{2}\left(M^2-t-m_b^2\right)$, $t=(p_a-p_c)^2$ and $\eta = p_b\cdot(p_a+p_c) = \frac{1}{2}\left(s_{ab} - s_{b\bar{c}} - m_a^2+m_c^2\right)$ (see Fig. 1). Other quantities in Eq. (1) are defined as follows: τ_i, τ_j and τ_k are Regge signatures, and $\xi_i(t) \equiv \left(\tau_i+e^{-i\pi\alpha_i(t)}\right)/\sin\pi\alpha_i(t)$, $\beta^i_{a\bar{c}}(t)$ and $\beta^k_{b\bar{b}}(0)$ are the reduced residue functions familiar from two-body scattering, and $g^k_{ij}(t)$ is the vertex for the three Reggeons $\alpha_i(t)$, $\alpha_j(t)$ and $\alpha_k(0)$. In order that Regge exchanges in the t-channel dominate, the range of integration in (1) must be such that $\eta \gg N$, $|t|$.

Strictly speaking the sum rules should be evaluated at fixed t and fixed η. However experiments are normally done at discrete laboratory energies, and $p_a \cdot p_b = \frac{1}{2}(\eta+\nu)$, so that data does not exist at fixed η. However, evaluating $E_c \frac{d^3\sigma}{dp_c}(a+b \to c+\text{anything})$ at one fixed laboratory energy $\omega_a = p_b \cdot p_a/m_b$ and $E_a \frac{d^3\sigma}{dp_a}(c+b \to a+\text{anything})$ at another fixed energy $\omega_c = p_b \cdot p_c/m_b$ will yield an approximation to the sum rule accurate to orders $N/m_b\omega_a$, $N/m_b\omega_c$, provided $|\omega_a - \omega_c| = 0(N/m_b) \ll \omega_a, \omega_c$. This point will be discussed below.

II. DERIVATION

It has been made plausible[2, 3] that the inclusive differential cross section for the process $a+b \to c+\text{anything}$ (see Fig. 1) is related to a discontinuity in $M^2 = (p_a+p_b-p_c)^2$ of the forward $a+b+c \to a+b+c$ scattering amplitude $A(s_{ab}, M^2, t)$. This discontinuity must be evaluated on a sheet where the incoming subenergy s_{ab} is just above its physical cut, and the outgoing subenergy $s_{\bar{a}\bar{b}}$ just below its physical cut, as indicated in Fig. 2. Clearly at fixed t we may equally well regard the amplitude A as a function of η and ν rather than $s_{\bar{a}\bar{b}}$ and M^2.

Then

$$E_c \frac{d^3\sigma}{dp_c}(a+b \to c+\text{anything}) = \frac{1}{\lambda\left(s_{ab}, m_a^2, m_b^2\right)} \operatorname{disc}_{\nu > 0} A(\eta, \nu, t) \qquad (2)$$

where $\lambda\left(s_{ab}, m_a^2, m_b^2\right)$ is the usual flux factor. According to the Steinmann-like relations,[3] the locations relative to their cuts of variables overlapping the missing-mass variable $M^2 = (p_a+p_b-p_c)^2$ do not affect the value of the discontinuity appearing in Eq. (2). We choose the incoming subenergy s_{bc} just below its cut and the outgoing subenergy $s_{\bar{b}\bar{c}}$ just above its cut. Then A is on a sheet such that the inclusive cross section for $c+b \to a+\text{anything}$ is the discontinuity of A in $(p_c+p_b-p_a)^2$. In terms of ν:

$$E_a \frac{d^3\sigma}{dp_a}(c+b \to a+\text{anything}) = \frac{1}{\lambda\left(s_{bc}, m_c^2, m_b^2\right)} \operatorname{disc}_{\nu<0} A(\eta,\nu,t) \tag{3}$$

Regge analyses[4] of the (3-3) amplitude A suggest that in the limit $\eta \gg \nu$, t it may be approximated asymptotically as

$$A \sim \sum_{i,j} \eta^{\alpha_i(t)+\alpha_j(t)} f_{ij}(\nu,t) + \widetilde{A}(\eta,\nu,t) \tag{4}$$

[For notational simplicity, we have absorbed signature factors and residues into the definition of $f_{ij}(\nu,t)$.] The powers of η in the first term of Eq. (4) are determined by the leading helicity poles which are related to the Regge poles indicated in Fig. 3. The remainder term $\widetilde{A}$ has a different power behavior in η and is not expected[5] to contribute to the discontinuities in ν. The quantity $f_{ij}(\nu,t)$ is referred to loosely as the Reggeon-particle scattering amplitude — more precisely it is the analytic continuation of the maximum helicity flip amplitude in the center-of-mass of the crossed channel $b\bar{b} \to \overline{\alpha_i}\,\alpha_j$.

In order to write a dispersion relation for $f_{ij}(\nu,t)$, and therefore derive a sum rule, one must understand its singularity structure. Supposing $\widetilde{A}$ has no discontinuity in ν, we find for large η from Eqs. (2) and (4) that

$$E_c \frac{d^3\sigma}{dp_c}(a+b \to c+\text{anything}) \approx \sum_{i,j} \eta^{\alpha_i+\alpha_j-1} \operatorname{disc}_{\nu>0} f_{ij}(\nu,t) \tag{5}$$

and from Eqs. (3) and (4) that

$$E_a \frac{d^3\sigma}{dp_a}(c+b \to a+\text{anything}) = \sum_{i,j} \eta^{\alpha_i+\alpha_j-1} \operatorname{disc}_{\nu<0} f_{ij}(\nu,t) \tag{6}$$

Thus f_{ij} has a right-hand cut which may be interpreted as the absorptive part of the $\alpha_i+b \to \alpha_j+b$ Reggeon-particle scattering amplitude, and a left-hand cut which corresponds to the absorptive part of $\bar{\alpha}_j + b \rightarrow \bar{\alpha}_i + b$. Both cuts are required because if f_{ij} is continued to particle poles in positive t one expects to recover the usual analyticity properties of two-body scattering amplitudes.

The question arises whether, at negative values of t, $f_{ij}(\nu, t)$ has any other singularities in the complex ν plane. Such singularities do not occur in models that have been studied such as perturbation theory,[6] the Gribov Reggeon calculus,[7] and the dual resonance model.[8] Further there is no reason yet known from S-matrix theory why singularities should occur at complex values of ν. Thus normal threshold singularities of the (3-3) amplitude A in other physical region variables such as $(p_a+p_b+p_c)^2$ move off to ∞ as $\eta \rightarrow \infty$, and so could not appear in $f_{ij}(\nu, t)$. Most simple triangle and box diagram Landau singularities in A move off to ∞ as $\eta \rightarrow \infty$; those that occur at finite values of ν lie on the real axis, and are included[3] in the definitions of the discontinuities to be associated with inclusive cross sections. It therefore seems reasonable to follow previous authors in assuming that $f_{ij}(\nu, t)$ has the same analyticity properties in ν as a (2-2) particle scattering amplitude.

In order to derive useful sum rules it is necessary to know the asymptotic behavior of f_{ij} in ν. This is determined by Reggeons in the $b\bar{b}$ channel via the "triple Regge" formula[9]

$$\eta^{\alpha_i(t)+\alpha_j(t)} f_{ij}(\nu, t) \underset{\nu \rightarrow \infty}{\sim} \sum_k \left(\frac{\eta}{\nu}\right)^{\alpha_i(t)+\alpha_j(t)} \nu^{\alpha_k(0)} \left[\frac{\tau + e^{-i\pi\left(\alpha_k(0)-\alpha_i(t)-\alpha_j(t)\right)}}{\sin \pi \left(\alpha_k(0)-\alpha_i(t)-\alpha_j(t)\right)}\right]$$

$$\xi_i(t)\, \xi_j^*(t)\, \beta^i_{a\bar{c}}(t)\, \beta^{j*}_{a\bar{c}}(t)\, \beta^k_{b\bar{b}}(0)\, g^k_{ij}(t) \qquad (7)$$

All the symbols in Eq. (7) were defined in Section I, with the exception of τ, the triple Regge signature factor.

That τ is not simply the signature τ_k of the Regge trajectory α_k can be seen by imagining a case where $\alpha_i(t)$ and $\alpha_j(t)$ are unit-separated trajectories of opposite signatures. Proceeding to a double particle pole in the t-channel, f_{ij} is proportional to an amplitude with cross-channel helicity flipped by $\alpha_i+\alpha_j$, an odd number. It is well-known[10] that in such a case $\tau = -\tau_k$. A choice of τ consistent with this observation is $\tau = \tau_i \tau_j \tau_k$. This is indeed the form taken by τ in Feynman tree graph models with elementary particle exchanges, and, more realistically, in the dual resonance model as shown in the appendix.

We have now motivated the required analyticity properties of the Reggeon-particle scattering amplitudes $f_{ij}(\nu, t)$, related (Eqs. (5) and (6)) its cuts to inclusive cross sections and know the behavior of $f_{ij}(\nu, t)$ as $\nu \to \infty$. Therefore we can derive a sum rule for each $f_{ij}(\nu, t)$ using the contour shown in Fig. 4. Adding these sum rules together with weights $\eta^{\alpha_i+\alpha_j-1}$ the finite mass sum rules (1) for inclusive reactions are immediately obtained. Note that (2-2) processes such as elastic reactions should be included in the missing mass integrals, and will give important contributions.

As mentioned in Section I, the finite mass sum rules should strictly speaking be evaluated using data on inclusive cross sections at fixed values of $\eta = p_b \cdot (p_a + p_c)$. As $\nu = p_b \cdot (p_a - p_c)$ varies this corresponds to varying the incident laboratory energies ω_a and ω_c over a range $O(N/m_b)$. But in the approximation where a t-channel Regge description is used this variation in energy makes a fractional change in the cross section of $O(N/\eta) \ll 1$. Hence if we insert data at fixed laboratory energies into the sum rules (1) they should still be accurate to order N/η. In fact to the same accuracy it is not necessary that the laboratory energies ω_a and ω_c be exactly equal, as long as $|\omega_a - \omega_c| = O(N/m_b)$. However this does mean that terms down by $O(N/\eta)$ relative to the leading terms in the Regge pole expansions (5), (6) or

(7) cannot be evaluated reliably using the sum rules (1). For example, in processes where both the Pomeron and ordinary meson trajectories can be exchanged in the t-channel, sum rules at fixed laboratory energies will permit the determination of Pomeron-Pomeron and Pomeron-Reggeon contributions, but not Reggeon-Reggeon contributions.

Schwarz-like[11] sum rules can be written down for other combinations of integrals over the inclusive cross sections $a+b \to c+\text{anything}$ and $c+b \to a+\text{anything}$, but then nonsense wrong signature fixed pole residues $R_{ij}^{(n)}(t)$ must also be included on the right-hand sides:

$$\int_0^N d\nu \; \nu^n \left[E_c \frac{d^3\sigma}{dp_c}(a+b \to c+\text{anything}) + (-1)^n \, E_a \frac{d^3\sigma}{dp_a}(c+b \to a+\text{anything}) \right]$$

$$= \sum_{i,j} \; \eta^{\alpha_i(t)+\alpha_j(t)-1} \; \xi_i(t) \; \xi_j^*(t) \; \beta_{a\bar{c}}^i(t) \; \beta_{a\bar{c}}^{j*}(t)$$

$$\left[R_{ij}^{(n)}(t) + \sum_k \left(1+(-1)^n \, \tau_i \tau_j \tau_k\right) \frac{g_{ij}^k(t) \, \beta_{b\bar{b}}^k(0) \; N^{\alpha_k(0)+n+1-\alpha_i(t)-\alpha_j(t)}}{\alpha_k(0)+n+1 - \alpha_i(t)-\alpha_j(t)} \right] \tag{8}$$

III. DISCUSSION

The most interesting applications of the finite mass sum rules (1) are likely to be in the estimation of triple Reggeon vertices by integrating over data at relatively low values of the missing mass. Data in the triple Regge region $\eta \gg \nu \gg m_b^2$ are not likely to be available until there are results from NAL; precise evaluations of triple Regge vertices from fits will have to wait until then. It would also be interesting to use the sum rules to investigate whether the Harari-Freund[12] conjecture can be generalized in the natural way to

Reggeon-particle scattering. One would evaluate resonance production contributions to the inclusive cross section integrals to see whether they built up Regge exchanges in the $b\bar{b}$ channel. A similar analysis could help resolve the controversy[13] on the duality properties of Pomeron-particle scattering.

It has been argued[14] that certain of the fixed pole residues $R_{ij}^{(n)}(t)$ appearing in Eq. (8) may vanish at t=0 because of crossed channel unitarity. This suggestion could in principle be checked by using the sum rules (1) to evaluate the triple Regge vertices $g_{ij}^{k}(t)$, and substituting them into the Schwarz sum rules (8) to evaluate the residues $R_{ij}^{(n)}(t)$. The evaluation of fixed pole residues in the Reggeon-particle amplitude for a range of t is quite interesting physically, since the fixed pole residues set the scale of Reggeon-Reggeon cut contributions to two-body scattering. Thus measurements of single-particle inclusive cross sections in principle determine the magnitude of cuts in two-body scattering. Thus the Regge pole description we have assumed could be checked for self-consistency.[15]

In all the above work the Pomeron has been treated as an ordinary Regge trajectory (possibly with zero slope), on the assumption that this is a reasonable first approximation to its nature. The sum rules give a lower limit to the rate of fall-off of Pomeron-particle scattering amplitudes. Consider the special case of the sum rule (1) with particles a and c identical, n=1, and η so large that non-diffractive processes may be ignored. Then

$$\int_0^N d\nu\ \nu\ \ E_a \frac{d^3\sigma}{dp_a}(a+b\rightarrow a+\text{anything}) \simeq \sum_k \left(\frac{\eta}{N}\right)^{2\alpha_P(t)-1} (N)^{\alpha_k(0)+1}\ \frac{2\left|\beta_{a\bar{a}}^{P}(t)\right|^2}{1-\cos\pi\alpha_P(t)}$$

$$\times\ \frac{g_{PP}^{k}(t)\ g_{b\bar{b}}^{k}(0)}{\alpha_k(0)+2-2\alpha_P(t)} \qquad (9)$$

where $\alpha_P(t)$ is the Pomeron trajectory, and the sum is over trajectories k with positive signature. The left-hand side of the sum rule (9) has a nonzero positive contribution from the elastic process $a+b \rightarrow a+b$. Because the inclusive cross section is positive this contribution cannot be cancelled, so that the coefficient of $s^{2\alpha_P(t)-1}$ on the right-hand side of (9) cannot fall to zero for large N. Hence there must be a nonzero coupling g_{PPk} for the Pomeron to some positive signature J-plane singularity with $\alpha_k \geq 0$. This could either be the f^0 or Pomeron trajectory,[17] or some other cut, trajectory or fixed pole with $J \geq 0$. Under reasonable assumptions[14] about the residues of wrong signature fixed poles in Pomeron-particle scattering, this restriction on multi-Pomeron couplings can be strengthened to establish[16] a lower bound on the triple-Pomeron coupling at $t \neq 0$.

The term $\sin \pi(\alpha_k(0)-\alpha_i(t)-\alpha_j(t)$ in the denominator of Eq. (7) might appear to give an unphysical singularity in t in the full 3-3 amplitude. It has been pointed out[18] that the apparent singularity can be cancelled by terms in $\widetilde{A}(s_{ab}, M^2, t)$ for $\alpha_i(t)+\alpha_j(t)-\alpha_k(0) = 0, -1, -2 \ldots$ and suggested that for $\alpha_i(t)+\alpha_j(t)-\alpha_k(0) = 1, 2, 3 \ldots$ the poles are cancelled by zeroes in the vertices $g_{ij}^k(t)$. This latter proposal is clearly required by sum rules (1) (continued if necessary to positive t) for the cases $\alpha_i(t)+\alpha_j(t)-\alpha_k(0) = \iota > 0$ such that $(-1)^\iota = \tau_i \tau_j \tau_k$. These zeroes arise because right signature fixed poles were assumed to be absent in the triple Regge formula (7). If nonsense wrong-signature fixed pole residues were nonsingular, then the finite mass Schwarz sum rules (8) could be used to prove the existence of the zeroes at integers $\iota : (-1)^\iota = -\tau_i \tau_j \tau_k$. A recent paper by Mueller and Trueman[19] reaches a similar conclusion on the basis of Feynman diagram calculations.

APPENDIX

In the dual resonance model, the helicity pole limit can be investigated explicitly.[5] We multiply each cyclically inequivalent ordering of the external momenta by the appropriate Chan-Paton factor. Then, in the limit, $\eta \to \infty$, $\nu \to \infty$, $\eta \gg \nu$ (fixed t) the asymptotic behavior of the amplitude is determined by the sum of eight cyclically inequivalent terms[20] (Fig. 5).

$$A \sim \sum_{i,j,k} \gamma_{ij}^{k} \left\{ \left[(-s_{ab})^{\alpha_i} + \tau_i (-s_{b\bar{c}})^{\alpha_i}\right] \left[(-s_{\bar{a}\bar{b}})^{\alpha_j} + \tau_j (-s_{\bar{b}c})^{\alpha_j}\right] (-s_{ab\bar{c}})^{\alpha_k - \alpha_i - \alpha_j} \right.$$

$$\left. + \left[(-s_{a\bar{b}})^{\alpha_i} + \tau_i (-s_{\bar{b}\bar{c}})^{\alpha_i}\right] \left[(-s_{\bar{a}b})^{\alpha_j} + \tau_j (-s_{bc})^{\alpha_j}\right] \tau_k (-s_{a\bar{b}\bar{c}})^{\alpha_k - \alpha_i - \alpha_j} \right\} \tag{A.1}$$

The residue, $\gamma_{ij}^{k} = \Gamma(-\alpha_i)\Gamma(-\alpha_j)\Gamma(\alpha_i + \alpha_j - \alpha_k)$, is the same for all eight terms. To the expression above must be added another term which has no discontinuity in the missing mass and is of no interest for the present discussion. In the limit of interest, we have

$$s_{ab} = s_{\bar{a}\bar{b}} \approx -s_{b\bar{c}} = -s_{\bar{b}c} \approx -s_{a\bar{b}} = -s_{\bar{a}b} \approx s_{\bar{b}\bar{c}} = s_{bc} \approx \eta > 0$$

$$s_{ab\bar{c}} \approx -s_{a\bar{b}\bar{c}} \approx \nu > 0$$

A complete definition of the expression requires a specification of the phases of each of the powers $(-\eta)^{\alpha}$. To obtain the inclusive cross section $d\sigma(a + b \to c + \text{anything})$, the prescription is to choose[2,3]

(α) $\quad \mathrm{Re}\, s_{ab} = \mathrm{Re}\, s_{\bar{a}\bar{b}} \approx \eta \qquad \mathrm{Im}\, s_{ab} = +i\epsilon \quad \mathrm{Im}\, s_{\bar{a}\bar{b}} = -i\epsilon$

Notice that the inclusive cross section d (a + b → c + anything) could equally well be obtained by the alternative prescription

(β) $\quad \mathrm{Re}\, s_{ab} = \mathrm{Re}\, s_{\bar{a}\bar{b}} \approx \eta \qquad \mathrm{Im}\, s_{ab} = -i\epsilon \quad \mathrm{Im}\, s_{\bar{a}\bar{b}} = +i\epsilon$

In general, the two prescriptions correspond to taking the discontinuity in $s_{ab\bar{c}}$ on different Riemann sheets. Of course, the complete determination of the sheet requires the specification of other energy variables, such as s_{bc} and $s_{\bar{b}\bar{c}}$, relative to their physical cuts; however, according to the Steinman-like relations,[3] the discontinuity in $s_{ab\bar{c}}$ is independent of the specification of the overlapping variables. (Furthermore, the equality of the discontinuity on sheets having either prescription (α) or (β) is a consequence of time reversal invariance.) Although the discontinuity in $s_{ab\bar{c}} > 0$ is independent of the choice of phase of s_{bc} and $s_{b\bar{c}}$, the discontinuity in $s_{a\bar{b}\bar{c}}$ will not be. The inclusive cross section $d\sigma(c+b \rightarrow a+\text{anything})$ will be obtained from either of the following choices

$$(\gamma) \quad \operatorname{Re} s_{bc} = \operatorname{Re} s_{\bar{b}\bar{c}} \approx \eta \qquad \operatorname{Im} s_{bc} = -i\epsilon \qquad \operatorname{Im} s_{\bar{b}\bar{c}} = +i\epsilon$$

$$(\delta) \quad \operatorname{Re} s_{bc} = \operatorname{Re} s_{\bar{b}\bar{c}} \approx \eta \qquad \operatorname{Im} s_{bc} = +i\epsilon \qquad \operatorname{Im} s_{\bar{b}\bar{c}} = -i\epsilon$$

On a sheet satisfying prescriptions (α) and (γ), the expression given above, Eq. (A.1), can be written simply as

$$A \sim \sum_{ijk} \gamma_{ij}^{k}\, \eta^{\alpha_i+\alpha_j}\, \xi_i \xi_j^* \left[(-\nu)^{\alpha_k-\alpha_i-\alpha_j} + (\tau_i\tau_j\tau_k)\, \nu^{\alpha_k-\alpha_i-\alpha_j}\right] \qquad (A.2)$$

If however we choose (α) and (δ) we find

$$A \sim \sum_{ijk} \gamma_{ij}^{k}\, \eta^{\alpha_i+\alpha_j} \left[\xi_i \xi_j^* (-\nu)^{\alpha_k-\alpha_i-\alpha_j} + \xi_j \xi_i^*\, \tau_i\tau_j\tau_k\, \nu^{\alpha_k-\alpha_i-\alpha_j}\right] \qquad (A.3)$$

Using the fact that $\gamma_{ij}^{k} = \gamma_{ji}^{k}$ the two expressions are precisely equal.

No doubt the asymptotic forms (A.2) and (A.3) follow from the assumption of Regge behavior on all the sheets and are not peculiar to the dual resonance model. In the general case, the residue would be $\gamma_{ij}^{k} = \beta_{a\bar{c}}^{i}(t)\, \beta_{\bar{c}a}^{j\,*}(t)\, \beta_{b\bar{b}}^{k}(0)\, g_{ij}^{k}(t)$. The symmetry property $\gamma_{ij}^{k} = \gamma_{ji}^{k}$ follows quite generally from the requirement that the discontinuity for $\nu > 0$ is the same no matter whether prescription (α) or

(β) is taken. The reality condition $\gamma_{ij}^{k} = \gamma_{ij}^{k*}$ is assured by the requirement that the discontinuity be real.

Assuming Regge asymptotic behavior on each of the many sheets, other sum rules could be written down; however, it seems that only on the sheets discussed above can the discontinuities be actually determined experimentally. For example, imagine choosing all subenergies above their cuts as for the physical $3 \rightarrow 3$ amplitude. The discontinuity in $s_{ab\bar{c}}$ would then be quite complicated, and multibody S-matrix elements would be needed for its evaluation.

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20. For this discussion, it is convenient to label the external particles as if they were all incoming.

FIGURE CAPTIONS

1. Kinematics for $a + b \rightarrow c$ + anything.

2. Discontinuity related to the inclusive cross section.

3. Representation in the limit $s_{ab} \gg M^2$, t of term in (3-3) amplitude with M^2 discontinuity.

4. Contour used in deriving sum rules.

5. Cyclically inequivalent terms contributing in triple Regge limit.

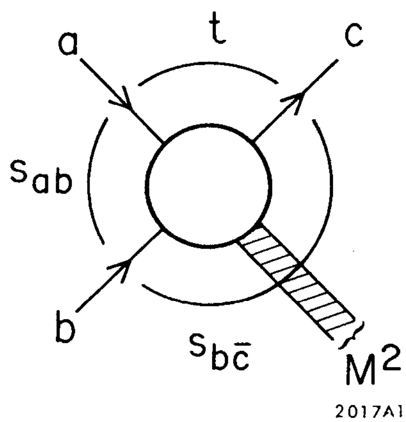

Fig. 1

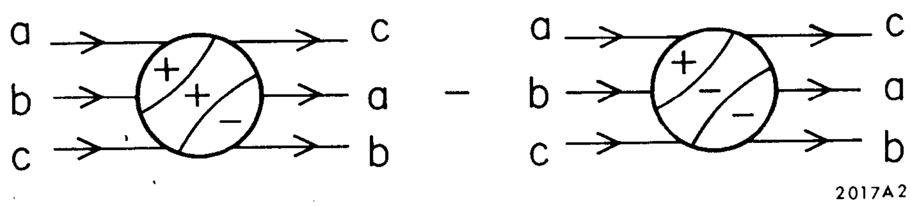

Fig. 2

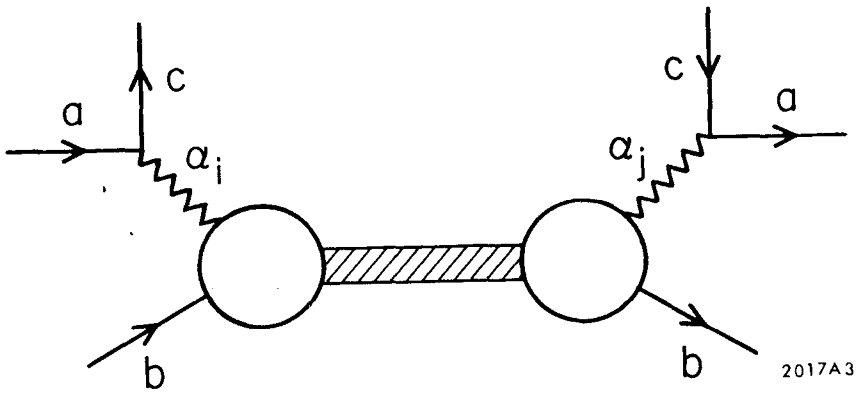

Fig. 3

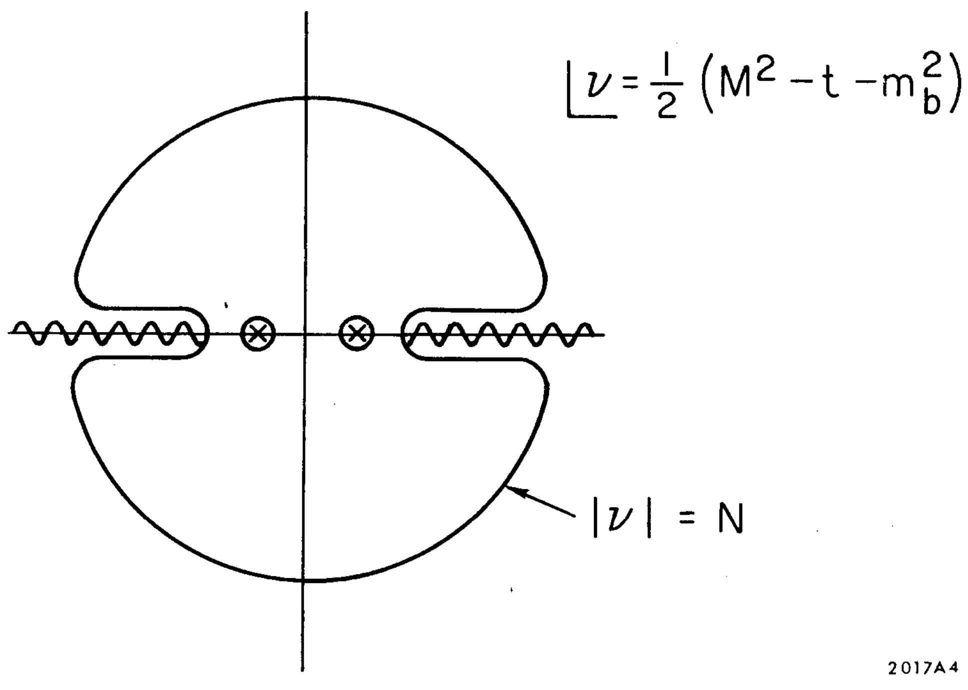

Fig. 4

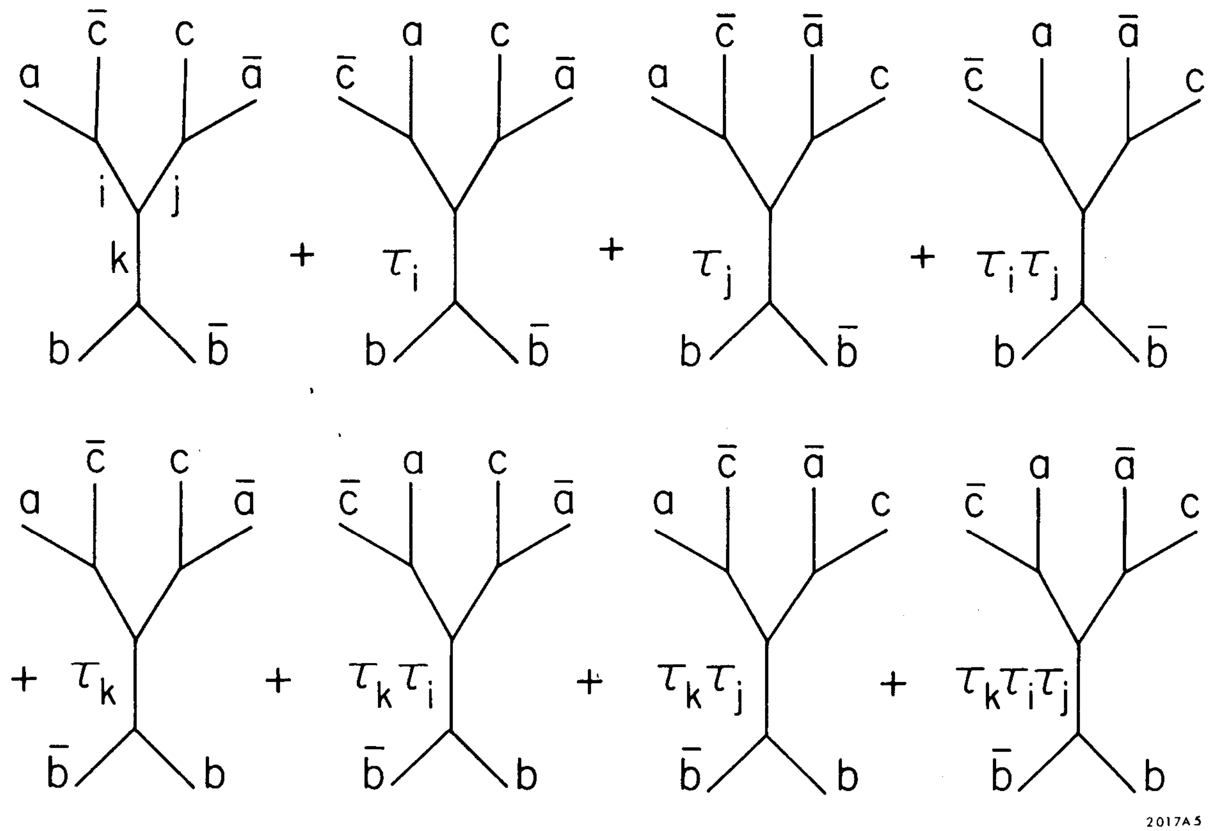

Fig. 5