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THE RENORMALIZED σ MODEL*†

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ABSRACT

A renormalization procedure of the boson σ model based on the finite field equations of Brandt-Wilson is given. We first show that the current operators appearing in the field equations, which are finite local limit of sums of nonlocal field products and suitable subtraction terms, can be chosen to be the same form as the one given for the symmetric limit except for the symmetry breaking constant source term itself. The set of integral equations derived from the field equations is shown to be equivalent to the usual Bogoliubov-Parasiuk-Hepp renormalization theory, and gives us immediately all the renormalized Green's functions in each order of perturbation theory in clear and straightforward fashion. We then analyze the structures of the model in detail. In particular, Ward identities are shown to be satisfied to all orders of perturbation theory. The Goldstone theorem is a particular consequence of these identities.

I. INTRODUCTION

The object of our interest in this paper is the so-called σ-model studied by Schwinger (1), Polkinghorne (2), and Gell-Mann and Levy (3). For simplicity, we shall neglect the baryonic fields and imagine a world composed of a pseudo-scalar meson $\phi(x)$ and a scalar meson $\sigma(x)$.[1] The formal Lagrangian of the world is then

$$\mathscr{L}(x) = \frac{1}{2} : \left[(\partial_\mu \phi)^2 + (\partial_\mu \sigma)^2\right] : - \frac{1}{2}\mu^2 : (\sigma^2 + \phi^2) : + \frac{1}{4} g : (\sigma^2 + \pi^2)^2 : + c\sigma \tag{I.1}$$

Except for the source term $c\sigma$,[2] (I.1) possesses the chiral symmetry which is in the present case an 0_2 symmetry in the two-dimensional space (ϕ,σ). We may define the axial vector current $A_\mu(x)$ associated with this symmetry formally as follows

$$A_\mu(x) = \phi(x)\ \partial_\mu\ \sigma(x) - \sigma(x)\ \partial_\mu\ \phi(x) \tag{I.2}$$

Because of the simple form of the symmetry breaking term $c\sigma(x)$ in (I.1), the model acquires the following very interesting features

a. We expect $\langle 0|\sigma|0\rangle = F_0 \neq 0$, since we can create the σ particle freely from the vacuum. If $F_0 \neq 0$ in the limit $c \rightarrow 0$, then the situation is precisely the spontaneously broken symmetry.[3] That is, we expect that the Goldstone mechanism would be in operation.

b. We also expect that the axial vector current $A_\mu(x)$ would be conserved "partially." Indeed, if we manipulate with the formal expression (I.2), we get the P.C.A.C. condition for the Lagrangian model,

$$\partial_\mu A_\mu(x) = c\ \phi(x) \tag{I.3}$$

Now both of the above features are very interesting in connection with the recent development of particle physics. It is usually assumed that the Lagrangian of the real world is approximately chiral symmetric with a small symmetry breaking parameter ϵ.[4] If one further assumes that the limit $\epsilon \rightarrow 0$ is the Goldstone limit, then one can show that the usual statement of the P.C.A.C. must be true (6).

We shall study the σ model in renormalized perturbation theory and examine in detail the properties a and b in this paper, our primary task will be to make the various results derived from formal argument rigorous and precise. We shall prove the Goldstone theorem to all orders in perturbation theory. As for the P.C.A.C., instead of constructing an explicit renormalized axial vector current operator, we shall prove a set of identities known as Ward identities. In formal Lagrangian theory, these identities follows from the P.C.A.C. condition (I.3). Conversely, we can prove the usual P.C.A.C. relations from these identities.

Most of the results we find here are known already from some lower order calculation, or from a different approach. The first work along such a line was done by Lee (7, 8), who has examined the model to some orders in g. The model was also studied by Symanzik (9, 10) to all orders in perturbation theory in the framework of BPH procedure.[5] Our approach in this paper is that of Brandt and Wilson,[6] who based the renormalization procedure on finite field equations. By employing this technique, we shall be able to exhibit the problems in a clear and rigorous fashion.

We now briefly summarize the content of this paper. In Section II, we prepare ourselves by discussing the symmetric model which corresponds to the formal Lagrangian (I.1) with c=0. This will serve as an introduction to the B-W renormalization theory, and at the same time will enable us to get used to the diagrammatic representation which we shall employ extensively throughout this paper. This diagrammatic technique is very useful for problems of such complexity. With the tools thus obtained, the σ model is defined in the framework of the B-W renormalization theory in Section III. The finite local field equations of the model are first assumed to be of the same form as for the symmetric model except for the source term in the σ equation. Then its correctness and consistency is established explicitly by considering equivalent BPH theory. In Section IV, the Ward identities of the σ model are introduced and proved to all order in perturbation series. By means of these identities, we examine some of the physical properties of the model in Section V. We first prove the Goldstone theorem in all orders of perturbation theory. Then by trivially extending the model to include a triplet of fields corresponding to pions, we shall see that Weinberg's low energy theorem as well as Adler's consistancy condition are simple consequences of Ward identities. Appendix A is devoted to low order calculations to illustrate our general discussion. Finally in Appendix B, diagrammatic Lemma's which is necessary for the proof of Ward identities are proved.

II. THE RENORMALIZATION OF THE SYMMETRIC THEORY

As an introductory preparation for the method we shall adopt throughout this paper, we shall study the renormalized perturbation theory of the symmetric model in this section, which corresponds to the formal Lagrangian (I. 1) with c=0

$$\mathscr{L}_s = \frac{1}{2} : \left[(\partial_\mu \phi)^2 + (\partial_\mu \sigma)^2 \right] : - \frac{1}{2} \mu^2 : (\sigma^2 + \phi^2) : + \frac{1}{4} g : (\sigma^2 + \phi^2)^2 : \qquad \text{(II. 1)}$$

We first explain briefly the finite field equation approach to the renormalization procedure. The field equations corresponding to the Lagrangian (II. 1) is then written down. In order to derive integral equation for the Green's function, we must establish some notations and definitions, in particular, the connectivity concepts. By means of these tools, we write down the set of the coupled integral equations connecting all the Green's functions of our theory with simple diagrammatic notations. The symmetry properties of their solutions are briefly discussed.

A. B-W Renormalization Procedure

The Lagrangian (II. 1) dictates, via standard method, the rules of calculation for the unrenormalized nth order scattering amplitude corresponding to a Feynman diagram $D(V_1 \ldots V_n; \mathscr{L})$ with vertices $V_1 \ldots V_n$ and a set of lines $\mathscr{L} = \left\{ \ell_1 \ldots \ell_L \right\}$. The Feynman rules for our theory are described in Fig. 1.

Instead of standard renormalization procedure established most completely by Bogoliubov, Parasiuk (11), and Hepp (12), we shall employ in this paper the elegant formalism initiated by Wilson (13, 14). Their idea is to note that the infinities arise in quantum field theory because of the meaningless field products at same points as in the Lagrangian (and in the derived field equations).

Therefore if we work with the field equations of the form

$$(\Box + \mu^2)\ \phi(x) = j_\phi(x) = \lim_{\xi \cdot \eta \to 0} g\left[(\phi(x-\xi)\ \phi(x-\eta) + \sigma(x-\xi)\sigma(x-\eta))\ \phi(x) - q(\eta, \xi, x)\right] \tag{II.2}$$

(and similar equations for σ), where $q(\eta, \xi, x)$ is determined in such a way that the resulting $j_\phi(x)$ is finite, then we might be able to get the finite answers directly in terms of the field equations. The work along this line was completed recently by Brandt.[7] Brandt has shown that Wilson's prescription (14) of constructing the finite current operator $j_\phi(x)$ gives a finite and consistent result by proving that the field equation is equivalent to a BPH theory.[8]

This procedure of basing renormalization procedures on finite field equations has several definite advantages over the usual BPH renormalization procedure. This approach allows one to work with a meaningful and rigorous expression of the current operator. The importance of a finite and well-defined expression of a current operator is clear from the recent development of particle physics. By means of these expressions of currents, one can, for example, evaluate rigorously the current-current or current-field commutation rules in perturbation theory.[9]

Another significant advantage of this formulation lies in the fact that we can impose symmetries of the theory directly on the expressions of the current. In the usual formulation of the renormalization, this symmetry condition cannot be incorporated into the theory in such a simple fashion, and we must constantly remove infinite quantities in order to maintain the symmetries. In our analysis of the symmetric model and the σ model, this symmetry of the current operator will play an essential role.

B. Field Equations and Current Definitions

The B-W field equations for the symmetric model are

$$
\begin{aligned}
(\Box + \mu^2)\,\phi(x) &= j_\phi(x) = :(\phi^2(x) + \sigma^2(x))\,\phi(x): \\
(\Box^2 + \mu^2)\,\sigma(x) &= j_\sigma(x) = :(\phi^2(x) + \sigma^2(x))\,\sigma(x):
\end{aligned}
\qquad \text{(II. 4)}
$$

with the current definitions;

$$
j_\phi(x) = :(\phi^2(x) + \sigma^2(x))\,\phi(x): = \lim_{\xi,\eta\to 0} \Big\{ T\big[(\phi(x-\xi)\,\phi(x-\eta) + \sigma(x-\xi)\,\sigma(x-\eta)\big]\,\phi(x)
$$

$$
- R_1(\xi,\eta)\,\phi(x) - R_2^{\mu}(\xi,\eta)\,\partial_\mu\,\phi(x) - R_3^{\mu\nu}\,\partial_\mu\,\partial_\nu\,\phi(x) - R_4(\xi,\eta)\,j_\phi(x)\Big\}
$$

$$
j_\sigma(x) = :(\phi^2(x) + \sigma^2(x))\,\sigma(x): = \lim_{\xi,\eta\to 0} \Big\{ T\big[(\phi(x-\xi)\,\phi(x-\eta) + \sigma(x-\xi)\,\sigma(x-\eta)\big]\,\sigma(x)
$$

$$
- R_1(\xi,\eta)\sigma(x) - R_2^{\mu}(\xi,\eta)\,\partial_\mu\sigma(x) - R_3^{\mu\nu}(\xi,\eta)\,\partial_\mu\,\partial_\nu\sigma(x) - R_4(\xi,\eta)\,j_\sigma(x)\Big\}
$$

(II. 4)

Here the symbol : : represents the fact that we are taking out the finite part of the operator product occurring inside the symbol. This finite part will be called a generalized Wick product because it is the generalization of ordinary Wick products for free fields. The limit in (II. 4) are assumed to exist in a weak sense which need not be specified here. The R's are covariant functions of ξ and η with singularities at $\xi = \eta = 0$. This form of the currents, first given by Wilson (14), is suggested by the result of the lower order perturbation theory. In perturbation theory, the leading singularities of products of field operators at small distance occur (within logarithms) as mass independent coefficients of local field operators. Since $\dim\phi = \dim\sigma = 1$ in mass unit, we shall have the following expansion as $\xi,\eta \to 0$:

$$
T\big[\phi(x-\xi)\,\phi(x-\eta) + \sigma(x-\xi)\,\sigma(x-\eta)\big]\,\phi(x) \simeq \sum_i E_i(\xi,\eta)\,0_i(x) + f(x) \qquad \text{(II. 5)}
$$

Here the sum is over all local fields $0_i(x)$ (which can be Lorentz tensors of any rank, and which can be generalized Wick products themselves) with dimensions less than or equal to 3. $E_i(\xi,\eta)$ has singularity of form (within log)

$$E_i(\xi,\eta) \sim \frac{1}{\xi^s \eta^t}$$

with

$$s+t = 3-\dim(0_i(x)) \quad .$$

f(x) is finite field operator and has no singularity in the limit $\xi,\eta \to 0$. The substitution terms in (II.4) are then exactly the terms $E_i(\xi,\eta)\, 0_i(x)$ in (II.5).

It must be noted that the allowed terms like $R(\xi,\eta) : \phi(x)\, \sigma(x) :$ are absent in (II.4). This is because of the internal symmetry of our theory. Firstly we want to impose the chiral symmetry

$$\begin{aligned} \phi(x) &\rightarrow \phi(x) - i\epsilon\, \sigma(x) \\ \sigma(x) &\rightarrow \sigma(x) + i\epsilon\, \phi(x) \end{aligned} \qquad \text{(II. 6)}$$

in our theory. Then the current j_ϕ (or j_σ) should transform in the same way as the field ϕ (or σ). Secondly, we assume that parity of $\phi(x)$ (or $\sigma(x)$) is positive (or negative) so that $j_\phi(x)$ (j_σ) has odd (even) parity. In view of the described prescription of substracting out the singular parts together with the above two conditions from symmetries, one can easily convince oneself that the form (II.4) is the only possibility.

The exact forms of the functions $R_i(\xi,\eta)$ are not determined yet. They are to be determined iteratively as we solve the field equations themselves.

To solve (II.3), we first derive integral equations relating various Green's functions. These integral equations are the renormalized version of those studied by Schwinger (18) and Dyson (19). In terms of these integral equations, an unambiguous scheme of perturbation processes can be established, giving fully renormalized Green's functions of any order.

In our analysis of the σ model to be discussed in the later chapters, we need not determine the explicit form of the subtraction functions $R_i(\xi,\eta)$. Our main concern will be the algebraic properties of the form of the current operator (II. 4) dictated by Wilson's prescription and the symmetry requirement.

An exactly analogous procedure as given by Brandt (15) for a simpler model can be employed for our model to derive from (II. 3) and (II. 4) the set of the coupled integral equations for the Green's functions.. This can be solved order by order in g and determine all of the Green's functions of our model completely. Their equivalence to the BPH Green's functions can be seen by showing that we can also derive the integral equations in BPH theory and they are exactly the same as those obtained by the B-W procedure. These derivations and proof of the equivalence are also along the same line as in Brandt's work (15) except for the slight complication due to the form of our four-point coupling given by the Lagrangian (II. 1). Here we shall accept the correctness of the field equations (II. 3) - (II. 4), and proceed to consider the consequency of these equations.

C. Some Notations and Definitions, Integral Equations

In order to write down the integral equations derived from the field equations, it is best to introduce some definitions and notations which we will adopt throughout this work. We begin with the notations for the Green's functions.

1. Green's Functions

We write[10]

$$\langle 0|T(\phi(x_1)\,\phi(x_2)\ldots\phi(x_n)\,\sigma(y_1)\,\sigma(y_2)\ldots\sigma(y_m))|0\rangle$$

$$\equiv \widetilde{G}(x_1, x_2, \ldots x_n;\, y_1, y_2, \ldots y_m)$$

$$= \int_{q_i p_j} \delta\left(\sum q_i + \sum p_j\right) e^{-i\left(\sum q_i \cdot x_i + \sum p_j \cdot y_j\right)}\, G(q_1 \ldots q_n;\, p_1 \ldots p_m)$$

$$\left(\int_{q_i p_j} = \int \prod_{i=1}^{n} \frac{dq_i}{(2\pi)^4} \prod_{j=1}^{m} \frac{dp_i}{(2\pi)^4}\right) \qquad \text{(II. 7)}$$

For the case of two-point functions, we have

$$\langle 0|\phi(x)\,\phi(y)|0\rangle \equiv i\widetilde{G}_\phi(x-y) \equiv \widetilde{G}(x, y;) \equiv \int_k e^{-ik\cdot(x-y)}\, iG_\phi(k)$$

$$\equiv \int_k e^{-ik\cdot(x-y)}\, G(k, k;) \quad , \qquad \text{(II. 8)}$$

and similarly for the σ propagator.

Notice that in (II. 7), the external propagators are included in the definition. Sometimes we want to play with the Green's functions with their external propagators amputated. This amputation will be represented by a bar below the appropriate momentum or space level.

Example: $$G(p_{\underline{1}};p_{\underline{2}}, p_3) = \frac{G(p_1;p_2, p_3)}{iG_\phi(p_1)\ iG\ \ (p_2)} \quad .$$

When there is no ambiguities, we simplify (II. 7) as follows;

$$G(q_1 \ldots q_n;\, p_1 \ldots p_m) = G(N;M) \qquad \text{(II. 8)}$$

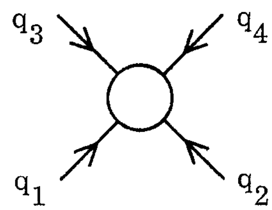
Here N denotes the set of the ϕ lines with momentum $q_1 \ldots q_n$ and M is for the σ lines. Again a bar will represent the amputation of the external lines. For example, $G(\underline{N};M)$ is the momentum space Green's function with all the external lives belonging to N amputated. We sometimes further simplify the notation by

$$G(N;M) = G(A), \qquad \text{where A denotes } N^{U}M \ . \tag{II.9}$$

The notations given above are very useful and will be needed in some occasions. Frequently, however, it is more convenient for our purposes to introduce a diagrammatic notation for the Green's functions. The following correspondence is obvious:

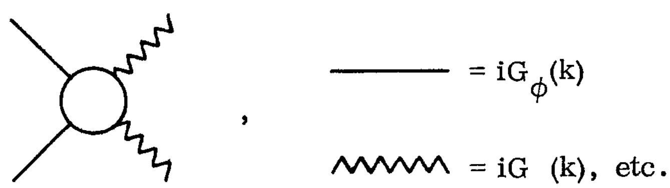
Example 1:

$G(q_1, q_2, q_3, q_4;)$ =



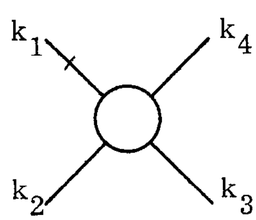
Other Green's functions shall be similarly represented.

Example 2:



Amputation of external propagators is expressed by a slash on the leg.

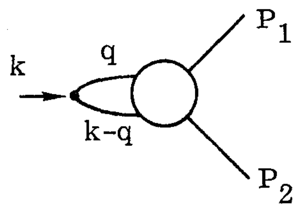
Example 3:



$= G(\underline{k_1}, k_2, k_3, k_4;)$

Frequently some of the external lines are combined together to form a closed loop; then the integration over the loop momenta is to be understood.

Example 4:



$$\equiv \int_q G(q, k-q, p_1, p_2;)$$

(The sense of this correspondence is understood if we consider the configuration space function $\tilde{G}(x, x, x_1, x_2 \ldots)$.) In many cases, we shall neglect the momentum levels when they are not important for our consideration.

2. Definition of Connectivity

We now turn to the connectivity concept of the renormalized Green's functions. This will turn out to be very important for our later analysis. First we define the disconnected part $G^D(N)$ of $G(N)$ with the set of external lines $N = \{\ell_1, \ell_2, \ldots, \ell_n\}$. Let a partition P decompose N into the subsets $N_1^P, \ldots, N_{ip}^P$ with $ip \neq 1$. Then

$$G^D(N) = \sum_P G^C(N_1^P)\, G^C(N_2^P) \ldots G^C(N_{ip}^P)\, \delta\left(\sum_{j=1}^{ip} Q_j\right) \qquad \text{for } n > 1$$

$$G_1^D = 0 \qquad \text{for } n = 1$$

$$G^C(N) = G(N) - G^D(N) \quad . \tag{II.10}$$

Here Q_j is sum of the all momentum of the lines in N_j^P.

The above definition includes the general case where one of the fields, say $\sigma'(x)$, has nonvanishing vacuum expectation value F. In this case, $G_1^C = G_1 = \langle 0|\sigma'|0\rangle = F$. If we introduce new fields by $\sigma' = \sigma + F$, and denote $\tilde{G}_{\{\sigma'\}}$ and

$G_{\{\sigma\}}$ the Green's function associated with σ' and σ respectively, then from the above definition it can be shown easily that

$$G^C_{\{\sigma\}}(N) = G^C_{\{\sigma'\}}(N) \qquad \text{for } n > 1 . \qquad (II.11)$$

This will be relevant for our σ model analysis in the next chapter.

We also define one particle irreducibility (1PI) concept. It is defined as the part of the Green's function in which one particle poles do not appear in the intermediate state. This differs slightly from the BPH definition (12) in the case where some of the fields can disappear into the vacuum through a vertex c. But in the framework of the B-W integral equation, our present definition is the only possible one. This point will be relevant in the next chapter.

One can give the algebraic definition of the 1PI part of $G^\Gamma(N)$ of $G(N)$ as follows. Pick up one external line ℓ from $N = N' \cup \{\ell\}$. Let a partition P decompose N' into subsets $N_1'^{\phi} \ldots N_{ip}'^{P}$ with $i_p > 1$. Then for $\omega(N) > 2$ ($\omega(A)$ = number of the elements in A)

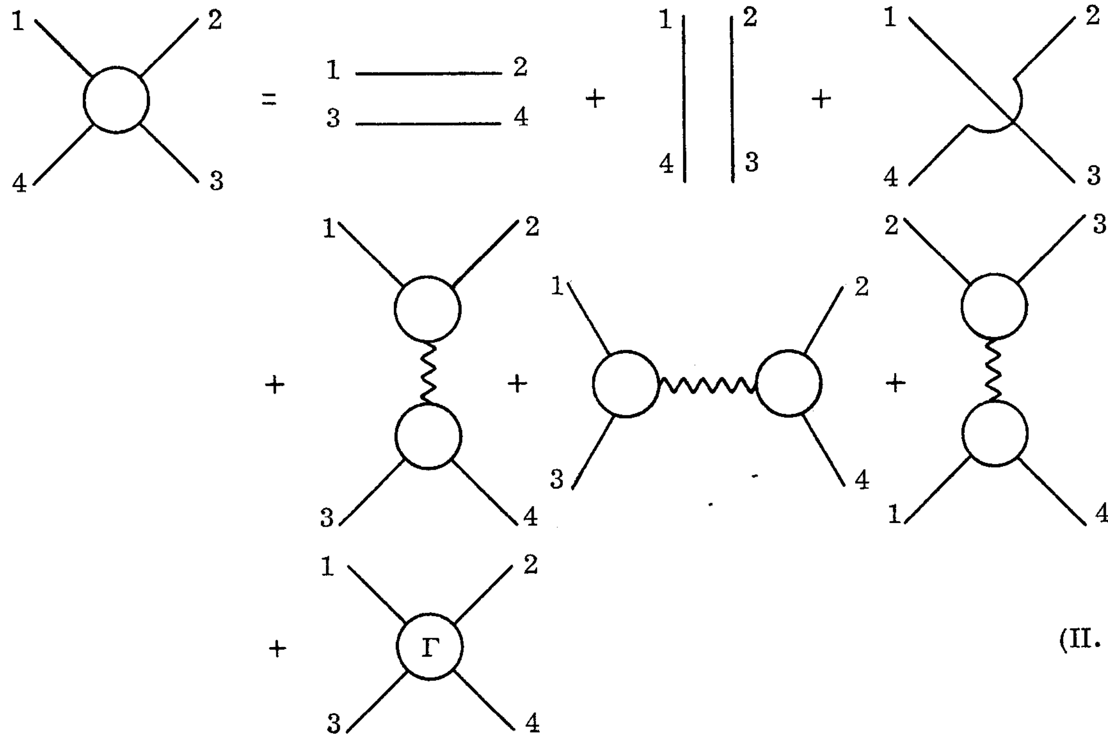
$$G^C(N) = \sum_P G^\Gamma\left(\{\ell, \underline{\ell_1}, \underline{\ell_2}, \ldots, \underline{\ell_{ip}}\}\right) G^C\left(N_1'^P \cup \{\ell_1\}\right)$$

$$\times\, G^C\left(N_2'^P \cup \{\ell_2\}\right) \ldots G^C\left(N_{ip}'^P \cup \{\ell_{ip}\}\right) \qquad (II.12)$$

For $\omega(N) = 2$ the 1PI parts $\Sigma_\phi(p)\left(\Sigma_\sigma(p)\right)$ of $G_\phi(p)\left(G_\sigma(p)\right)$ are defined as

$$G_\phi(p) = \frac{1}{p^2 - \mu^2 - i\,\Sigma_\phi(p)} \qquad (\phi \longleftrightarrow \sigma) \quad . \qquad (II.13)$$

Example 5: For the symmetric model where $\langle\phi\rangle_0 = 0\ \langle\sigma\rangle_0 = 0$,[11]



(II. 14)

The consistency and the correctness of the definition (II. 14) can be seen most easily from the corresponding Feynman diagram and the BPH definition. Consider the sum of all connected Feynman diagrams of order n with external lines $\ell, \ell_1, \ldots \ell_n$. For each of these diagrams, we can find a maximal 1PI part attached to line ℓ, which corresponds to the blob Γ in (II. 14). The rest of the diagram is of the form given by (II. 14). Now for different partition P, the resulting diagrams are also distinct. Since the renormalized Green's function is obtained by a linear operation on the Feynman diagram, this establishes the consistency of the definition (II. 14).

Another important definition we shall employ throughout this paper is the type X part $G^{X_M}(N)$ of G(N) with respect to a subset M of N. This is defined as the part which does not have poles in any of the momentum variables $Q_u = P_M + P_U$, $U \subset N' = N-M$ and $P\ \ = \sum_{\ell\in}\ \ P_\ell$. U can be null set or N' itself. In general, type X part of a Green's function could involve some of the terms occurring in the disconnected part of the function. But the separation of the function into its connected and disconnected parts are not hard, and therefore one need only to define a connected and type X part (CX). We may define a CX part algebraically as follows.

$$G^{CX_M}(N) = G^C(N) - \sum_U{}' G^{CX_M}\left(M^U U\{\underline{\ell}\}\right) G^C\left(\{\ell\}^U (N'-U)\right) \qquad \text{(II. 15)}$$

Here the sum extends over all subset U of N' with $\omega(N'-U) \geq 2$. For $\omega(N') = 1$, $G^{CX_M}(N) = G^C(N)$.

The correctness of above definition can also be seen from the corresponding Feynman diagram analysis. Let $D_F^{\Omega}(\mathscr{L})$ denote the sum of all Feynman diagrams with the set of external lines $\mathscr{L}$ and with connectivity Ω. Then any diagram which has pole in the variable $Q_\mu = P_M + P_U$ belongs to

$$D_F^{X_M}\left(M^U U'^U\{\ell\}\right) * D_F^C\left(\{\ell\}^U N'-U'\right)$$

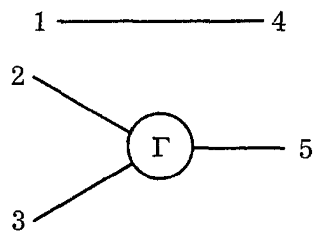
for some $U' \leq U$. Furthermore for different U', the set defined above is different. Therefore the decomposition (II. 15) follows. For $\omega(N-M) = 1$, we obviously have

$$G^{CX_M}(N) = G^C(N) \quad . \qquad \text{(II. 16)}$$

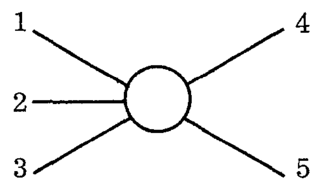
It is important to realize that the type X part $G^{X_M}(M^U \underline{N})$ becomes 1PI part when the lines belonging to M are joined together. In fact, this property is our motivation for the definition of type X diagram, and is very important in writing down the integral equation in the next subsection.

In the graphical notation we adopted, the various parts of a Green's function will be represented by the letters, C, Γ, X, etc. When considering the type X diagram, we shall either put the reference set on the left side of the blob, or we shall have already closed all the lines in M. In such a case, the set M is clear and will not be specified.

Example 6:



belongs to the type X parts of the



The integral equations to be discussed in the next subsection relate various 1PI functions and type X functions to each other. The complete Green's functions can, of course, be constructed out of them by means of the above definitions.

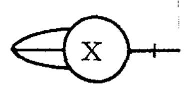
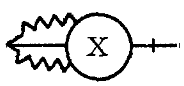
In Section IV these definitions of the various parts of a Green's function are very important. There we will use diagramatic notations instead of the Symanzik's notation to help our intuition.

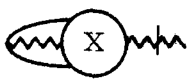
3. Integral Equations

With these notational preparations, we can write down the integral equations obtained from the B-W field equations (II.3, II.4) in a concise and intuitively appealing fashion. They can be derived in exactly the same manner as given by Brandt and Wilson (15). The procedure is as follows. We multiply both sides of (II.3) by any combination of fields $\phi(x_1)\ldots\phi(x_n)\sigma(y_1)\ldots\sigma(y_n)$ and take the vacuum expectation value of its time ordered product. In view of (II.4), we will get an integral equation involving various Green's functions. With the help of the definition given in the previous subsection, this can be transformed into a simpler form. In order not to lengthen this already too long a section, we will not go into the details of this procedure. Here we will merely write down the result of this process for our later convenience.

In the following, we will make extensive use of the graphical representation for the integral equations. It is possible to write them in terms of the Symanzik's notation $G(p_1 \ldots ; q_1 \ldots)$. For some of the cases following, the latter notation might be more comfortable to some readers. However, it is the author's opinion that by using this diagram technique, we can develop an intuitive feeling on the complicated jargon of the integral equations.

Now we begin with the 1PI parts of the propagators.

$\Sigma_\phi(p) = ig\Big[$  $p +$  $p - \{R\} - \frac{R_4}{ig}\, \Sigma_\phi(p)\Big]$. (II. 17)

$\Sigma_\sigma(p) = ig\Big[$  $p +$  $p - \{R\} - \frac{R_4}{ig}\, \Sigma_\sigma(p)\Big]$. (II. 18)

$$\{R\} = R_1(\xi,\eta) - i p_\mu R_2^\mu(\xi,\eta) - p_\mu p_\nu R_3^{\mu\nu}(\xi,\eta)$$

The limit $\xi, \eta \to 0$ in (II. 17), (II. 18) was taken by

$$\lim_{\xi,\eta \to 0} R(\xi,\eta) = \int_{k_1 k_2} \lim_{\xi,\eta \to 0} e^{-i\xi k_1 - i\eta k_2} R(k_1,k_2)$$

and the integral symbols are to be shared with the other terms in the right-hand side of (II.17), (II.18). In other words, (II.17) really means

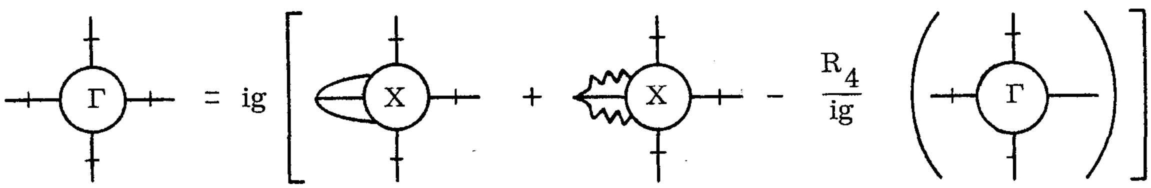
$$\Sigma_\phi(p) = ig \int_{k_1}\int_{k_2} i^3 G_\phi(k_1)\, G_\phi(k_2)\, G_\phi(p-k_1-k_2)\; G^X(\underline{k_1}, \underline{k_2}, p, \underline{p-k_1-k_2};)$$

$$+ i^3 G_\sigma(k_1)\, G_\sigma(k_2)\, G_\phi(p-k_1-k_2)\; G^X(\underline{p-k_1-k_2}, p; k_1, k_2)$$

$$- R_1(k_1, k_2) + i p_\mu R_2^\mu(k_1, k_2) + p_\mu p_\nu\, R_3^{\mu\nu}(k_1, k_2)$$

$$- \frac{R_4}{ig}(k_1, k_2) \cdot \Sigma_\phi(p) \quad .$$

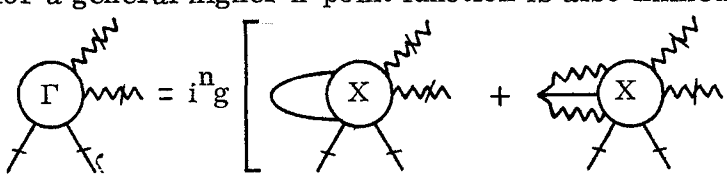
For higher n-point functions, we have first

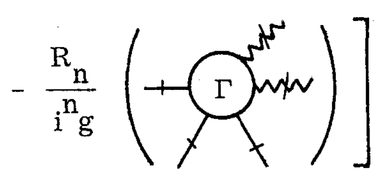


(II.19)

This form looks slightly different from the one given by Brandt (15). They are actually the same, and the simpler form (II.19) is obtained because of our definition of the type X diagram.

The case for a general higher n-point function is also immediate:





(II.20)

for certain integer n.

Equations (II.17) - (II.20), together with the renormalization conditions to determine the subtraction points, form the complete set of integral equations for the symmetric model. They can be solved order by order in g, and give exactly the same result as the BPH procedure (12). Perhaps it is important to remark that the type X diagrams in Eq. (II.17) - (II.20) have in general disconnected contributions. (C.f. Example 6, Section II.) For practical calculations, it would be in general convenient to separate out these disconnected contributions and deal only with connected contributions. These procedures would be extensively used in later sections. In Appendix A, some of these are illustrated by lower order calculations.

4. Properties

Before concluding this section, we will briefly discuss the symmetry properties of the integral equations (II.17) - (II.20). Since the chiral symmetry is built into the field equations (II.3), (II.4), the solution of the integral equation should be chiral symmetric. One should have for example:

$$\Sigma_{\phi}(p) = \Sigma_{\sigma}(p) \tag{II.21}$$

$$G^{\Gamma}(\underline{0},\underline{0},\underline{0},\underline{0};) = G^{\Gamma}(;\underline{0},\underline{0},\underline{0},\underline{0},) = \frac{1}{3}\, G^{\Gamma}(\underline{0},\underline{0},;\underline{0},\underline{0}) \tag{II.22}$$

These relations can be easily confirmed in lower order examples and are required by the general rule of the chiral invariance. The proof of these will be postponed until we treat the σ model in the next sections. There, we will see in general that

$$\begin{aligned} & G^{\Gamma}(\underline{p_1},\underline{p_2},\cdots\underline{p_n},\underline{q_1};\underline{q_2},\underline{q_3},\cdots\underline{q_m}) + G^{\Gamma}(\underline{p_1},\underline{p_2},\cdots\underline{p_n},\underline{q_2};\underline{q_1},\underline{q_3},\cdots\underline{q_m}) + \cdots \\ & -G^{\Gamma}(.\underline{p_2},\underline{p_3},\cdots\underline{p_n};\underline{q_1},\underline{q_2},\cdots\underline{q_m},\underline{p_1}) + G^{\Gamma}(\underline{p_1},\underline{p_3},\cdots\underline{p_n};\underline{q_1},\underline{q_2},\cdots\underline{q_m},\underline{p_2}) + \cdots \\ & = 0 \quad . \end{aligned} \tag{II.23}$$

Equations (II.21) and (II.22) are a direct consequency of these identities.

Note that in a BPH scheme, the symmetry condition (II.21) and (II.22) are constraints on the subtraction constants. If we do not impose this constraint on the subtraction procedure, we will get a finite result but the symmetry would be completely lost in general. For our B-W approach, as soon as we have written down the equations and current definition in the form of (II.3), (II.4), we are guaranteed to have chiral symmetry. This, as mentioned in the introduction of this section, is a significant advantage of our present approach.

III. THE σ MODEL

A. Introduction

In this chapter, we start to analyze the σ model mathematically. In order to deduce a finite and meaningful consequence, we must renormalize the formal theory described by the Lagrangian (I. 1). Some orientation to this purpose was already made in the previous section, where we have considered the renormalization problem of the symmetric model. Now the Lagrangian for these two models (the symmetric and the σ) differ only by an extremely simple term, i.e., by a source term. This fact leads one to expect that the renormalization of the model can be carried out in such a way that much of the symmetry present in the symmetric model could be maintained. We shall see in this section that this is indeed the case.

What do we mean by "maintain much of the symmetry" in a renormalized perturbation theory? In BPH scheme, the Lagrangian is only a formal device to get Feynman rules, and the symmetry present in the Lagrangian does not immediately guarantee the symmetry of the resulting theory. The underlying approximate symmetry must also be reflected on the BPH subtraction procedure which renders the theory finite. In such a situation, we must invoke some other principles which would supply us a definite and precise statement of a "partial symmetry." Such was the procedure taken by Symanzik (10). In order to determine constraints upon various subtraction constants, he makes use of the P.C.A.C. relation (I. 3) (in renormalized form). By means of this, he derives Ward identities of the σ model, which then determines the subtraction procedure uniquely. But the consistency of these identities with the Green's functions thus obtained remained to be shown.

If one chooses the B-W procedure which was discussed in the previous section and which we shall employ throughout this work, however, one does not need any extra principles to proceed. The condition of approximate symmetry envisaged by the Lagrangian (I. 1) can be imposed directly on the B-W field equations themselves.[12] We shall see that the form of the field equations which "break the symmetry only by a source term" (a constant term in the field equations) is unique. The form of the field equations thus determined, all of the Green's functions are completely determined according to the general procedure of B-W. Furthermore, the B-W integral equations exhibit the relations between various Green's functions in a clear and rigorous fashion. One can in fact establish the equivalence of our approach to the Symanzik's one by showing that these Green's functions satisfy the Ward identities. We shall discuss these matters in the later sections.

We have actually generalized slightly the B-W procedure in the preceding paragraph. The statement "breaking symmetry only by a source term" was derived from the formal Lagrangian language. Whether we could use this statement to obtain the form of the field equation is not completely obvious because of the necessary procedure of renormalization. Therefore the correctness of our procedure will be proved in detail by showing that our theory is equivalent to a BPH theory of Lagrangian (I. 1).

In B, we discuss the BPH theory of the σ model. This will be proved to be useful for our later discussion of the consistency of the B-W field equation. In C, we discuss the B-W field equations and derived integral equations. The form of the current operators are first written down and the consistency is proved in the later parts of this section. Finally, the perturbation schemes (the way we expand amplitudes in perturbation series) are explained in D.

B. BPH Description

From Lagrangian (I. 1), we can derive in the usual manner the Feynman rules, which are identical to those of the symmetric model except for the additional vertex

C (III. 1)

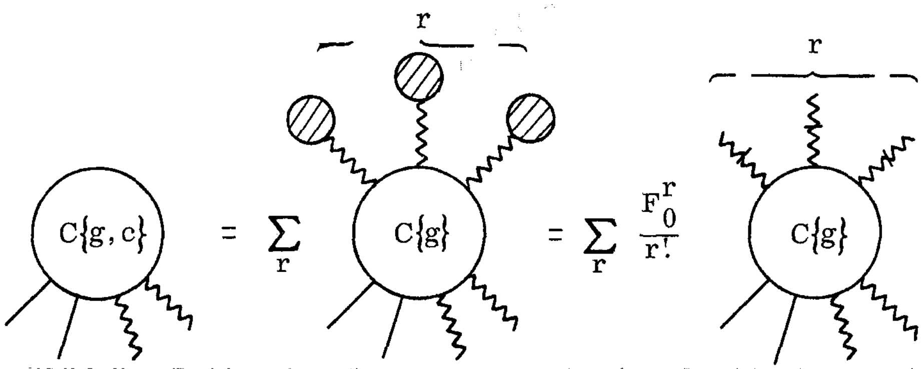
In the following vertices of (III. 1) will be called the c vertex, and the vertices of Fig. 1 will be called g vertices. The renormalized BPH theory with vertices $\{V_i\}$ will be called a BPH $\{V_i\}$ theory. Thus the theory described in the previous section is equivalent to a BPH $\{g\}$ theory, and we are going to describe a BPH $\{g.c\}$ theory in this subsection. (Of course, it should be understood that the Feynman rule does not specify completely a BPH theory. In addition, one must also specify the subtraction conditions.)

In this section, scalar field will be named as σ' to distinguish it with the translated field $\sigma = \sigma' - F$, $\langle 0|\sigma|0\rangle = 0$. As remarked in II.B.2, there is no difference between these two definitions if we consider only connected parts of a graph. Now in a BPH $\{g.c\}$ theory, a σ' line can disappear into the vacuum through a c-vertex. It will generally have self energy corrections before its disintegration. Such a line with all its radiative correction will be called a ξ line.[13] According to the BPH definition of the reducibility (12), every vertex that belongs to a ξ line is weakly connected[14] to the rest of the diagram. Therefore, the renormalization of a ξ line can always be carried out separately from the rest of the diagram. We shall call the value of this renormalized ξ line, that is the vacuum expectation value of σ', as F. At this point, we can proceed in two different ways.

a. First, sum over all such ξ line. Then the resulting theory is equivalent to a theory with modified propagators and additional 3 particle vertices as shown in Fig. 2.[15] Now we do not have any c vertex and the definition of irreducibility, given by BPH and the one given in Section II.C coincides. By usual counting rule, the class of 1PI superficially divergent (primitively divergent) diagrams is determined as shown in Fig. 3. This is the approach taken by Lee (7, 8) and Symanzik (9, 10). Lee proves, by means of the conventional renormalization theory (20) that all of the infinities can be absorbed into g, c, μ^2. Symanzik, on the other hand, determines the constant on the BPH subtraction constants by means of the Ward identities and shows that these identities completely specify the subtraction procedure. However, the consistency of the resulting theory with the Ward identities themselves remained to be explicitly shown.

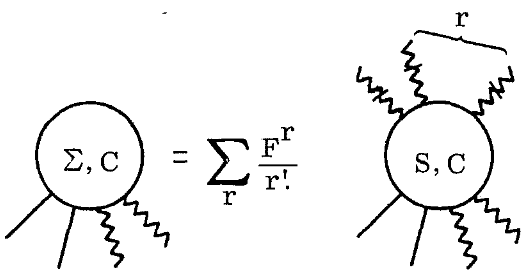
b. There is another approach by means of which we shall see the equivalence of the BPH and the B-W procedure. This is as follows. As for the definition of the irreducibility, neglect the ξ line as above. (The value of ξ line is, of course, assumed to be F.) Now in order to renormalize the diagram, we keep the points where the ξ line shot out and attach each of these points with an external σ line with 0-momentum. In this way, we have a correspondence between a BPH $\{g.c\}$ diagram and a BPH $\{g\}$ diagram. Then the renormalization of the latter diagram can be carried out symmetrically.

The precise form of the above mentioned correspondence can be expressed diagramatically as follows. In the following we represent a Feynman diagram (not renormalized B-W diagram) of a BPH $\{V_i\}$ theory by writing in the symbol $\{V_i\}$ inside the blobs of a diagram. The connectivity of the diagram is expressed by the usual letter C, Γ, X, CX, etc. Then



(III. 2)

Here the slash on the external legs means the amputation of the external lines with all its self energy corrections,  is a ξ line, F_0 the unrenormalized value of a ξ line. r is the number of ξ lines present in the diagram. The factor $1/r!$ is given by the combinatorics of the σ model. Equation (III. 2) is actually a part of the theorem given by Lee and Gervais (8). We renormalize (III. 2) in the fashion explained above to get



(III. 3)

This is now the relation between a renormalized B-W function of Σ model and that of symmetric model. The letters Σ and S signigy the σ model and the symmetric model respectively. Equation (III. 3) will be the starting point of our discussion of the equivalence of the B-W and BPH approach in the next section.

Before ending this subsection, one should note the following extremely important point. That is, procedures a and b are only a very limited class of possible choice of subtraction procedure. As a matter of fact, there need not be any constraint on the various subtraction constants in a. The reason why we want the Ward identities to determine the constraints on the subtraction constants in a or the reason why we want the very particular procedure of b is, of course, to ensure the chiral symmetry: Even though the symmetry is broken by the source term, we want to maintain most of it in order to have a "partial" symmetry. The fact that the theory resulting from this particular renormalization process, the case b, indeed breaks the symmetry only partially will be proved in the next section.

C. Field Equations and Their Correctness

1. Field Equations

From the fact that the chiral symmetry is broken only by a source term which is a constant in the formal field equation of σ', we guess that the correct form of the field equation from the general B-W procedure is

$$\begin{aligned} (\Box^2+\mu^2)\,\phi(x) &= g\, j_\phi(x) \\ (\Box^2+\mu^2)\,\sigma'(x) &= g\, j_{\sigma'}(x) + \gamma \end{aligned} \qquad \text{(III. 4)}$$

With the form of j_ϕ and $j_{\sigma'}$ remaining the same as (II. 4). γ is a constant related to the source strength c of the η field. Introducing $\sigma = \sigma' - F$, we get

$$\begin{aligned} (\Box^2+\mu^2)\,\phi(x) &= g\, J_\phi(x) \\ (\Box^2+\mu^2)\,\sigma(x) &= g\, J_\sigma(x) - \mu^2 F + \gamma \qquad \text{with } \langle\sigma\rangle_0 = 0\ . \end{aligned}$$

Current definitions now take the following form:

$$J_\phi(x) = \lim_{\xi,\eta\to 0} \left[(\sigma^2+\phi^2)\,\phi - R_1\,\phi - R_2^\mu\,\partial_\mu\,\phi - R_3^{\mu\nu}\,\partial_{\mu\nu}\,\phi + (F^2+2F\sigma)\,\phi - R_4 J_\phi\right]$$

$$J_\sigma(x) = \lim_{\xi,\eta} \left[(\sigma^2+\phi^2)\sigma - R_1\sigma - R_2^\mu\,\partial_\mu\sigma - R_3^{\mu\nu}\,\partial_{\mu\nu}\sigma + 3\sigma^2 F + \phi^2 F + 3\sigma F^2 + F^3 - R_1 F - R_4 J_\sigma\right].$$

(III. 6)

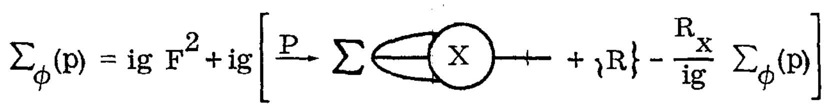
In the above, we omitted the obvious argument of the functions. For example;

$$(\phi^2+\sigma^2)\sigma = [\phi(x-\xi)\,\phi(x-\eta)+\sigma(x-\xi)\,\sigma(x-\eta)]\sigma(x), \text{ etc.}$$

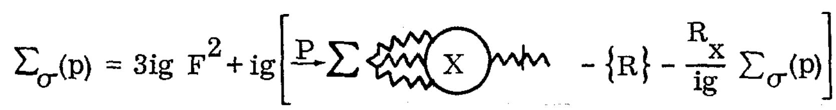
As a matter of fact, the field equations (III.4) - (III.6) are not a completely obvious result of the B-W prescription. The point is that since the symmetry is broken anyway, the subtraction functions can have many nonsymmetric contributions. For example, terms like $B(\xi)\ \phi(x-\xi)\ \sigma(x)$ cannot be excluded a priori from the current definition of (III.5), (III.6).

Nevertheless, only with the form (III.5), (III.6) could one give the statement "breaks a symmetry by a source term only" a definite meaning. We shall explicitly show the correctness and consistency of these equations in the later part of this section by establishing the equivalence of these equations with the BPH theory discussed in the previous section. But before doing this, let us first write the integral equations derived from the field equations (III.4) - (III.6). Again, the derivation will be along the same line as given by Brandt (15).

We first have for the 1PI part of the propagators:

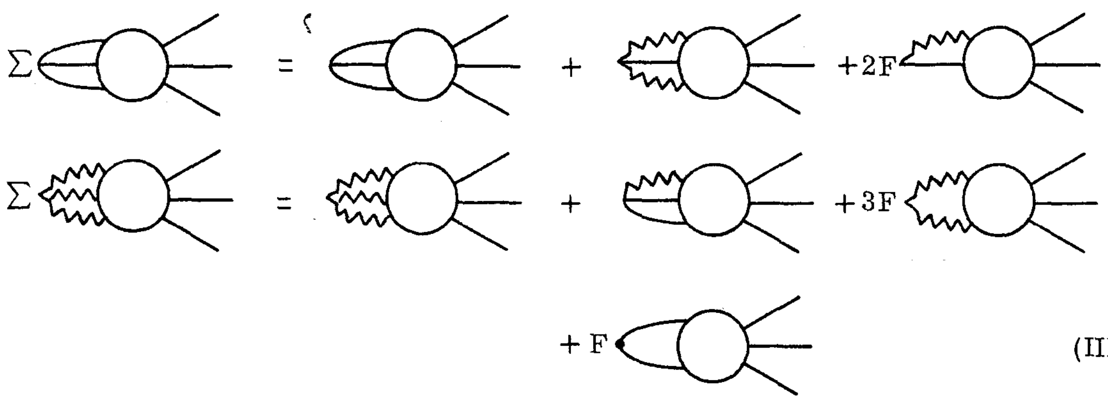


(III.7)



(III.8)

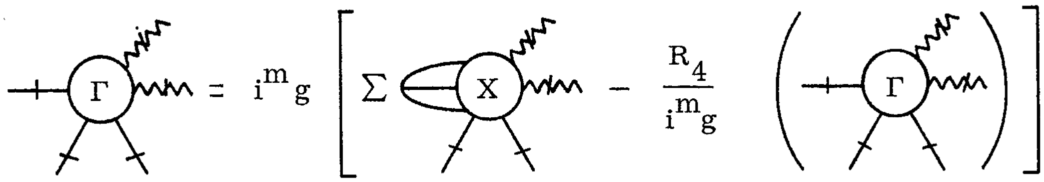
Here we have introduced the simplifying notations.



(III.9)

Other notations were introduced previously.

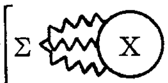
For the case of higher n-point functions, we have[16]



(III. 10)

for some integer m.

There is one more equation resulting from the consistency condition $\langle 0|\sigma|0\rangle = 0$. Taking the vacuum expectation value of (III. 4) we have

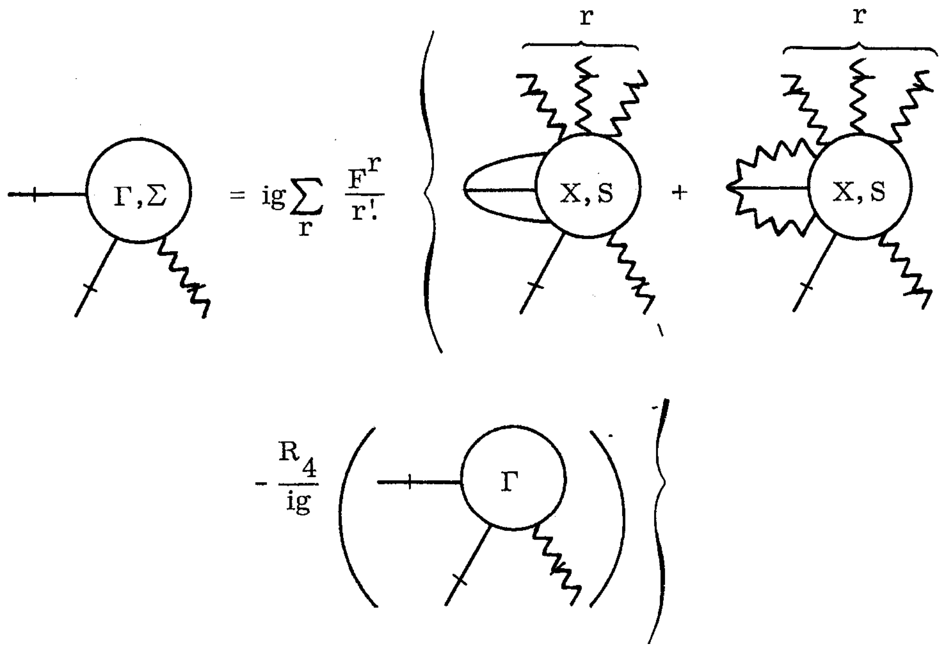
$$\mu^2 F - \gamma = g\Big[\Sigma$$  $$+ F^3 - R_1 \cdot F - R_4 \frac{\mu^2 F - \gamma}{g}\Big] \qquad \text{(III. 11)}$$

All of the Green's functions of our theory are determined from the set of the coupled integral equation (III. 7) - (III. 10). The vacuum expectation value of the σ' field is determined by (III. 11). They can be solved iteratively order by order in the perturbation theory.

2. Proof of Consistency

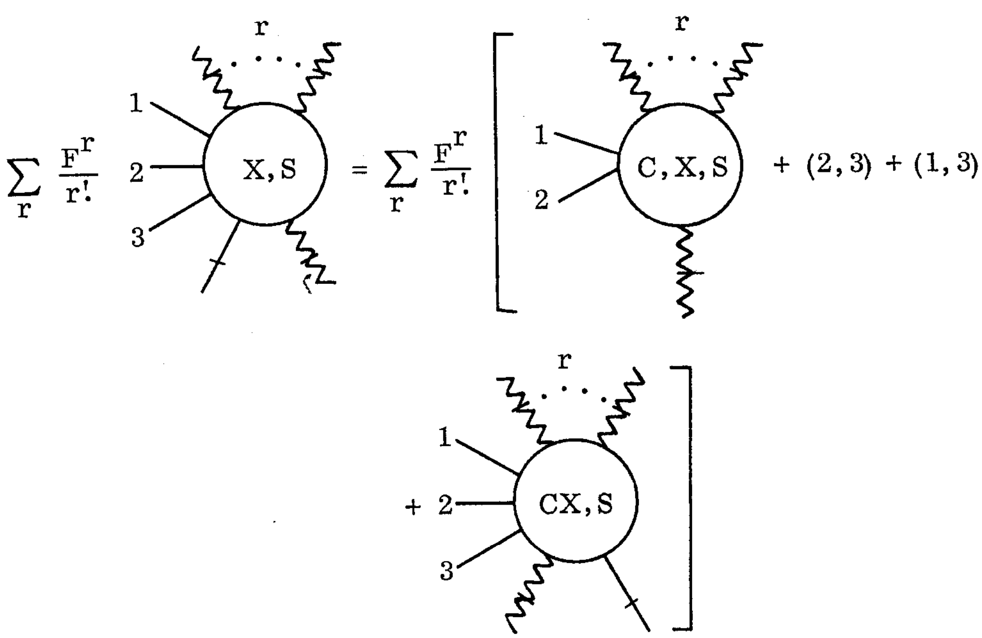
We now proceed to show the correctness of our field equations. We shall see explicitly that the set of the integral equations (III. 7) - (III. 11) is equivalent to the renormalized BPH theory defined in the previous section. The basis of our argument will be the fact that the BPH $\{g\}$ theory, as explained in Section II, is completely equivalent to the B-W field equation (II. 3), (II. 4) and therefore to the derived integral equations (II. 17) - (II. 20). The proof of this fact is exactly analogous with the one given by Brandt for pseudoscalar coupling theory (15) and will not be given here.

For simplicity, we consider the 3-point function. In view of (III.2) and (II.20), we have



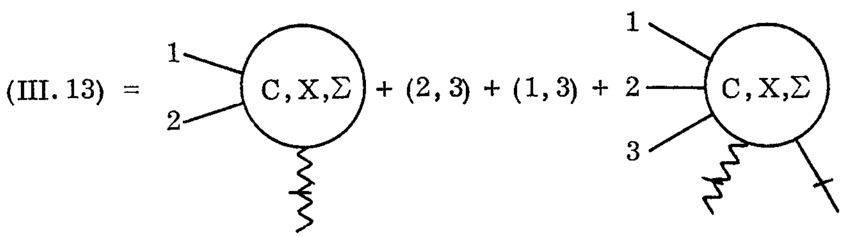
(III.12)

We now separate out the disconnected parts from the type X diagram in the right-hand side of (III.12). In the following, the diagrams are type X with respect to the external lines on the left side of the blobs. First we consider

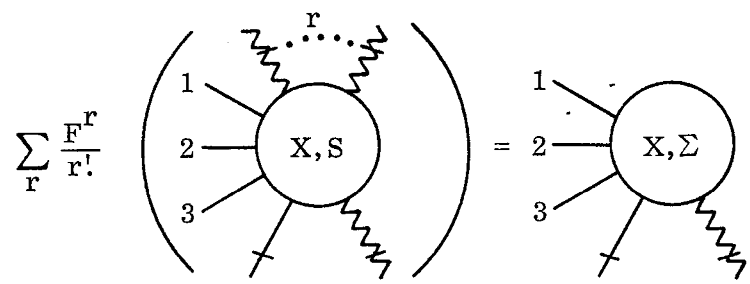


(III.13)

Use (III.2) to reduce this in the following form

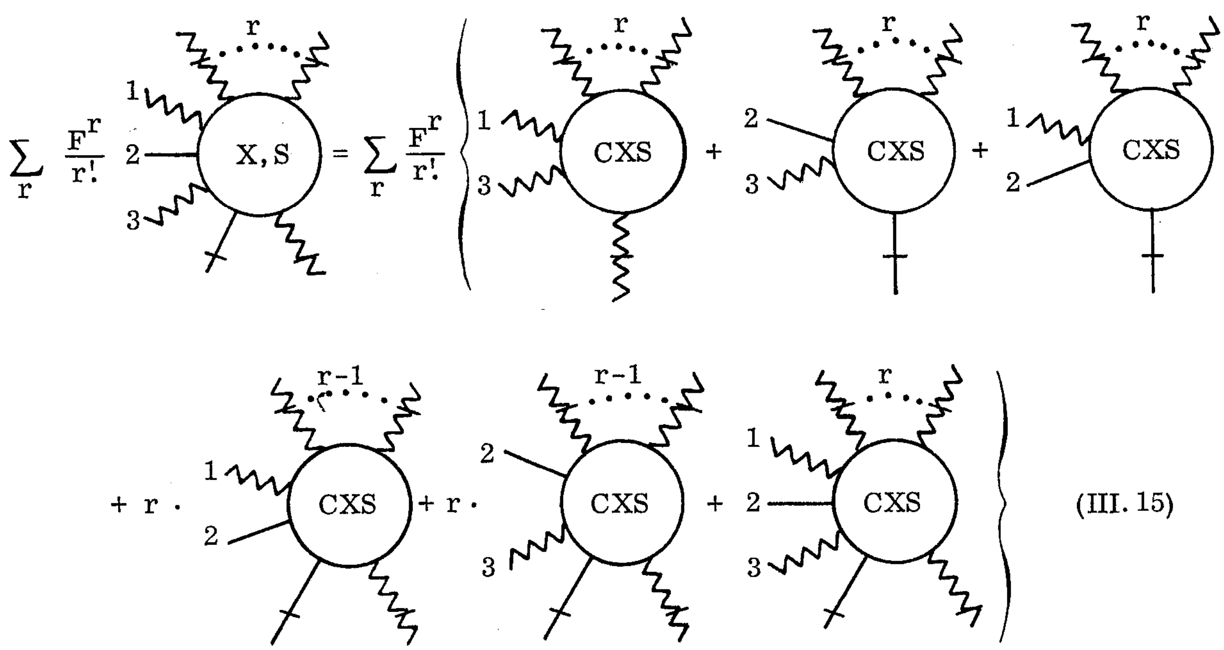


Therefore we have



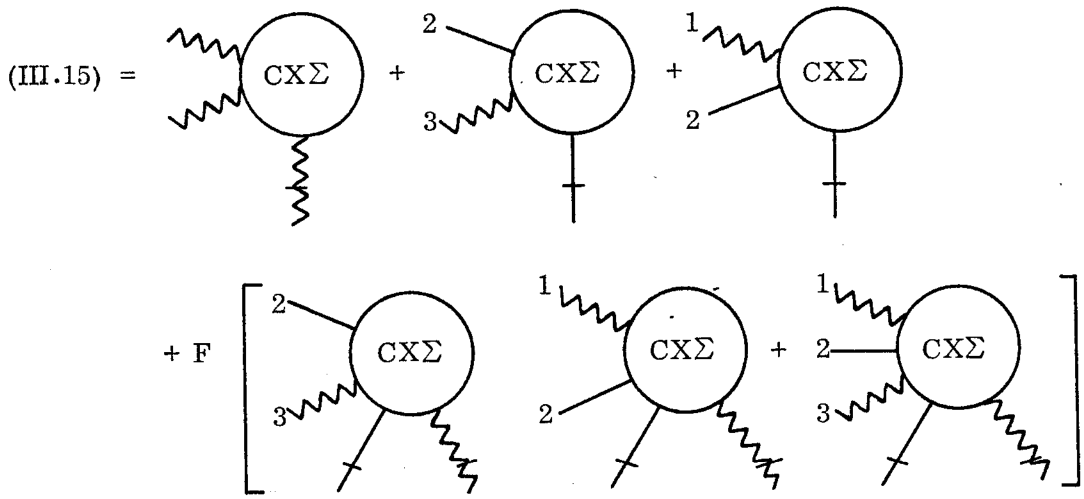
(III.14)

Next, we consider



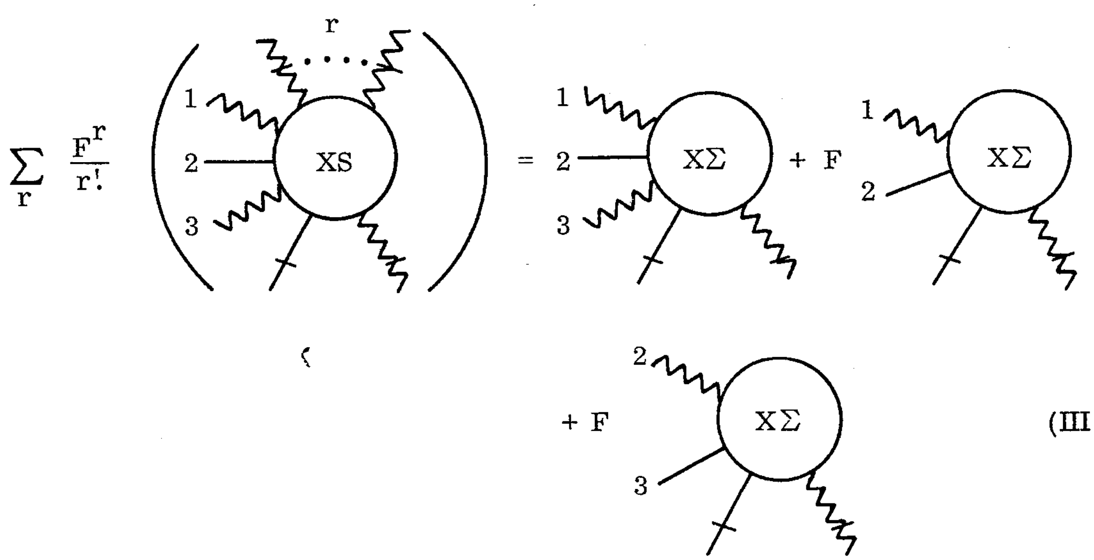
(III.15)

Again in view of (III.2) this is reduced to



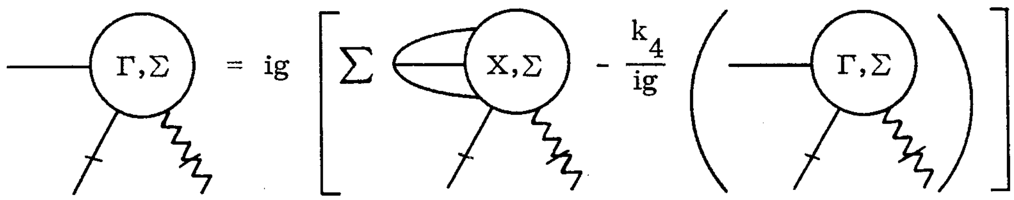
(III.16)

Therefore



(III.17)

by adding the disconnected parts together. After closing the lines 1, 2, 3 in (III.16) and (III.17), and from (III.12), we finally have



(III.18)

This is exactly the same as (III.10).

It is clear that a procedure similar to the above could be employed to obtain all of (III.7) - (III.11). We shall not repeat the arguments here.[17]

D. Perturbation Scheme

How should one expand the Green's functions in perturbation series? For the symmetric model, no ambiguity arises, and we will expand the amplitude in the coupling constant g. For the σ model we must first decide how we should treat F in the series. One obvious possibility is to expand in g regarding F fixed. We call it "g-scheme." Another very interesting procedure suggested by the experiences coming from the chiral dynamics (21) is to expand in g regarding gF^2 fixed. In BPH language, this corresponds to the procedure a of Section III.B and is to expand in the number of loops appearing in the diagram (7) The zeroth order diagrams are all trees, the first order one loop, etc. We call the latter the "loop scheme."

Both of the above schemes are completely legitimate, and we will see in the next chapter that the Ward identities hold to all orders in each of the above schemes. In the appendix A, we will explain these and other points of this chapter by means of some lower order calculation in both schemes.

IV. WARD IDENTITIES

This section is devoted for the discussion and proof of the Ward identities of the σ model. We expect these identities to be true since we have partially conserved axial vector current. We first state the identities by using the argument of Symanzik (9, 10) and prove these identities in all orders of perturbation theory. Some proofs of diagrammatic theorems are relegated to the Appendix B.

A. The Identities

In this subsection, we assume the existence of the partially conserved (renormalized) axial vector current $A_\mu(x)$

$$\partial_\mu A_\mu(x) = f\ \phi(x) \tag{IV.1}$$

in order to guess the identities we want to establish in the next section. The arguments are due to Symanzik (9, 10). We further assume that the current generates the chiral group, i.e.,

$$\left[A_0(0), \phi(x)\right]_{ET} = + i\,\delta(x)\,\sigma'(x) + \text{S.T.} \tag{IV.2}$$

$$\left[A_0(0), \sigma'(x)\right]_{ET} = - i\,\delta(x)\quad \phi(x) + \text{S.T.} \tag{IV.3}$$

where S.T.'s are possible Schwinger terms which need not be specified here.

Consider the following identities:

$$\int dx\, \partial_\mu \langle 0|T\, A_\mu(x)\, \phi(x_1), \ldots, \phi(x_n)\sigma'(y_1), \ldots, \sigma'(y_m)|0\rangle = 0 \tag{IV.4}$$

The left-hand side can be expanded as follows:

$$\int dx \langle 0|T\, \partial_\mu A_\mu(x)\, \phi(x_1), \ldots, \phi(x_n)\sigma'(y_1), \ldots, \sigma'(y_m)|0\rangle$$

$$+ \sum_i \int dx\, \delta(x^o - x_i^o) \langle 0|T \left[A_0(x), \phi(x_i)\right] \phi(x_1), \ldots, \hat{\phi}(x_i), \ldots, \phi(x_n)$$

$$\times\, \sigma'(y_1), \ldots, \sigma'(y_m)|0\rangle$$

$$+ \sum_j \int dx\, \delta(x^o - y_i^o) \langle 0|T \left[A_0(x); \sigma(y_i)\right] \phi(x_1), \ldots, \phi(x_n)$$

$$\times\, \sigma'(y_1), \ldots, \hat{\sigma}'(y_j), \ldots, \sigma'(y_m)|0\rangle \qquad \text{(IV.5)}$$

By means of (IV.1), (IV.3) and (IV.5), and assuming that the Schwinger terms do not contribute, we have then

$$0 = -f \int dx \langle 0|T\, \phi(x)\, \phi(x_1), \ldots, \phi(x_n)\sigma'(y_1), \ldots, \sigma'(y_m)|0\rangle$$

$$-i \sum_i \langle 0|T \sigma'(x_i)\, \phi(x_1), \ldots, \phi(x_i), \ldots, \phi(x_n)\sigma'(y_1), \ldots, \sigma'(y_m), 0\rangle$$

$$+i \sum_j \langle 0|T\, \phi(x_j)\, \phi(x_1), \ldots, \phi(x_n)\sigma'(y_1), \ldots, \sigma'(y_j), \ldots, \sigma'(y_m)|0\rangle$$

From the definition of the Green's function in momentum space, it then follows

$$(-i)\; fG(0, p_1, p_2, \ldots, p_2; q_1, \ldots, q_m) = \sum_i G(p_1, \ldots, p_i, \ldots, p_n; p_i q_1, \ldots, p_m)$$

$$- \sum_j G(q_j p_1, \ldots, p_n; q_1, \ldots, q_j, \ldots, q_m) \quad .$$

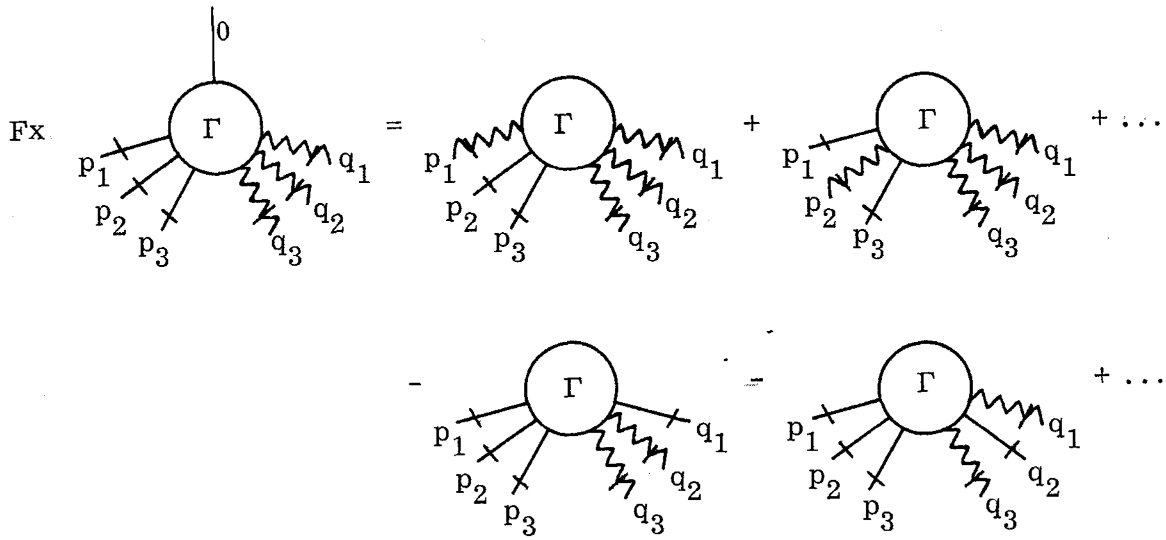
From the definition of the connected parts, it then follows that

$$-if\; G^c(0, p_1, p_2, \ldots, ; q_1, \ldots, q_m) = \sum_i G^c(p_1, \ldots, p_i, \ldots, p_n; p_1, q_1, \ldots, q_m)$$

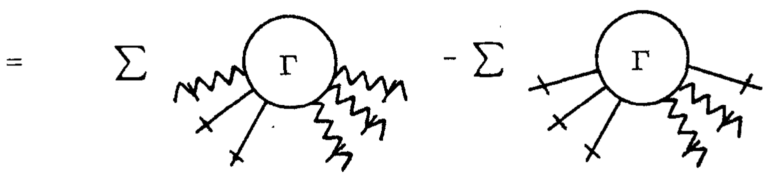
$$- \sum_j G^c(q_j p_1, \ldots, p_n; q_1, \ldots, q_j, \ldots, q_m) \quad . \qquad \text{(IV.6)}$$

This is one of the desired identities.

We now state this and related identities in diagram notations. They will be proved rigorously in the next subsection, and for this purpose, the diagrammatic representation is very convenient. First we have

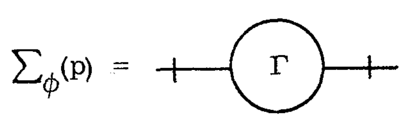


which we shall denote concisely as

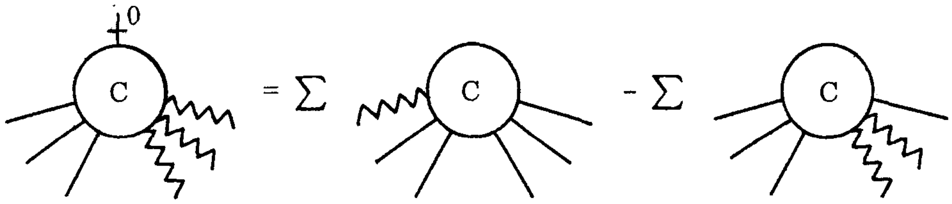


(IV.7)

When the left-hand side of (IV.7) is a 3-point function, the following definitions are understood:



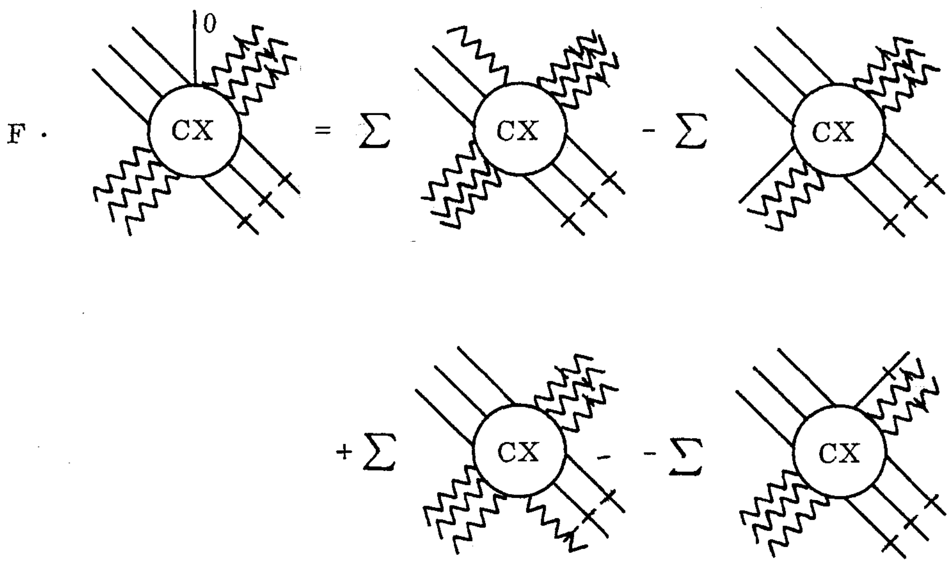
For a connected part, we have



(IV.8)

Notice the difference between (IV.7) and (IV.8). In (IV.7), all of the external lines are amputated, while we keep them in (IV.8) except for the zero momentum line on the left-hand side. We call this particular line "0-line."

Finally, we have



(IV.9)

Here the diagrams are type X with respect to the lines attached on the left side of the blobs. Notice that (IV.8) is the same as (IV.6) if we identify

$$F = -f\, G_{\phi}(0) \quad . \tag{IV.10}$$

We shall see in Section V that

$$F\, G_{\phi}^{-1}(0) = -\gamma \quad . \tag{IV.11}$$

Therefore the identification (IV.10) leads to

$$f = \gamma \quad , \qquad \text{(IV.12)}$$

i.e., γ is the symmetry breaking parameter as we expected.

Note also that if we take $\gamma=0$, $F\neq 0$ then we have the symmetric model, and in this case the identity (IV.7) is the same as (II.23) given in Section II.

B. Proof of the Identities

Now we want to prove (IV.7) - (IV.8) to all orders in the perturbation theory. The method of our proof will be inductive. We first need the following lemmas:

Lemma 1: If (IV.7) is true then (IV.8) is true.

Lemma 2: If (IV.7) and (IV.8) are true then (IV.9) is true.

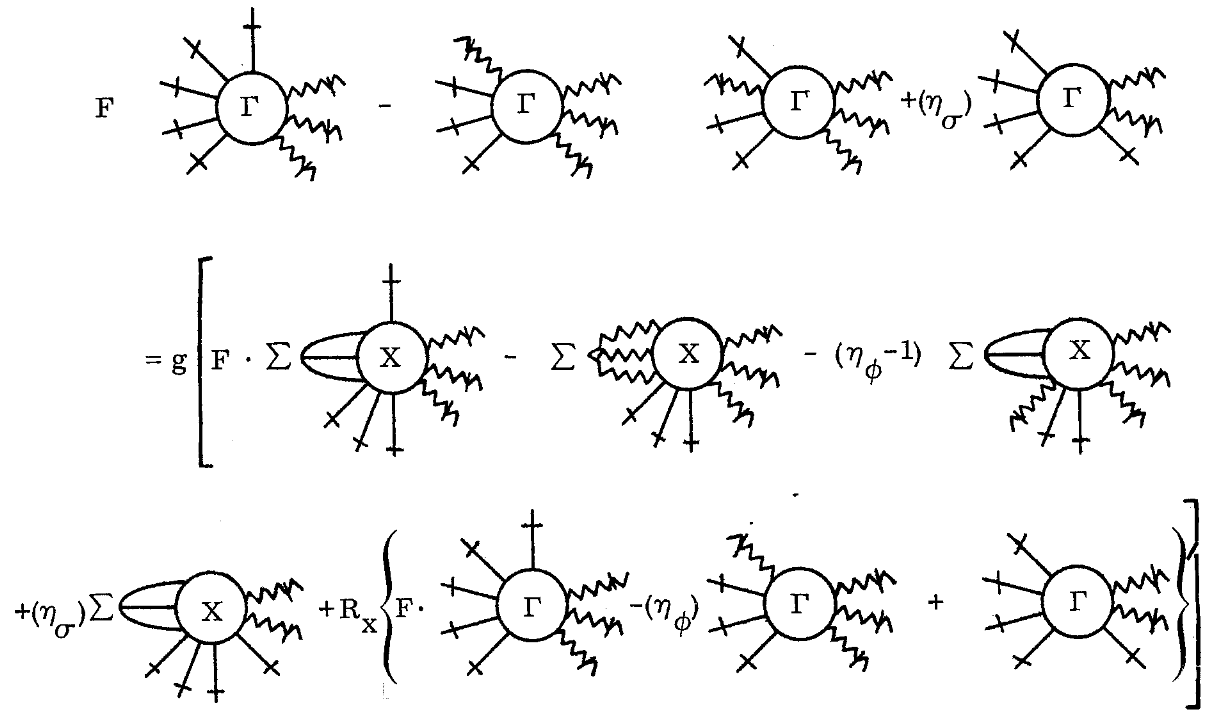
The proof of the above lemmas involves elaborate and somewhat tedious analysis of diagrams. In order not to interrupt the flow of the argument, we will postpone this proof to Appendix B.

As the basis of our induction argument, we first note that (IV.7) is true in the lowest order as is shown in the appendix. It is true in both of the perturbation schemes explained in the previous section.

We will first show the proof for the g scheme. The proof for the other scheme will be obtained by the time we finish our proof for the first scheme.

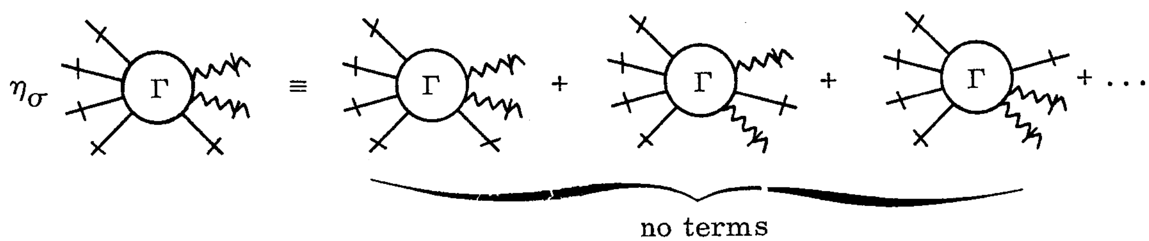
We then assume (IV.7) and therefore, by Lemma 1 and 2, (IV.8) and (IV.9) are true to nth order in g. We want to show that they are true to n+1th order in g.

Consider the left-hand side of (IV.7). Let η_ϕ be the number of external ϕ-lines except for the 0-line and η_σ be the number of external σ lines. We are going to consider only the case $\eta_\phi \geq 2$, and $\eta_\sigma \geq 1$. The other simple cases can be treated separately using the method described below. We have, from (III.10),

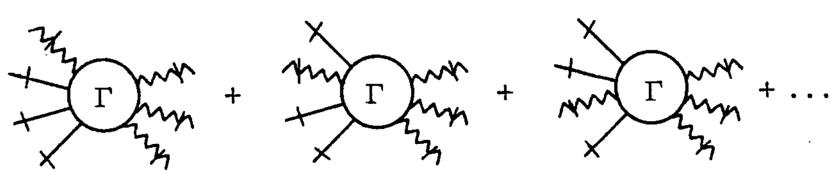


(IV.13)

The notations (η_σ) or $(\eta_\phi - 1)$ represent the number of the terms belonging to the group of the diagrams the typical form of which was shown. For example,

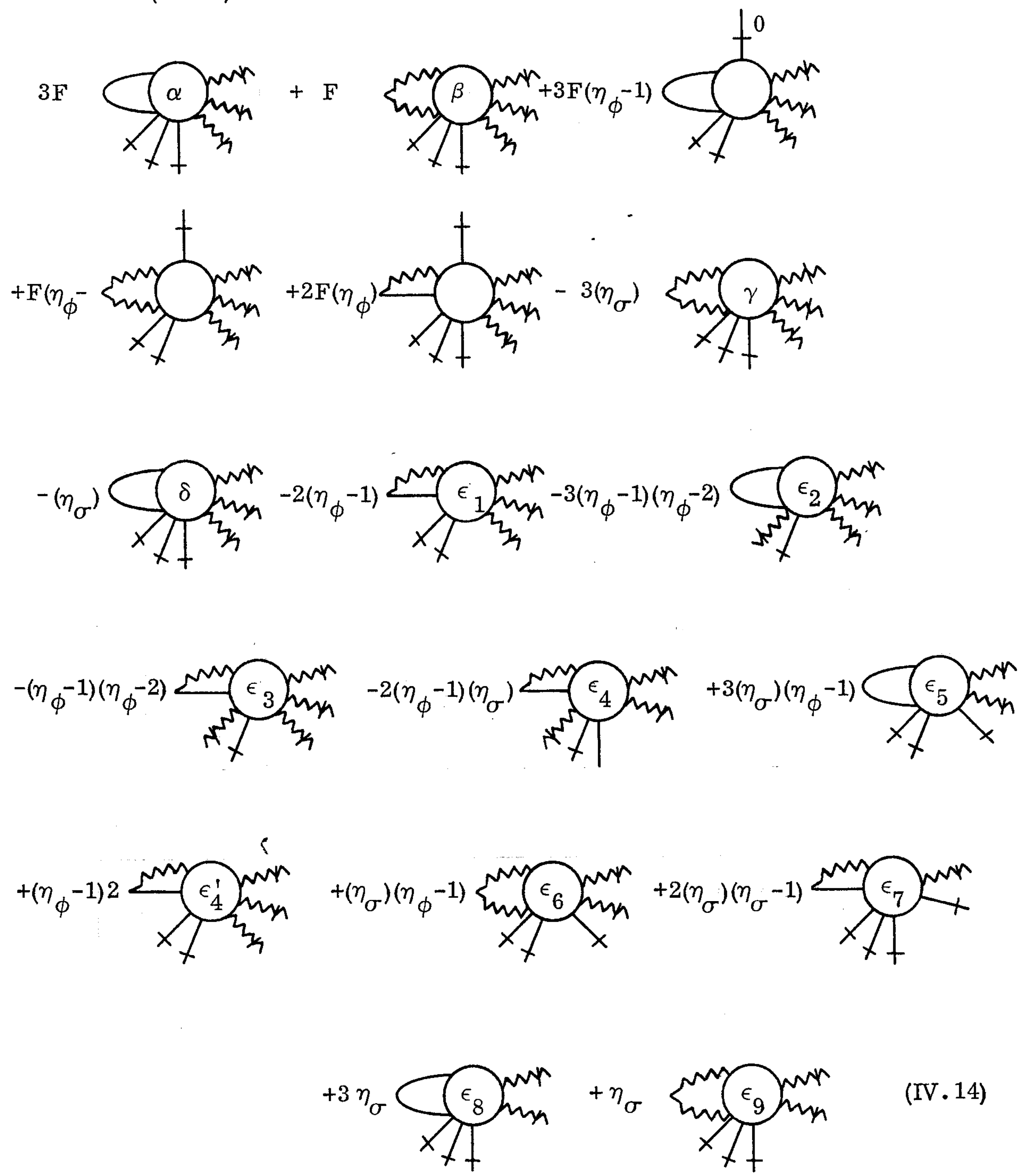


Note that (IV.13), we divided the sum of the terms



in two groups: One with first term only, the rest of the terms composing the second group.

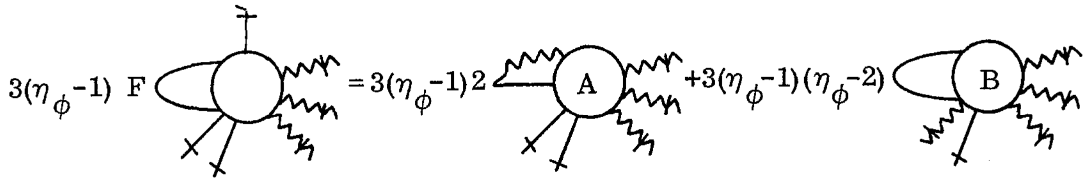
Now if we could show the right-hand side of (IV.13) vanishes in order n,[18] then we would have shown that (IV.7) is true in order n+1. We now proceed to this end. To do this, we must separate out the disconnected contributions from the type X diagrams in order to be able to apply (IV.9). The disconnected contribution of (IV.13) is[19]

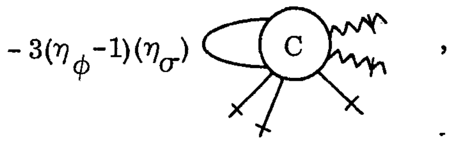


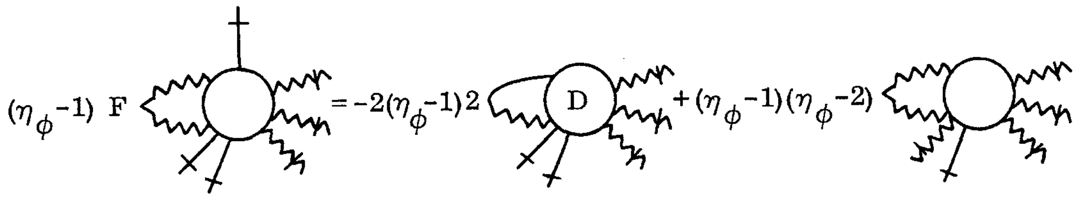
(IV.14)

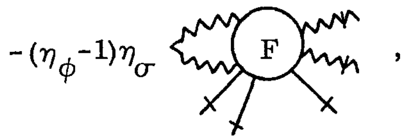
(In each of the above diagrams the level X, C is dropped.)

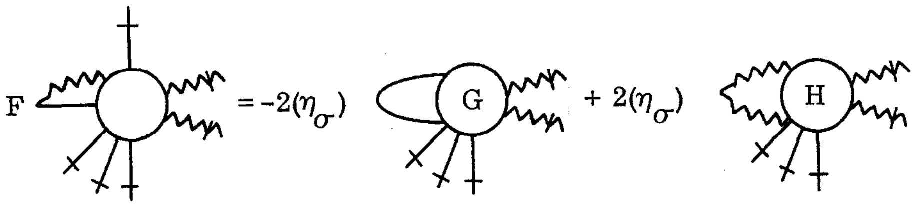
For the diagrams which have 0 lines, we use (IV.9) to get

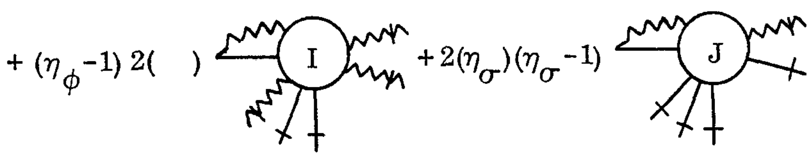












putting this back into (IV.14) we observe that the following cancellations occur.

$D + A + \epsilon + \epsilon'_4 = 0$

$B + \epsilon_2 = 0$

$C + \epsilon_5 = 0$

$E + \epsilon_3 = 0$

$\delta + G + \epsilon_8 = 0$

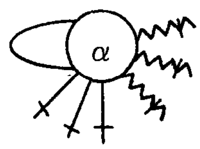
$F + \epsilon_6 = 0$

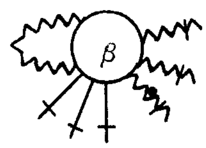
$\gamma + H + \epsilon_9 = 0$

$I + \epsilon_4 = 0$

$J + \epsilon_7 = 0$

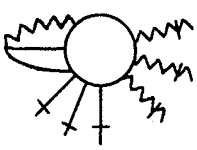
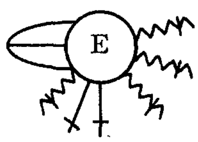
So the only remaining terms are

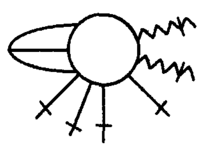
(IV.14) = 3F 

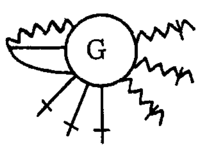
+ F 

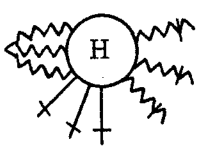
(IV.19)

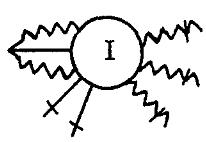
Next, consider the connected contribution of (IV.13). Again by means of lemma 2, we have

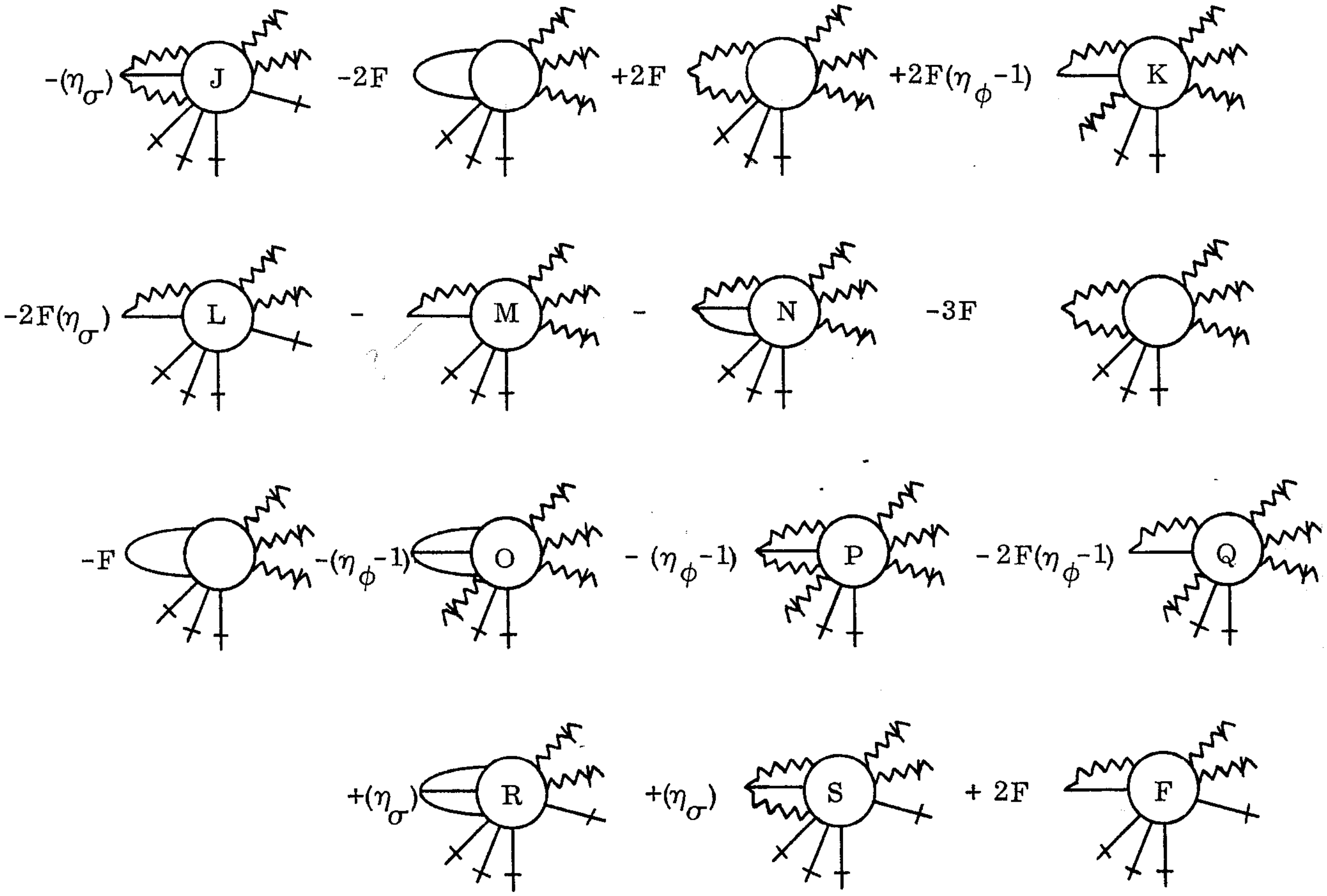
3  + $(\eta_\phi - 1)$ 

$- \eta_\sigma$ 

-2 

+ 

+ $(\eta_\phi - 1)$ 



(IV.20)

The blobs without any letters cancel the disconnected contribution (IV.19) and also

$$D + G + N = 0$$

$$E + O = 0$$

$$F + R = 0$$

$$H + M = 0$$

$$I + P = 0$$

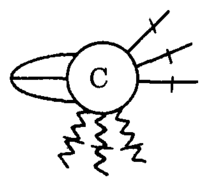
$$J + S = 0$$

$$K + Q = 0$$

$$L + T = 0$$

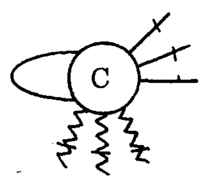
So we have indeed shown that the right-hand side of (IV. 13) vanishes in nth order. With similar calculation for simpler cases for $\eta_\phi=1$ or $\eta_\sigma=0$, we have seen that (IV. 7) is true to all orders in g.

If we examine our proof shown above, we find that terms of the form,



(IV. 21)

and of the form



(IV. 22)

do not mix together and cancel among themselves separately. Since the blobs contain only connected diagrams, terms of the form (IV. 21) will increase the number of loops by 2, (IV. 22) by 1. This means that (IV. 7) and therefore (IV. 8) and (IV. 9) are true in each order in the loop expansion. (For the case $\eta_\phi=1$ or $\eta_\sigma=0$, we come across a constant term, e.g., 2Fg. This corresponds to the tree term in the usual perturbation theory. In this case, those tree terms cancel among themselves.)

V. THE GOLDSTONE THEOREM AND P.C.A.C.

In this section, we discuss some consequences of Ward identities of previous sections. First we prove the Goldstone theorem to all orders of perturbation theory. This is achieved by means of Ward identities of the previous section together with the integral equation (III.11). Next we study P.C.A.C. relations in our model. Adler consistency condition is easy to recognize and Weinberg's relation is also simple consequence of Ward identities.

A. The Goldstone Theorem

In view of the expected relation

$$\partial_\mu A^\mu(x) = +\gamma\,\phi(x)$$

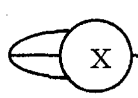
we shall have the situation of spontaneously broken symmetry in the limit

$$\begin{array}{c} \gamma \to 0 \\ F \neq 0 \end{array} \qquad \text{(V.1)}$$

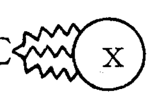
Now the zero mass particle must have the quantum number of the axial vector current, i.e., that of the ϕ-particle. So we may expect that the ϕ mass vanishes in the limit (V.1). Of course, the question whether this limit prevails must be settled. It may be that whenever $\gamma \to 0$ then $F \to 0$. In such a circumstance, we do not have the Goldstone phenomena. This point was discussed by Lee and Basdevant (22) and here we shall merely assume that we have the limit (V.1).

By combining (III.7) and (III.11) we get

$$F\left(\Sigma_\phi(0) - i\mu^2\right) + i\gamma = ig \left[F \cdot \Sigma \right.$$



$$+ \quad -\Sigma$$

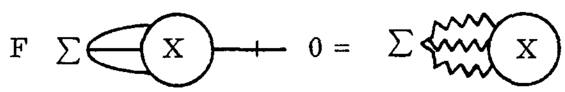


$$\left. - R_4 \frac{F\left(\Sigma_\phi - i\mu^2\right) + i\gamma}{ig} \right] \qquad \text{(V.2)}$$

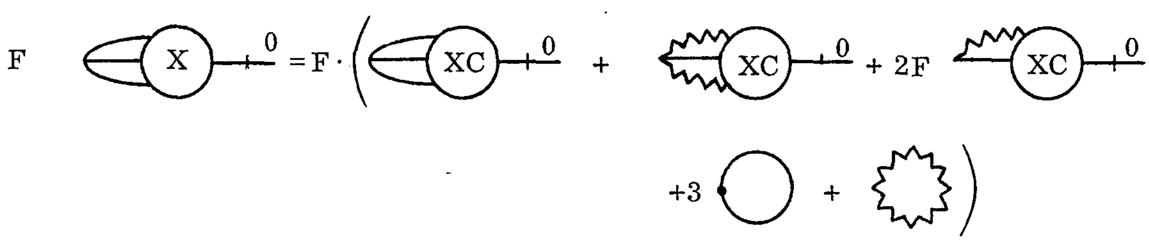
We shall now show that

$$F\left(\Sigma_{\phi}(0) - i\mu^{2}\right) \equiv + iF\, G_{\phi}^{-1}(0) = -i\gamma \quad . \tag{V.3}$$

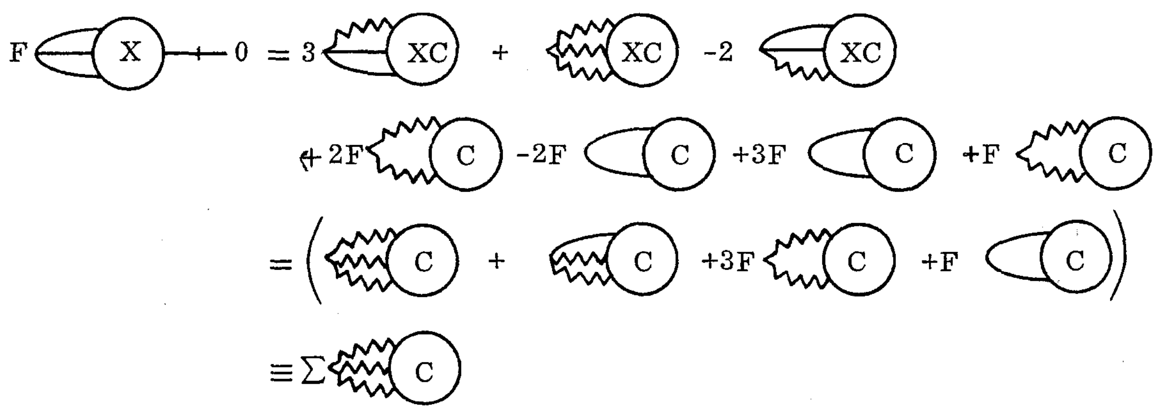
This relation was remarked earlier in connection with the Ward identities. In order to prove (V.3), we note that they are true in lower orders as is clear from (V.2). So we want to show



The proof goes along the similar line as in Section IV.[20] First separate out the disconnected and connected contribution to type X diagram.



The last two terms are the disconnected contributions. In view of (IV.6), one gets



Therefore, we have seen the correctness of (V.4).

Now for the case the limit (V.1) holds, we have

$$G_\phi^{-1}(0) = 0 \ .$$

This completes the proof of the Goldstone theorem.

B. P.C.A.C.

An obvious way to proceed would be to construct explicitly the axial vector current of the form

$$A_\mu(x) = \lim_{\xi \to 0} \left[\phi(x)\, \partial_\mu \sigma(x+\xi) - \sigma(x)\, \partial_\mu \phi(x+\xi) - q(\xi, x) \right] \quad ,$$

where $q(x, \xi)$ is the appropriate operator subtraction term explained in Section II. Then we can explicitly see whether the P.C.A.C. relation holds,

$$\partial_\mu A_\mu(x) = \gamma\, \phi(x)$$

and whether various commutation relations of $A_\mu(x)$ with itself and other fields agree with the rule of the algebra of the current. In view of Brandt's analysis of quantum electrodynamics (13), this procedure is expected to be a very complicated and intricate one and we want to present a complete analysis of P.C.A.C. in a future paper. In order to analyze the soft pion theorems, however, it turns out that we do not need such a detailed preparation and the Ward identities stated and proved in Section IV are sufficient machineries.

For example, by dividing out by propagators in (IV.6) and putting particles on the mass shell, one immediately recognizes Adler's self-consistency condition (23).[21] In the rest of this subsection, we shall see how the low energy theorem of Weinberg (24) works.

First, we must generalize (IV.6) for the present case where the pions are isotriplet. It is obvious that our generalization takes the following form (for the four-point function);

$$\lim_{p_1 \to 0} F \cdot G(\underline{p_1}a, p_2b, p_3c, p_4d;) = \delta_{ab} G^c(p_3c, p_4d; p_2) + \delta_{ac} G^c(p_2b, p_4d; p_3c)$$

$$+ \delta_{ad} G^c(p_2b, p_3c; p_4) \qquad (V.5)$$

In (V.5), we treated the special case for the π-π scattering, and we put isospin indices right next to the momentum level in the definition of the connected amplitude.

The identities for a general configuration are also immediate.

Now since the limit (V.5) is well defined, the result should not depend upon the way in which the limit $p_1, p_3 \to 0$ is taken. We will take the $p_1 \to 0$ limit first, and let

$$p_2 = p+\epsilon \qquad p_4 = -p+\epsilon \qquad p_3 = -2\epsilon \quad . \qquad (V.6)$$

Then we want to show,

$$\lim_{\epsilon \to 0} T_{ab,cd}(0, p+\epsilon | 2\epsilon, p-\epsilon) \equiv \lim_{\epsilon \to 0} G^c(\underline{0(a)}, \underline{p \quad (b)}, -\underline{2\epsilon(c)}, -\underline{p \quad (d)};)$$

$$= T0\, \delta_{ac}\delta_{bd} + (\delta_{ab}\delta_{cd} - \delta_{ad}\delta_{bc}) \frac{2\epsilon \cdot p}{F^2} i \quad .$$

(V.7)

Now we will show that (V.7) is a simple consequence of (V.5) and the Bose symmetry. We first rewrite (V.5) as follows:

$$\lim_{\epsilon \to 0} F \cdot T_{ab,cd}(0, p+\epsilon | 2\epsilon, p-\epsilon) = \lim_{\epsilon \to 0} G_\phi^{-1}(-2\epsilon)\, G_\phi^{-1}(p-\epsilon)\, G_\phi^{-1}(p+\epsilon)$$

$$\times \Big\{ \delta_{ab}\, G_\phi(-2\epsilon)\, G_\phi(p-\epsilon)\, G_\sigma(p+\epsilon)\, G^c(\underline{-2\epsilon\,(c)}, \underline{-p+\epsilon\,(d)}; \underline{p+\epsilon})$$

$$+ \delta_{ac}\, G_\phi(p+\epsilon)\, G_\phi(-p+\epsilon)\, G_\sigma(-2\epsilon)\, G^c(\underline{p+\epsilon\,(b)}, \underline{-p+\epsilon\,(d)}; \underline{-2\epsilon})$$

$$+ \delta_{ad}\, G_\phi(p+\epsilon)\, G_\phi(-2\epsilon)\, G_\sigma(+p-\epsilon)\, G^c(\underline{p+\epsilon\,(b)}, \underline{-2\epsilon\,(c)}; \underline{p-\epsilon}) \Big\}$$

$$= \delta_{ab}\, G_\phi^{-1}(p+\epsilon)\, G_\sigma(p+\epsilon)\, G^c(\underline{-2\epsilon\,(c)}, \underline{-p+\epsilon\,(d)}; \underline{p+\epsilon})$$

$$+ \delta_{ac}\, G_\phi^{-1}(2\epsilon)\, G_\sigma(2\epsilon)\, G^c(\underline{p+\epsilon\,(b)}, \underline{-p+\epsilon\,(d)}; \underline{-2\epsilon})$$

$$+ \delta_{ad}\, G_\phi^{-1}(p-\epsilon)\, G_\sigma(p-\epsilon)\, G^c(\underline{p+\epsilon\,(b)}, \underline{-2\epsilon\,(c)}; \underline{p-\epsilon}) \qquad \text{(V.8)}$$

Now, since

$$G_\phi^{-1}(p \pm \epsilon) \sim (p \pm \epsilon)^2 - \mu_\pi^2 = \pm 2p \cdot \epsilon + 0(\epsilon^2) \qquad \text{(V.9)}$$

Therefore for first order calculation, we can replace ϵ by zero for the first and the third term in (V.8) except the G_ϕ^{-1} factor for which we use (V.9).

For the middle term, we note that

$$G_\sigma(\epsilon) = f_\sigma(\epsilon^2) \qquad G_\phi(\epsilon) = f_\phi(\epsilon^2)$$

and

$$G^c(\underline{p+\epsilon\,(b)}, \underline{-p+\epsilon\,(d)}; -2\epsilon) = \delta_{bd}\, f((p+\epsilon)^2, (p-\epsilon)^2, p^2 - \epsilon^2)$$

$$= \delta_{bd}\, f((p-\epsilon)^2, (p+\epsilon)^2, p^2 - \epsilon^2) \quad . \qquad \text{(V.10)}$$

The first equality from the isospin conservation, and the second from the Bose symmetry of our theory.

From (V.10) we see that,

$$f((p+\epsilon)^2, (p-\epsilon)^2, (p^2-\epsilon^2)) = g(p^2, \epsilon^2, (\epsilon\cdot p)^2) \tag{V.11}$$

Therefore, the middle term of (V.8) cannot contribute to the first order term in ϵ, and we can replace ϵ by 0.

That is, to the first order in ϵ, (V.8) becomes

$$\begin{aligned} &= \delta_{ab}(2p\cdot\epsilon)\, G_\sigma(p)\, G^c(\underline{0(c)}, \underline{-p(d)}; \underline{p}) \\ &+ \delta_{ac}\, G_\phi^{-1}(0)\, G_\sigma(0)\, G^c(\underline{p(b)}, \underline{-p(d)}; \underline{0}) \\ &+ \delta_{ad}(-2p\cdot\epsilon)\, G_\sigma(p)\, G^c(\underline{p(b)}, \underline{0(c)}; \underline{p}) \end{aligned}$$

Now from (IV.6) (its generalized version) and the form

$$G^c(\underline{p(b)}, \underline{-p(d)}; \underline{(0)}) = \delta_{bd}\, c'(p)$$

We finally get

$$\lim_{\epsilon\to 0} F\cdot T(0, p+\epsilon \mid 2\epsilon, p-\epsilon) = \delta_{ac}\delta_{bd}\, c'(p)\, G_\phi^{-1}(0)\, G_\sigma(0) + \frac{2\cdot p\cdot\epsilon}{F}\, i(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) \tag{V.12}$$

Therefore, we have seen that (V.12) is true with

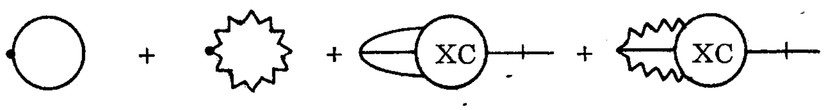
$$T^o = \left(\frac{1}{F}\right) c'(p)\, G_\phi^{-1}(0)\, G_\sigma(0) \quad .$$

APPENDIX A

In this appendix, we show explicitly how the subtraction functions determined from the symmetric model give finite results for the σ model by considering some lower order diagrams. We will carry out our calculation in both of the perturbation scheme defined at the end of Section IV.

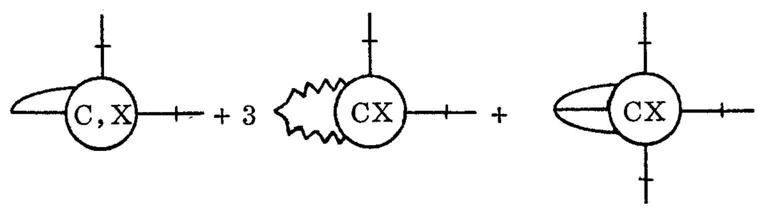
We first determine the subtraction function R's from the symmetric theory. We write (II.17) and (II.18) in the following form.

$\Sigma_\phi(p) = ig\Big[\,3$

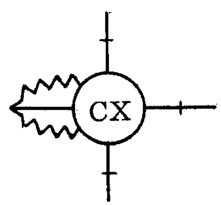


$$- R_1 + ip_\mu R_2^\mu - p_\mu p_\nu R_3^{\mu\nu} - \frac{R_4}{ig}\,\Sigma_\phi(p)\Big] \qquad (A.1)$$

$\Omega = ig\Big[\,6+9$



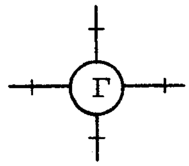
$+$

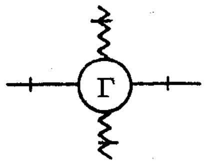


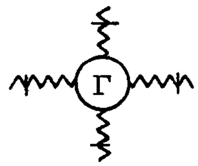
$$+ - \frac{R_4}{ig}\,(\Omega)\Big] \qquad (A.2)$$

Note that in the above, each blob is a connected one. In order to determine subtraction functions R'(x), we need the following renormalization condition.

$$\Sigma_\phi(0) = \partial\Sigma_\phi(0) = \partial\partial\,\Sigma_\phi(0) = 0$$
$$\Sigma_\eta(0) = \partial\Sigma_\eta(0) = \partial\partial\,\Sigma_\eta(0) = 0 \qquad (A.3)$$

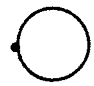
and $\Omega(0,0,0,0) = 6\,ig =$ 



$= 2ig$ 

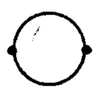
$= 6ig$ (A.4)

In zeroth order $\Sigma_\phi^{(0)} = \Sigma_\sigma^{(0)} =$ all 1PI vertex parts $= 0$. We put these values to the right-hand side of (A.1) and (A.2) to get first order values, and compare this with (A.3) and (A.4) to obtain R's. In this way, we obtain

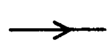
$R_1^{(0)} = 4$  $= i\,4 \int \frac{dk}{(2\pi)^4} \frac{1}{k^2-\mu^2}$

$$R_{2\mu}^{(0)} = R_{3\mu\nu}^{(0)} = 0\ , \qquad \text{(A.5)}$$

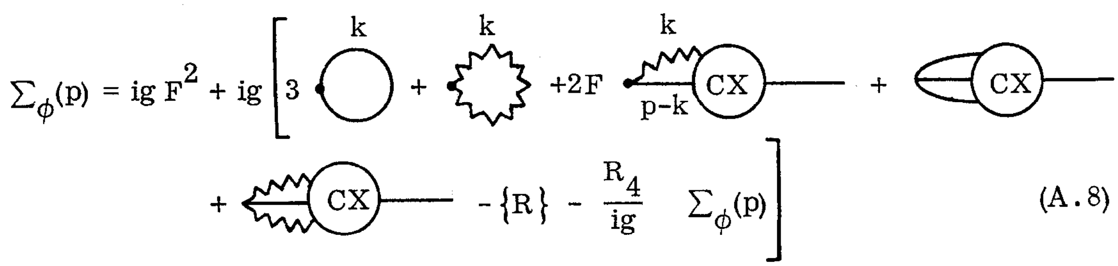
and

$k_4^{(1)} = 10\,ig$  $= -10\,ig \int \frac{dk}{(2\pi)^4} \frac{1}{(k^2-\mu^2)^2}$ (A.6)

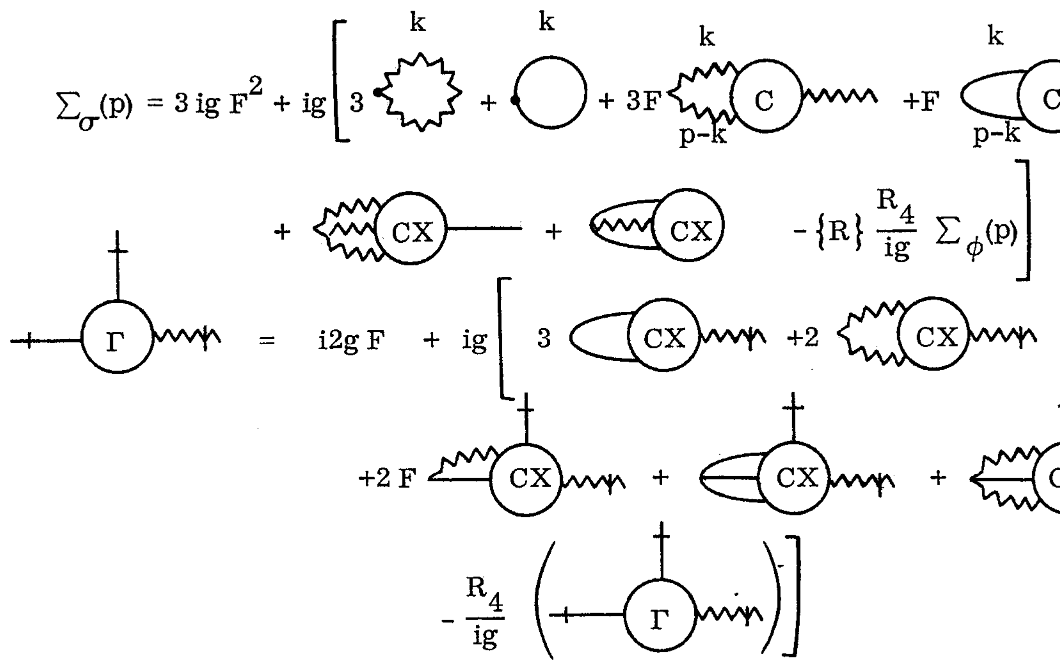
In deriving (A.5) and (A.6), we have used the fact that for zeroth order, the following is true;

k  = k  $= \frac{i}{k^2-\mu^2}$. (A.7)

With the lowest order value of R's thus determined, we now calculate some lower order value of the σ model functions. We now write down the integral equations (II.17) - (II.19).



(A.8)



(A.9)

(A.10)

We now consider each of two schemes separately.

1. <u>The g-scheme</u>

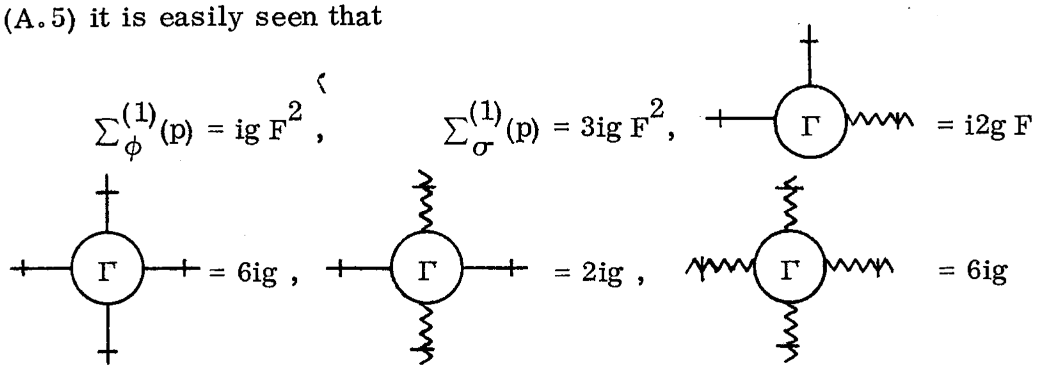
a. Lowest order

It is immediately seen that $\Sigma_\phi^{(0)}(p) = \Sigma_\sigma^{(0)}(p) =$ All 1PI

$$\text{vertex parts}^{(0)} = 0 \quad . \tag{A.11}$$

b. First order in g

We substitute the result (A.11) to the right-hand side of (A.8) - (A.10). With (A.5) it is easily seen that



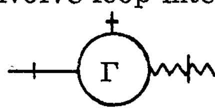
(A.12)

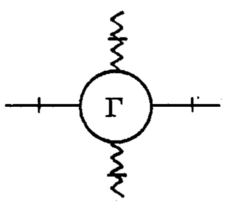
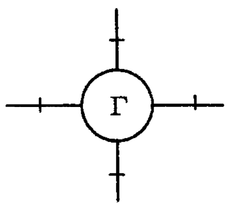
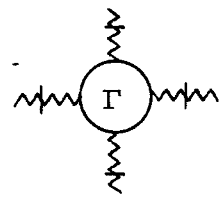
Other 1PI vertex parts = 0. Notice that (A.11) satisfies the Ward identities (IV.7) trivially. For first order (A.12), these identities are easily seen to be correct.

2. The loop-scheme

a. Lowest order

Calculating only those terms which do not involve loop integration, we get,

$\Sigma_{\phi}^{(0)}(p) = ig\,F^2$, $\Sigma_{\sigma}^{(0)}(p) = 3ig\,F^2$,  $= 2ig\,F$,

 $= 2ig$,  $= 6ig$,  $= 6ig$

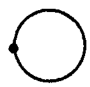
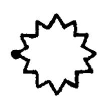
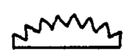
others $= 0$ (A.13)

Note that the lowest value of this scheme is equal to the first order value of the g scheme. They satisfy the Ward identities (IV.7).

b. First order

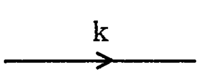
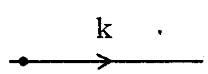
We calculate only those terms which involve only one loop integral. We consider $\Sigma_{\phi}(p)$ as example and show that it is finite to this order.

For first order calculation, we can write

$\Sigma_{\phi}(p) = ig\,F^2 + ig\Big[3$  $+$  $+ 2F$  $(2ig\,F)$

$- \{R\} - \frac{R_4}{ig}\Sigma_{\phi}\Big]$ (A.14)

In view of (A.13), we note,

 $= \frac{1}{k^2-\mu^2+3g\,F^2}$  $= \frac{1}{k^2-\mu^2+g\,F^2}$.

(A.15)

We also recall that R_1 and R_4 have one loop contributions (A.5) and (A.6). Then we have

$$\Sigma_\phi(p) = ig\,F^2 + \frac{ig}{(2\pi)^4} \int dk \left[\frac{3i}{k^2-\mu^2+gF^2} + \frac{i}{k^2-\mu^2+3gF^2} - \frac{4igF^2}{(k^2-\mu^2+3gF^2((p-k)^2-\mu^2+gF^2)} \right.$$

$$\left. - \frac{4i}{k^2-\mu^2} + \frac{10igF^2}{(k^2-\mu^2)^2} \right] . \tag{A.16}$$

Terms in (A.16) can be easily seen to combine to give the finite result,

$$\Sigma_\phi(p) = igF^2 + \frac{(-g)}{(2\pi)^4} \int dk \left[\frac{3gF^2}{(k^2-\mu^2)(k^2-\mu^2+gF^2)} - \frac{3gF^2}{(k^2-\mu^2)^2} \right]$$

$$+ \left[\frac{3gF^2}{(k^2-\mu^2)(k^2-\mu^2+3gF^2)} - \frac{3gF^2}{(k^2-\mu^2)^2} \right]$$

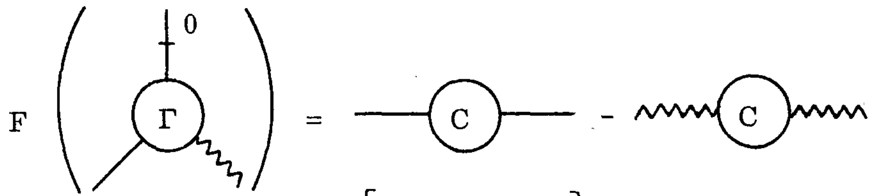
$$- 4gF^2 \left| \frac{1}{(k^2-\mu^2+3gF^2)} \frac{1}{((p-k)^2-\mu^2+gF^2)} - \frac{1}{(k^2-\mu^2)^2} \right] \circ$$

APPENDIX B

Here we prove two lemmas introduced in Section IV.

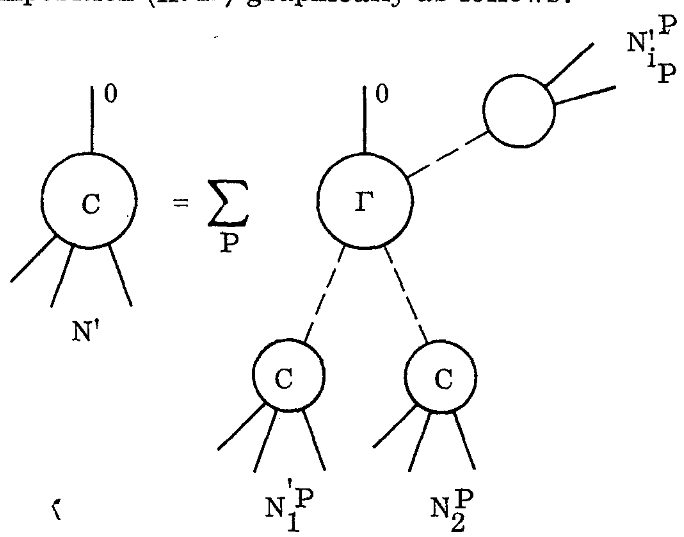
Proof of lemma 1:

We first consider the 3-point function. In this case $\Gamma = c$ and the result



$$= i\left[G_{\phi}(p) - G_{\sigma}(p)\right] \qquad \text{(B.1)}$$

is obtained readily from the definition of the $\sum$'s. With this result, we now consider the general case. For simplicity, let's assume that all external lines are ϕ lines. A more general case can be treated in exactly the same manner. One can write the decomposition (II. 12) graphically as follows:

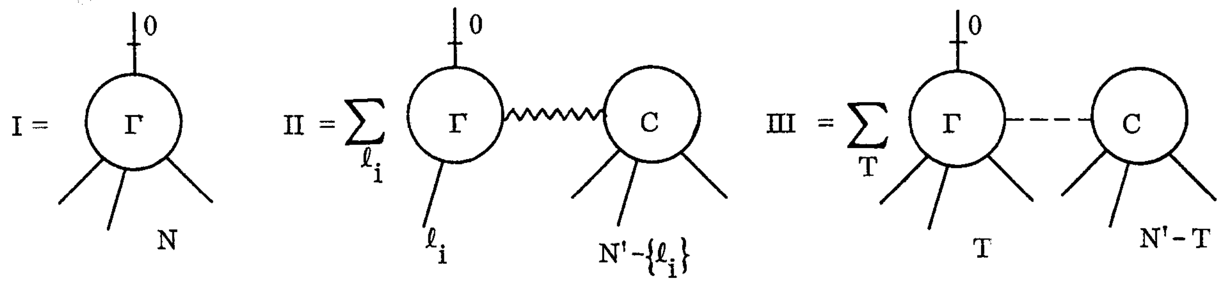


(B. 2)

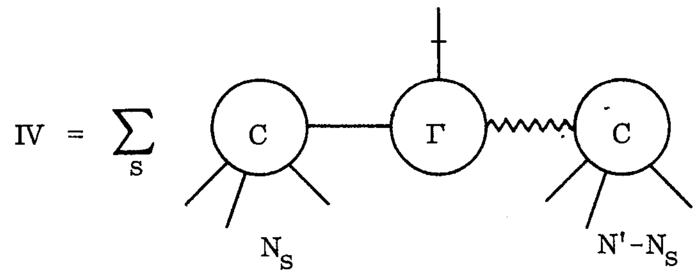
One can classify the terms occurring in (B. 2) as follows:

$$\text{(B. 2)} = \text{I} + \text{II} + \text{III} + \text{IV} + \text{V} + \text{VI}$$

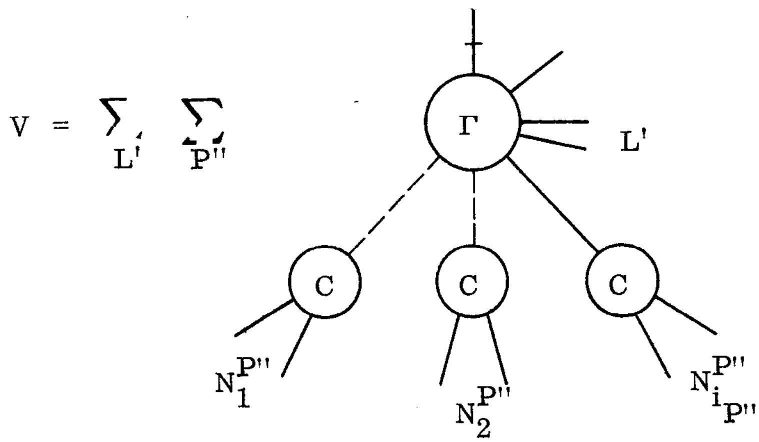
Here



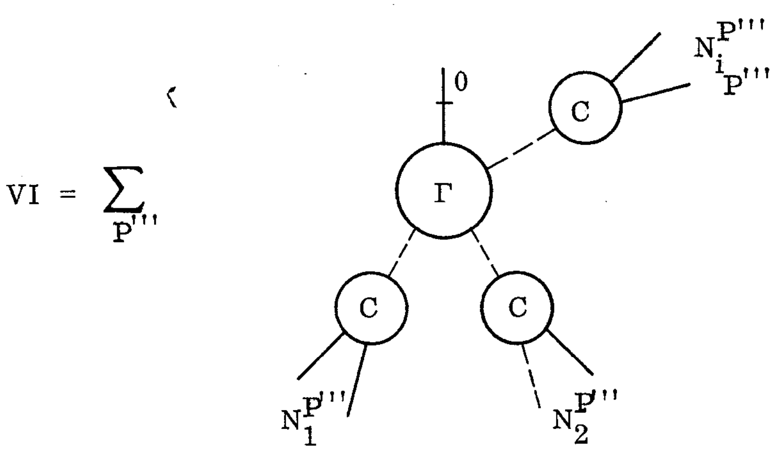
I =
0
Γ
N
II = Σ
ℓi
0
Γ
C
ℓi
N' - {ℓi}
III = Σ
T
0
Γ
C
T
N' - T



IV = Σ
s
C
Γ
C
Ns
N' - Ns



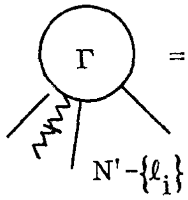
V = Σ Σ
L' P''
Γ
L'
C
C
C
N1P''
N2P''
NiP'' P''



VI = Σ
P'''
0
Γ
C
C
C
NiP''' P'''
N1P'''
N2P'''

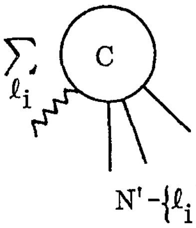
We will denote the number of the elements of a set A by $\omega(A)$. In (B.3) T is the subset of N' with $\omega(T) \geq 2$. S is a cut (Schnitt) which divides N' into N_s and $N'-N_s$. L' is a subset of N' with $1 < \omega(L') < \omega(N')$. P'' decomposes N'-L' into $N_1^{P''}, \ldots, N_{i_{P''}}^{P''}$ with $\omega\left(N_{j_{P''}}^{P''}\right) \geq 2$, $j=1,\ldots,i$ and $i_{P''} \geq 2$. P''' divides N' into $N_1^{P'''}, \ldots, N_{i_{P'''}}^{P'''}$ with $\omega\left(N_{j_{P'''}}^{P'''}\right) \geq 2$, $j=1,\ldots,i$ and $i_{P'''} \geq 3$.

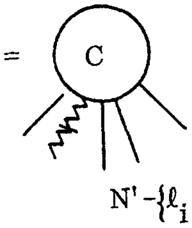
We now apply (IV.7) and (B.1) to each of (B.3) to get,

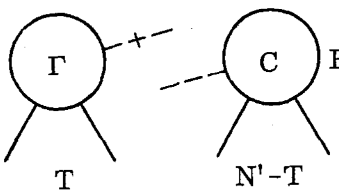
$\mathrm{I} = \sum_i$ 

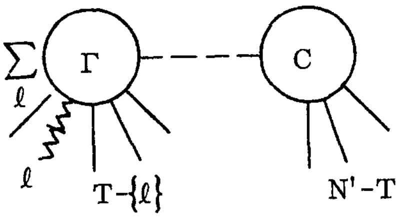
$= A_1$ $\qquad$ $\mathrm{II} = A_2 + B_2$ $\qquad$ $\mathrm{III} = A_3 + B_3$

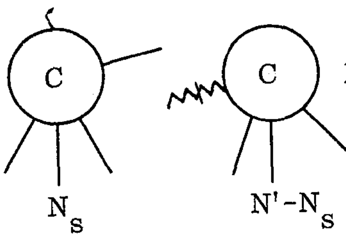
$\mathrm{IV} = A_4 + B_4$ $\qquad$ $\mathrm{V} = A_5 + B_5$ $\qquad$ $\mathrm{VI} = A_6$.

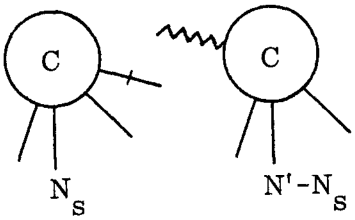
$A_2 = \sum_{\ell_i}$ 

$B_2 =$ 

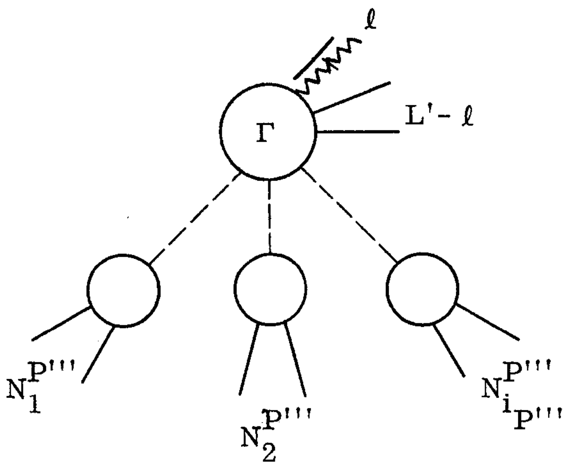
$A_3 = \sum_T (-1)^{\omega(T)}$ 

$B_3 = \sum_T \sum_\ell$ 

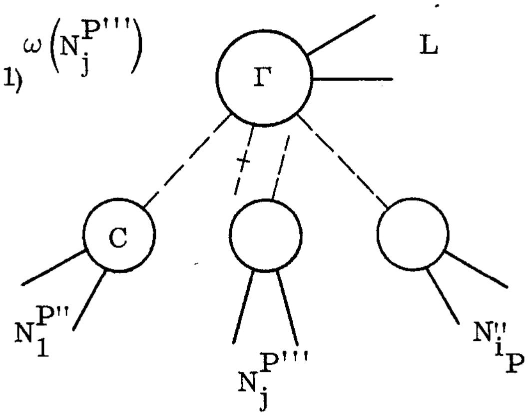
$A_4 = \sum_s (-1)$ 

$B_4 = \sum_s$ 

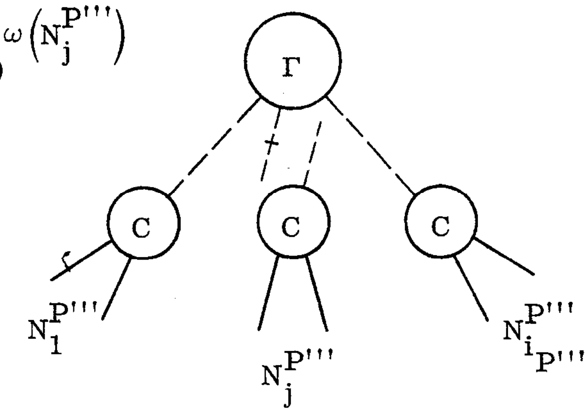
$$A_5 = \sum_{L'} \sum_{\ell} \sum_{P''}$$



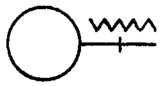
$$B_5 = \sum_{L'} \sum_{j} \sum_{P''} (-1)^{\omega\left(N_j^{P'''}\right)}$$



$$A_6 = \sum_{P'''} \sum_{j} (-1)^{\omega\left(N_j^{P'''}\right)}$$



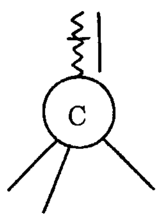
Here  reads "amputate σ propagator and multiply by ϕ propagator."
Similarly for



We first note that A_2 is the desired result, so the rest of the terms should cancel among themselves. Let's see how this cancellation comes about. It is easy to see that

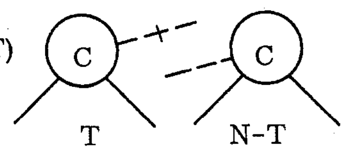
$$A_1 + B_2 + B_3 + A_5 = 0. \tag{B.4}$$

This follows from the decomposition of



into the form (B.2).

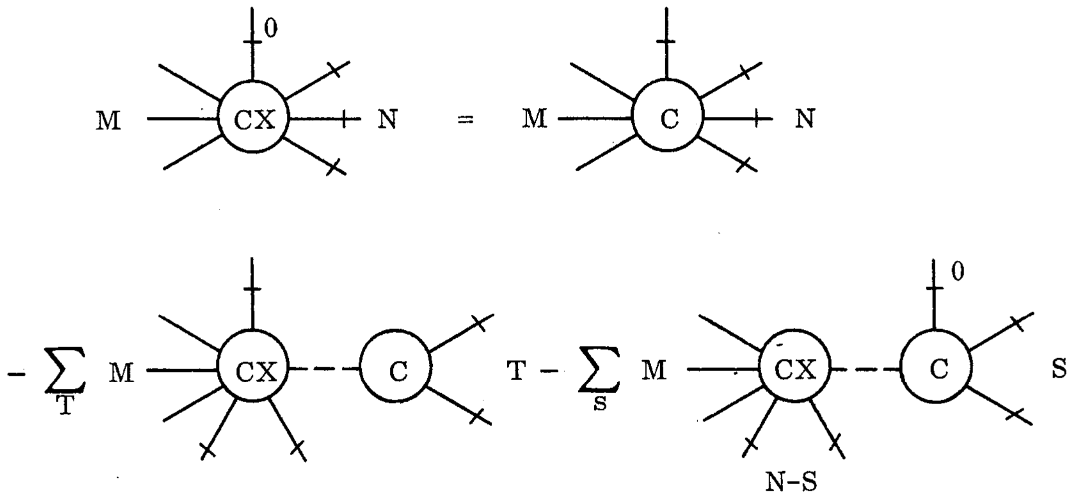
Next consider the rest of the terms. First note that $A_4 + B_4$ can be written as follows:

$$A_4 + B_4 = -\sum_T (-1)^{\omega(T)}$$ 

where T is defined similarly as above. Now decompose the blob external lines T into the form (B.2). Then we get precisely $-A_3-B_5-A_6$. Therefore the proof of the lemma 1 is now complete.

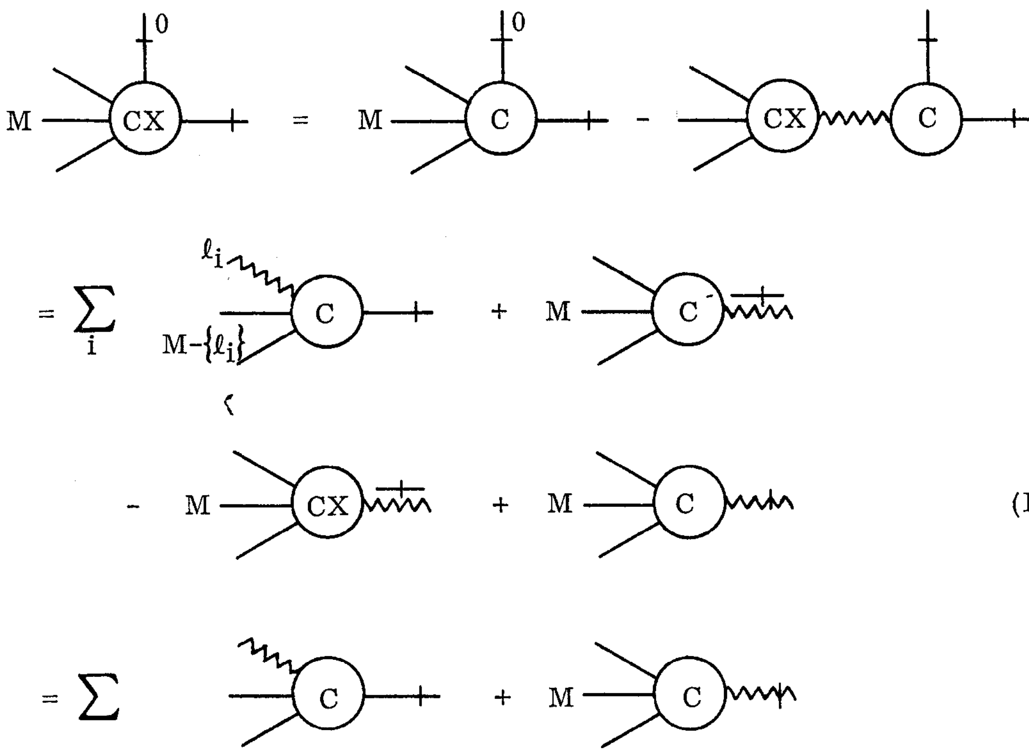
Proof of lemma 2:

Again we consider the special case were all external lines are ϕ lines. According to (II. 15), we have the following decomposition



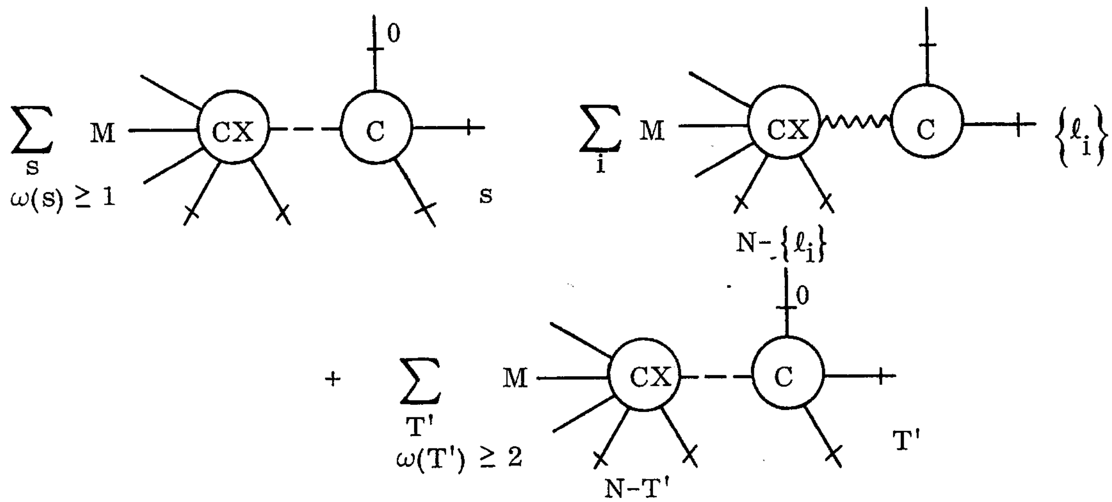
(B.5)

Here T is a subset of N with $\omega(T) \geq 2$ and S with $\omega(S) \geq 1$. We shall prove the lemma by induction on $\omega(N)$. First when $\omega(N)=0$, $C \cdot X = C$ and (B.5) is just (IV.8). Next when $\omega(N)=1$, we have

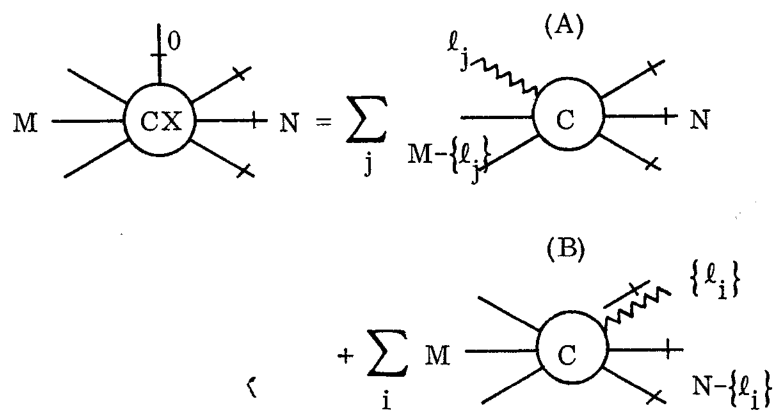


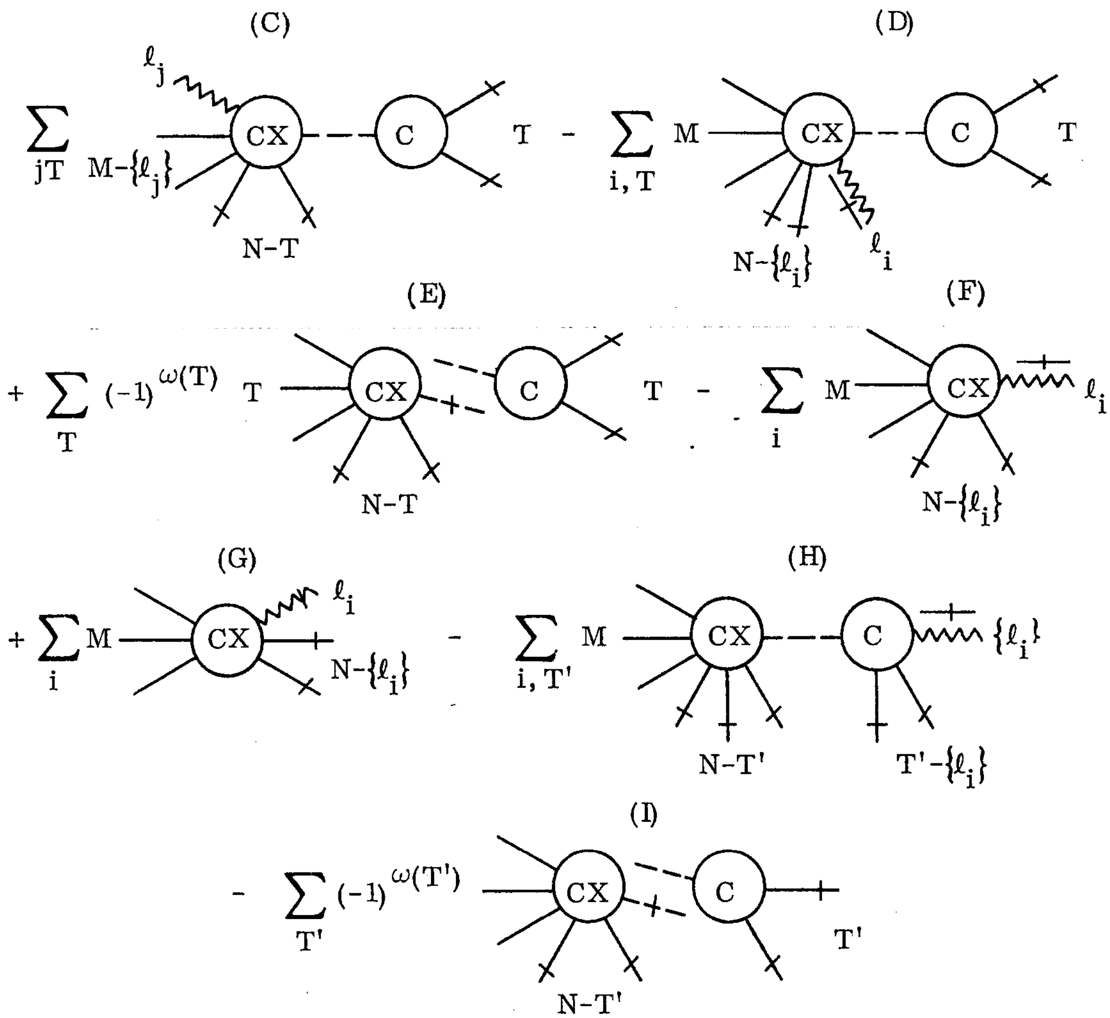
(B.6)

which is of the form (IV.9). Now we assume that (IV.9) is true for all $\omega(N) \leq n-1$. Consider an N such that $\omega(N) = n$. Then (B.5) $\omega(N-T) \leq n-2$ so we can apply (IV.9) to the second term of (B.5). The last term of (B.5) can be further decomposed as



Therefore, using (IV.8) and (IV.9), we have





We see easily see that

$$B + D + F + H = 0 , \qquad E + I = 0$$

and

$$A + C + G$$

is the desired result.

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FOOTNOTES

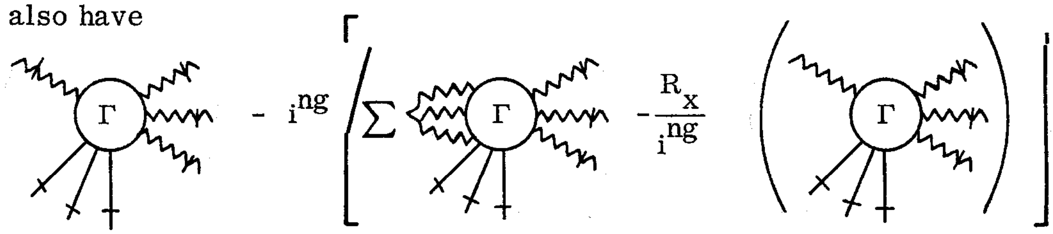
1. Since there are no nucleons in our model, the terms pseudoscalar and scalar are not really meaningful. But we shall stick to these names because the generalization of the model to the more realistic case is straightforward.
2. This terminology is derived from Schwinger. One might prefer to call it the "tadpole" term. But throughout this work, we shall refer a linear term in Lagrangian (and therefore constant term in field equation) a "source term".
3. See, for example, the review article by G. S. Guralnik, C. R. Hajen, T. W. B. Kibble, Ref. 4.
4. Recently the smallness of the symmetry breaking parameter ϵ was questioned by R. A. Brandt and G. Preparata, Ref. 5.
5. B. P. H. refers to Bogoliubov, Parasiuk, and Hepp. See Section II.
6. Their theory will be called B-W theory throughout this work. See Section II.
7. R. A. Brandt (Ref. 13) discussed in detail a model in which a pseudoscalar meson $\phi(x)$ and a fermion $\psi(x)$ are coupled through the interaction Lagrangian $\bar{\psi}(x)\psi(x)\phi(x)$.
8. See also Zimmermann, Ref. 16.
9. ETCR's thus calculated usually involve infinities and operator Schwinger terms. See R. A. Brandt, Ref. 17.
10. This notation was first introduced by Symanzik, Ref. 9.
11. For symmetric model, the 3 point functions actually vanish by the symmetry requirement.
12. These and other points of this paragraph will be made exact in subsection C of this section.

13. This is ϕ line defined by B. W. Lee, Ref. 7.

14. "Weakly connected" means "1PR and connected".

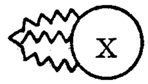
15. This is ϕ' language of B. W. Lee, Ref. 7.

16. We also have



The consistancy of this with (III.10) can be easily established.

17. Equation (III.11) comes from the vacuum expectation value of σ. We write this in a more suggestive form

$$F = \frac{\gamma}{\mu^2} + g \left[\frac{1}{\mu^2} \sum \right.$$

$$\left. + \frac{F^3}{\mu^2} - \frac{R_1 F}{\mu^2} - Rx \frac{F - \frac{\gamma}{\mu^2}}{g} \right]$$

The first term is zeroth order term in g, i.e.,


$$= \left(\frac{i}{-\mu^2}\right) \quad ic = \frac{C}{\mu^2}$$

therefore we have $C=\gamma$.

18. Not considering the g factor.

19. Because of the loop integration, (IV.14) is divergent. But it is obvious that all of the following identities are true before the loop integration is done.

20. See footnote 18.

21. This was first noticed by K. Symanzik, Refs. 9 and 10.

FIGURE CAPTIONS

1. Feynman rules for the symmetric model.
2. Feynman rules for the Σ model (ϕ language).
3. Primitive divergences for the Σ model. ν is degree of primitive divergence.

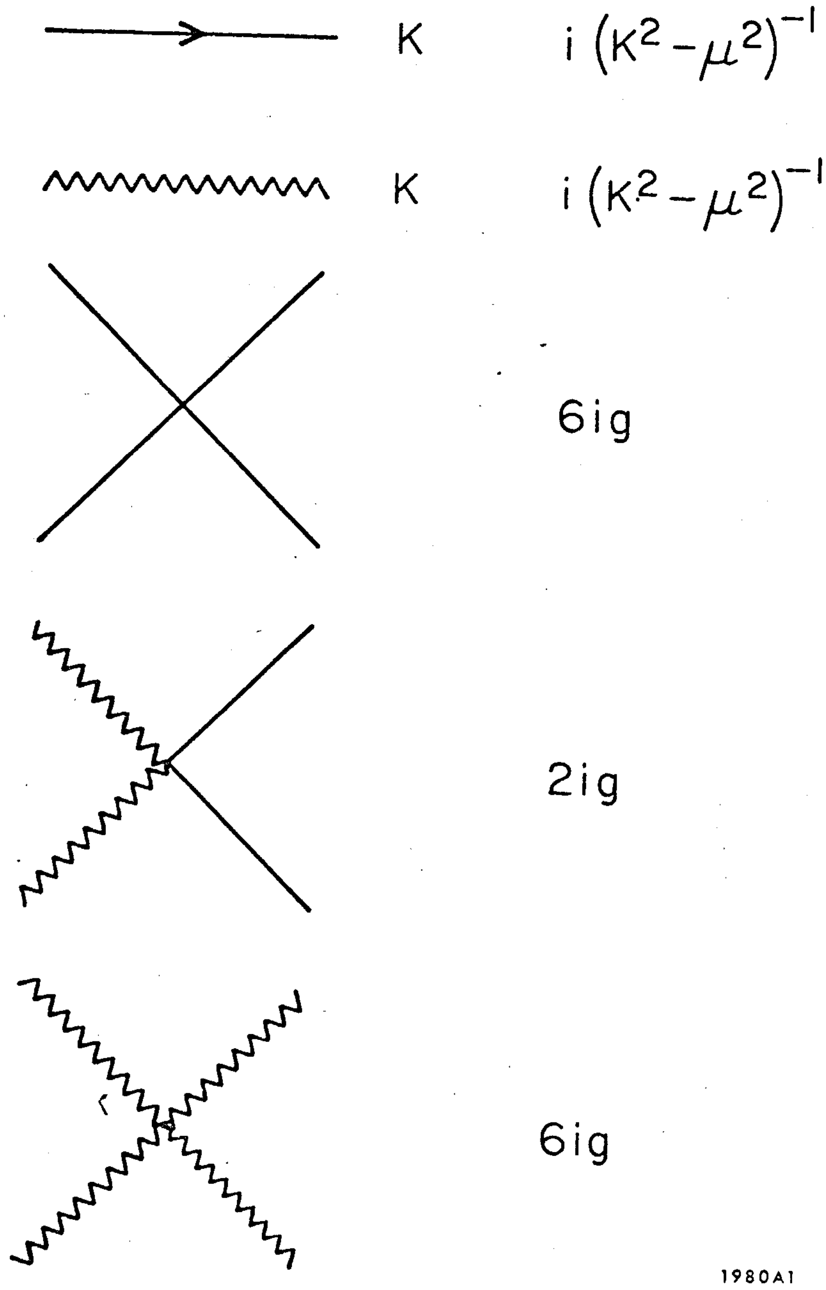


Fig. 1

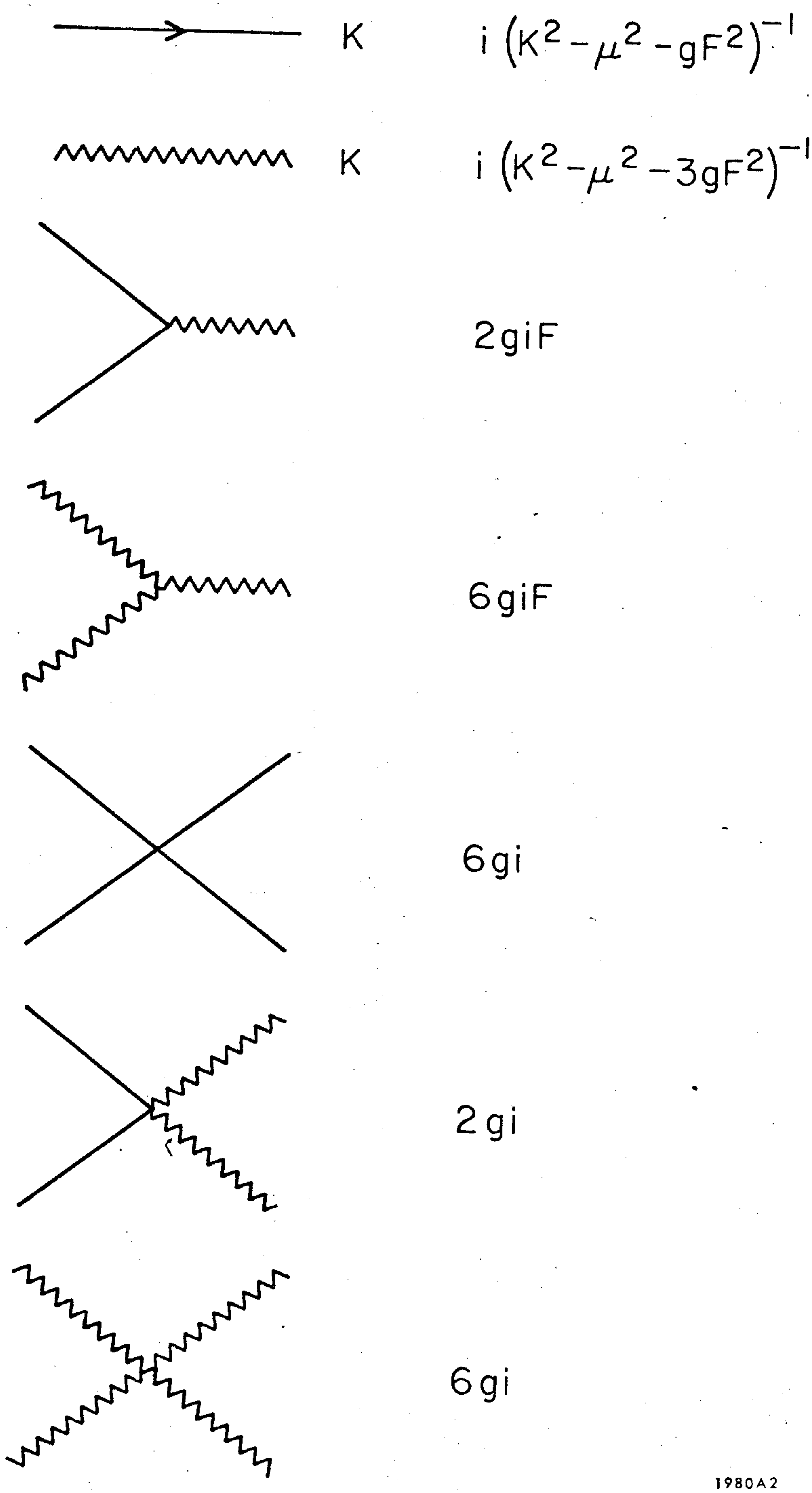

K
$i\left(K^2-\mu^2-gF^2\right)^{-1}$
K
$i\left(K^2-\mu^2-3gF^2\right)^{-1}$
2giF
6giF
6gi
2gi
6gi
1980A2

Fig. 2

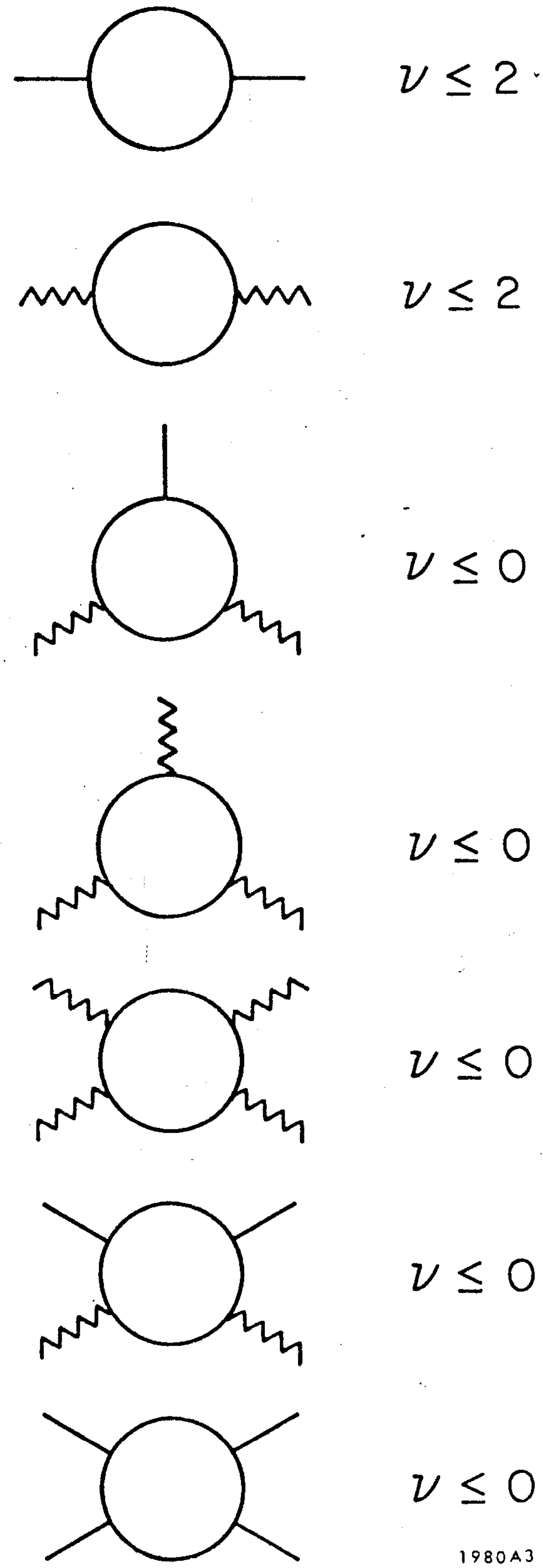


Fig. 3