SLAC-PUB-943
(TH) and (EXP)
August 1971

PROPERTIES OF AMPLITUDES THAT SATURATE THE FROISSART BOUND*

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ABSTRACT

Bounds are obtained on the near-forward scattering amplitudes that saturate the Froissart bound.

(Submitted to Phys. Rev. Comments and Addenda.)

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*Work supported by the U. S. Atomic Energy Commission.

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In recent papers, Eden and Kaiser[1,2] have obtained bounds on the near-forward scattering amplitudes that violate the Pomeranchuk theorem[3] and correspond to asymptotically constant total cross sections. The purpose of the present note is to apply the techniques of Refs. 1 and 2 to the class of amplitudes that saturate the Froissart bound.[3] For our purposes here, we will consider the symmetric scattering amplitude[3,4]

$$F(S, 0) = -C_1(-iS)\left[\ln(-iS)\right]^2 \quad . \tag{1}$$

We use the following assumptions:

(a) Scattering amplitudes have the analyticity and the asumptotic growth properties that have been proved from axiomatic quantum field theory.[5]

(b) Spin effects are neglected.

(c) The forward elastic scattering amplitude is given by Eq. (1).[6]

We list below the properties of the amplitude in Eq. (1). We follow as closely as possible the notation of Refs. 1 and 2.

I. Forward Scattering Amplitude

From Eq. (1), we obtain for large S

$$F(S, 0) \sim \pi C_1 S \ln S + iC_1 S(\ln S)^2 \quad . \tag{2}$$

The total and differential cross sections are

$$\sigma_T(S) \sim C_1(\ln S)^2 \, , \tag{3}$$

$$\left.\frac{d\sigma}{dt}\right|_{t=0} \sim \frac{C_1^2(\ln S)^4}{16\pi}\left[\frac{\pi^2}{(\ln S)^2} + 1\right] \quad . \tag{4}$$

Note that the differential cross section in the forward direction increases as $(\ln S)^4$. Since, $\sigma_{el} \leq \ _T$, the width of the forward peak must decrease at least as fast as $(\ln S)^{-2}$. See Section V below.

II. Bounds on the Elastic Cross Section

Using the fact that the partial-wave series can be truncated after L ($L = C\,S^{1/2}\ \ln S$) terms with negligible error as $S \to \infty$, if $|t| < t_0$, where t_0 is the nearest singularity in t,[7] we obtain the following bound on the elastic cross section,

$$\frac{1}{16\pi}\left(\frac{C_1^2}{C^2}\right)(\ln S)^2 \le \sigma_{el}(S) \le C_1 (\ln S)^2 \ . \qquad (5)$$

III. Upper Bound on the Modulus of the Amplitude

Let

$$|f(S,t)| \equiv \left|\frac{F(S,t)}{F(S,0)}\right| \ . \qquad (6)$$

We obtain the following bound on $f(S,t)$ for $|t| < t_0$,

$$|f(S,t)| \geq C_3^2\ e^{4C|t|^{1/2} \ln S} \ , \qquad (7)$$

where

$$C_3^2 = 16\pi\ \frac{C^2}{C_1}\Big) \ .$$

IV. Zeros of f(S,t)

We find that $f(S,t)$ has no zero when $|t| \le r_0(S)$ and has at least one zero when $|t| \le r_1(S)$,

$$r_0(S)\ (\ln S)^2 = C_4 \ , \qquad (8)$$

$$r_1(S)\ (\ln S)^2 = C_5 \ , \qquad (9)$$

where

$$(C_4)^{-1} = 4e^2 C^3 \left(\frac{\pi}{C_1}\right)^{1/2} \ ,$$

and

$$C_5 = \left(\frac{256\,C}{C_1}\right)^2 .$$

Likewise, we can show that for any finite integer N, there exists a constant D_N, such that there must be N zeroes of $|f(S,t)|$ in the range[8]

$$C_4 \leq t\,(\ln S)^2 \leq D_N . \tag{10}$$

The number of zeroes in the circle $|t| = b < t_0$ has the bound[9]

$$N(S,b) \leq C\,e(C_3')^{1/2}\,b^{1/2}\,\ln S . \tag{11}$$

V. Width of the Forward Peak

Let $\Delta(S)$ denote the width of the forward peak. The following bounds are obtained on $\Delta(S)$,

$$\left(\frac{C_4}{8}\right)\left[2C\,C_4^{1/2} + \ln C_3\right]^{-1} \leq \Delta(S)\,(\ln S)^2 \leq \frac{16\pi}{C_1} . \tag{12}$$

The author wishes to thank Leroy Murray for helpful discussions.

REFERENCES

1. R. J. Eden and G. D. Kaiser, Phys. Rev. D3, 2286 (1971).

2. R. J. Eden and G. D. Kaiser, Nucl. Phys. B28, 253 (1971).

3. See, e.g., R. J. Eden, High Energy Collisions of Elementary Particles (Cambridge University Press, New York, 1967).

4. We could for example consider the more general amplitude $F(S, 0) = F_+(S, 0) + F_-(S, 0)$ where $F_+ = -C_1(-iS)\left[\ln(-iS)\right]^2$ and $F_- = iC_2(-iS)\left[\ln(-iS)\right]^2$. However, none of our results are changed in an essential way by using the amplitude given by Eq. (1).

5. R. J. Eden, Rev. Mod. Phys. 43, 15 (1971).

6. Note that with the amplitude given by Eq. (1), the following results follow:

   (a) $\frac{\mathrm{Re}\,F(S,0)}{\mathrm{Im}\,F(S,0)} \sim \frac{\pi}{\ln S}$;

   (b) The Pomeranchuk theorem holds in the form

$$\lim_{s\to\infty} \frac{\sigma_T(A+B \to A+B)}{\sigma_T(\bar{A}+B \to \bar{A}+B)} = 1 \ .$$

7. A. Martin, Phys. Rev. 129, 1432 (1963).

8. For recent results on the distribution of these zeroes, see, e.g., G. Auberson, T. Kinoshita and A. Martin, Phys. Rev. D3, 3185 (1971).

9. See Ref. 1, Section 2.6.