ERRATA FOR SLAC-PUB-932

Yung-Su Tsai, "Decay Correlations of Heavy Leptons in $e^+ + e^- \rightarrow \ell^+ + \ell^-$," Phys. Rev. D 4, 2821 (1971).

1. A term, $-M_\ell^3(M_\rho^2 - 4M_\pi^2)/\omega_1$, was left out of the square bracket of Eq. (2.20); the complete expression for the square bracket of Eq. (2.20) should read

$$\left[16\, M_\ell^2 \left(\omega_1 - \frac{M_\ell^2 + M_\rho^2}{4M_\ell}\right)^2 + M_\rho^2 \left(1 - \frac{4M_\pi^2}{M_\rho^2}\right)\left(3M_\ell^2 - M_\rho^2\right) - \frac{M_\ell^2}{\omega_1}\left(M_\rho^2 - 4M_\pi^2\right)\right].$$

2. The right-hand side of Eq. (4.11) must be divided by 4; hence 16 in the denominator should be replaced by 64. Similarly, 8 in the denominator of Eq. (4.12) should be replaced by 32.

3. The right-hand side of Eqs. (4.26) and (4.33) must be multiplied by 4. Also, in the sentence following Eq. (4.33), "because C is equal to...." should be replaced by "because 4C is equal to...."

4. The two mistakes of factor 4 shown above cancel each other in the three examples of decay correlations given on pp. 2834 and 2835 if we ignore the corrections. However, after the corrections made in 2 and 3 above we must also multiply by 4 the left-hand side of Eqs. (4.34), (4.35), (4.37), (4.38), (4.41), and (4.42).

The author wishes to thank Dr. John Liu, Dr. T. Sanda, and Dr. S. Y. Tsai for pointing out the mistakes.

SLAC-PUB-932
July 1971
(TH) and (EXP)

DECAY CORRELATIONS OF HEAVY LEPTONS IN $e^+ + e^- \rightarrow \ell^+ + \ell^-$ *

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ABSTRACT

Assuming leptons heavier than muons exist in nature we consider their decay modes and the correlations between the decay products of ℓ^+ and ℓ^- in the colliding beam experiment: $e^+ + e^- \rightarrow \ell^+ + \ell^-$. Far above the threshold, the helicities of ℓ^+ and ℓ^- tend to be opposite to each other and near the threshold the directions of spins of ℓ^+ and ℓ^- like to be parallel to each other and the sum of the two spins likes to be either parallel or antiparallel to the direction of the incident electron. Because the parity conservation is violated maximally in the decays of ℓ^+ and ℓ^-, the angular distributions of decay products depend strongly on the spin orientation of the heavy leptons. Since the spins of ℓ^+ and ℓ^- are strongly correlated in the production we found a strong correlation between the energy-angle distributions of the decay products of ℓ^+ and ℓ^-. The decay widths of ℓ^- into channels $\nu_\ell \bar{\nu}_e e^-$, $\nu_\ell \bar{\nu}_\mu \mu^-$, $\nu_\ell \pi^-$, $\nu_\ell k^-$, $\nu_\ell \rho^-$, $\nu_\ell k^*$, $\nu_\ell A_1$, $\nu_\ell Q$ and ν_ℓ+ hadron continuum as functions of the mass of ℓ^- are estimated.

(Submitted to Phys. Rev. and also to the 5th International Symposium on Electron and Photon Interactions at High Energies, Cornell University, Ithaca, New York, August 23-27, 1971)

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* Work supported by the U. S. Atomic Energy Commission.

I. INTRODUCTION

Since muons exist in nature for no apparent reason, it is possible that other heavy leptons may also exist in nature. If one discovers heavy leptons, one may be able to understand why muons exist and obtain some clue as to why the ratio of the muon mass to the electron mass is roughly $m_\mu/m_e \approx 210$. Searches for these leptons have been attempted in the past[1, 2] and no doubt people will be looking for these particles in the $e^+ + e^-$ colliding beam experiments[2] $(e^+ + e^- \rightarrow \ell^+ + \ell^-)$, photo pair production experiments[3] $(\gamma + z \rightarrow \ell^+ + \ell^- + z^*)$ and neutrino experiments from the electron machine[4] $(\nu_\ell + z \rightarrow \ell^- + z^*)$. We have made extensive calculations for these cross sections. This paper deals mainly with the decay correlations in the reaction, $e^+ + e^- \rightarrow \ell^+ + \ell^-$.

We assume that if heavy lepton exists the leptonic current in the usual current-current effective Lagrangian[5] of the weak interaction is given by

$$J^\lambda_{\text{lept}} = \overline{\nu}_\mu \gamma^\lambda(1-\gamma_5)\, \mu + \overline{\nu}_e \gamma^\lambda(1-\gamma_5)\, e + \overline{\nu}_\ell \gamma^\lambda(1-\gamma_5)\ell$$

and the electromagnetic interaction of the heavy lepton is exactly like that of an electron or a muon. The major difference between the heavy lepton and the muon is that whereas the muon is lighter than any strongly interacting particle, the heavy lepton, if it exists, is expected to be heavier than the k meson; and hence the heavy lepton decays[6] into hadrons in addition to electron and muon.

In the electromagnetic <u>scattering</u> of an electron, it is well known that at high energies $(m/E \rightarrow 0)$ the helicity of the electron remains the same during the scattering, whereas at low energies $(m/E \rightarrow 1)$ the direction of the spin with respect to a fixed coordinate system in space is preserved during the scattering.[7]

In Section IV we show that analogous things happen in the reaction $e^+ + e^- \to \ell^+ + \ell^-$. Far above the threshold ($M_\ell/E \to 0$), the helicities of ℓ^+ and ℓ^- like to be opposite to each other, whereas near the threshold ($M_\ell/E \to 1$) the directions of spins of ℓ^+ and ℓ^- like to be parallel to each other and their total spin likes to be either parallel or antiparallel to the direction of the incident electron. To the lowest order in α, ℓ^+ and ℓ^- are not polarized <u>if only one of them is analyzed</u> and if neither the incident electron nor positron is polarized. This is very similar[8] to the ep scattering where in the lowest order in α the differential cross section is independent of the spin orientation of the target proton if the incident electron is unpolarized. Since ℓ^+ and ℓ^- decay via weak interaction where parity conservation is violated maximally, the angular distribution of decay products depends strongly on the spin orientation of the heavy lepton. Since the spins of ℓ^+ and ℓ^- are strongly correlated in the production, we expect the angular distributions of decay products of ℓ^+ to be strongly correlated to those of ℓ^-. In Section II, we discuss the decay widths and energy-angle distribution of the charged decay products from an arbitrarily polarized ℓ^- into $\nu_\ell + \bar{\nu}_e + e^-$, $\nu_\ell + \bar{\nu}_\mu + \mu^-$, $\nu_\ell + \pi^-$, $\nu_\ell + k^-$ and $\nu_\ell + \rho^-$. The invariance under CP is then used to relate the energy-angle distribution of the decay products of ℓ^+ to that of ℓ^-. In Section III, the hadronic decay width of $\ell^\pm$ is written in terms of an integration over the spectral functions of weak hadronic currents. Weinberg's sum rule is used to evaluate the decay width of $\ell \to \nu_\ell + A_1(1070)$. Das, Mathur and Okubo sum rules are used to evaluate the decay widths of $\ell \to \nu_\ell + k^*(892)$ and $\ell \to \nu_\ell + Q\ (\sim 1300)$. CVC and the result of the $e^+ + e^-$ colliding beam experiment from Frascati are used to evaluate the width of ℓ when its mass is large. If weak vector bosons exist and their mass is less than M_ℓ, ℓ will first decay into $W + \nu_\ell$ semiweakly rather than decay directly into hadrons and leptons. Subsequently W decays into $e\nu$, $\mu\nu$ and hadrons semiweakly. The total hadronic decay width of W is expected[9] to be about the same as its leptonic decay width. In Section IV we first obtain the spin correlation function for the reaction $e^+ + e^- \rightarrow \ell^+ + \ell^-$, then we fold the results of Section II to obtain the correlation of the decay products of ℓ^+ and ℓ^-. In Section V we summarize the general aspects of searching for the existence of $\ell^{\pm}$.

II. DECAY OF POLARIZED $\ell^{\pm}$

In this section we give the energy and angular distribution of the decay products of heavy leptons. We assume the heavy leptons to have an arbitrary polarization denoted by $\vec{w}$ in the rest frame of the heavy lepton. The three components of $\vec{w} = (W_x, W_y, W_z)$ have the usual meaning, for example:

$$W_x = \frac{\text{no. of } \ell \text{ with spin along + x direction - no. of } \ell \text{ with spin along -x direction}}{\text{no. of } \ell \text{ with spin along + x direction + no. of } \ell \text{ with spin along -x direction}} \quad .$$

We assume the existence of a neutrino ν_ℓ (and $\bar{\nu}_\ell$) which has a helicity - (+ for $\bar{\nu}_\ell$) and the same leptonic quantum number as ℓ^- (ℓ^+ for $\bar{\nu}_\ell$) in exact analogy with the properties of $\mu^{\pm}$, ν_μ and $\bar{\nu}_\mu$.

A. Leptonic Decay Modes

Similar to the decay $\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu$, heavy leptons decay leptonically via

$$\ell^- \rightarrow \mu^- + \nu_\ell + \bar{\nu}_\mu \qquad \text{and}$$

$$\ell^- \rightarrow e^- + \nu_\ell + \bar{\nu}_e \quad .$$

For antileptons we have

$$\ell^+ \rightarrow \bar{\nu}_\ell + \nu_\mu + \mu^+ \qquad \text{and}$$

$$\ell^+ \rightarrow \bar{\nu}_\ell + \nu_e + e^+ \quad .$$

As is well known from the muon decay, the energy and angular distribution of the electron from an arbitrarily polarized heavy lepton can be written in the rest frame of ℓ as[10]

$$\Gamma\begin{pmatrix}\ell^-\to\nu_\ell+\bar{\nu}_e+e^- \\ \ell^+\to\bar{\nu}_\ell+\nu_e+e^+\end{pmatrix}=\frac{G^2M_\ell^5}{3\times 2^7\pi^4}\int d\Omega_e\int_0^1 dx\, x^2\left[3-2x\mp(\vec{w}\cdot\hat{p}_e)(2x-1)\right] \tag{2.1}$$

where $G=1.02\times10^{-5}/M_p^2$, x is E/E_{max} of the electron with $E_{max}=M_\ell/2$, $\vec{w}$ is the polarization vector of the heavy lepton and $\hat{p}_e$ is the unit vector along the direction of the electron. We have ignored the mass of the electron in Eq. (2.1). The polarization dependent term $\vec{w}\cdot\hat{p}_e$ is due to the parity nonconservation. The relative magnitude of the parity violating term is maximum when the electron is at the maximum possible energy $(x = 1)$. Near $x = 1$, an electron likes to be emitted opposite to the direction of the spin of ℓ^-, whereas the positron likes to be emitted parallel to the direction of the spin of ℓ^+. Near the lower end of the energy spectrum $(x\to0)$ exactly the opposite holds. The easiest way to understand these qualitative features is to draw some pictures. Figure 1 shows why the decay of ℓ^+ can be obtained from that of ℓ^- by changing the sign of the polarization vector in Eq. (2.1). Since we have ignored the mass of the electron, e^-, ν_ℓ and ν_e have negative helicities, and e^+, $\bar{\nu}_\ell$ and $\bar{\nu}_e$ have positive helicities. Let us first consider a charge conjugate state, shown in Fig. 1b, of an arbitrary angular distribution of the ℓ^- decay shown in Fig. 1a. Figure 1b is unrealizable physically because e^+, ν_ℓ and $\bar{\nu}_\ell$ have wrong helicities. The mirror image of Fig. 1b shown in Fig. 1c, is physically realizable because the spins of all particles in Fig. 1b are flipped by taking the mirror image.

Since (c) is obtainable from (a) by the combined operation of CP, the probability of (a) must be equal to the probability of (c) if the decay is invariant under CP. This explains why the decay of ℓ^+ can be obtained from that of ℓ^- by changing the sign of the polarization vector in Eq. (2.1). In order to understand which sign belongs to which decay, we consider the case $x = 1$ as shown in Fig. 2. At $x = 1$ kinematics require that ν and $\bar{\nu}$ are both emitted in the opposite direction to the direction of the electron. Since the component of the orbital angular momentum is zero along the direction the electron (z axis) and the z components of two neutrino spins add up to zero, the z component of the spin of the electron must be parallel to the spin of the heavy lepton. Since the electron has a negative helicity and the positron has a positive helicity, the positron likes to be emitted along the spin of ℓ^+ whereas the electron likes to be emitted opposite to the spin of ℓ^- when $x \sim 1$. When $x \to 0$, the kinematics require that ν and $\bar{\nu}$ come out in the opposite direction to each other, hence their net spin is equal to unity and is pointing toward the direction of $\bar{\nu}$. In order to conserve angular momentum, e^- has to move in the direction of $\bar{\nu}$ and the spin of ℓ^- has to point in the direction of $\bar{\nu}$. Hence, near $x = 0$, e^- likes to come out along the direction of the spin of ℓ^- which is exactly the opposite to the case for $x = 1$.

Integrating Eq. (2.1) with respect to the solid angle and x, we obtain

$$\Gamma(\ell^- \to \nu_\ell + \bar{\nu}_e + e^-) = \Gamma(\ell^+ \to \bar{\nu}_\ell + \nu_e + e^+) = \frac{G^2 M_\ell^5}{3 \times 2^6 \pi^3} \quad . \tag{2.2}$$

It is convenient to write

$$\Gamma(\ell^- \to \nu_\ell + \bar{\nu}_\ell + e^-)/\hbar = 2.66 \times 10^{10} \text{ sec}^{-1} M_\ell^5/M_p^4 , \tag{2.3}$$

where M_ℓ and M_p are in units of GeV.

When the mass of the muon is not ignored, the energy-angle distribution of the muon can be written as

$$\Gamma\begin{pmatrix}\ell^- \to \nu_\ell + \bar{\nu}_\mu + \mu^- \\ \ell^+ \to \bar{\nu}_\ell + \nu_\mu + \mu^+\end{pmatrix} = \frac{G^2 M_\ell^5}{3\times 2^7 \pi^4}\, \frac{8}{M_\ell^4} \int_0^{p_{max}} p^2 dp \int d\Omega \tag{2.1'}$$

$$\left[3M_\ell - 4E - 2M_\mu^2/E + 3M_\mu^2/M_\ell \mp (\vec{w}\cdot\hat{p})\,\frac{p}{E}\,(4E - M_\ell - 3M_\mu^2/M_\ell)\right] ,$$

where p is the momentum of the muon, $E = (p^2 + M_\mu^2)^{1/2}$ and $p_{max} = (M_\ell^2 - M_\mu^2)/(2M_\ell)$. After carrying out the integration with respect to p and Ω we obtain

$$\Gamma\begin{pmatrix}\ell^- \to \nu_\ell + \bar{\nu}_\mu + \mu^- \\ \ell^+ \to \bar{\nu}_e + \nu_\mu + \mu^+\end{pmatrix} = \frac{G^2 M_\ell^5}{3\times 2^6 \pi^3}\left[1 - 8y + 8y^3 - y^4 - 12y^2 \ln y\right] , \tag{2.2'}$$

where $y = M_\mu^2/M_\ell^2$. As pointed out by Thacker and Sakurai,[6] the correction due to the finite muon mass can amount to 25% if $M_\ell = 0.6$ GeV.

B. $\ell^- \to \nu_\ell + \pi^-$ and $\ell^+ \to \bar{\nu}_\ell + \pi^+$

This decay mode can be calculated from the knowledge of $\pi^- \to \bar{\nu}_\mu + \mu^-$ and $\pi^+ \to \nu_\mu + \mu^+$. The easiest way to see the connection between the two reactions is to assume the existence of weak vector bosons and write two Feynman diagrams, as shown in Fig. 3. $e\mu\ell$ universality implies that W couplings to $\mu\nu_\mu$, $e\nu_e$ and $\ell\nu_\ell$ have the same strength g given by $g^2/M_W^2 = G/\sqrt{2}$. The coupling constant between W and π is given by $g f_\pi \cos\theta_c$ where $\theta_c \sim 15^\circ$ is the Cabibbo angle. From the Feynman diagram the decay width for $\pi \to \mu + \nu$ is given by

$$\Gamma_{\pi\to\mu+\nu} = \frac{G^2 f_\pi^2 \cos^2\theta_c}{8\pi}\, M_\pi M_\mu^2\,(1 - M_\mu^2/M_\pi^2)^2 .$$

Hence from the experimental lifetime of $\tau = \hbar/\Gamma = 2.6 \times 10^{-8}$ sec we obtain

$$f_\pi = 0.137\ M_p \quad . \tag{2.4}$$

The angular distribution of $\pi^\pm$ from the decay of a polarized $\ell^\pm$ can be computed from the Feynman diagram. We have

$$\Gamma\begin{pmatrix} \ell^- \rightarrow \nu_\ell + \pi^- \\ \ell^+ \rightarrow \bar{\nu}_\ell + \pi^+ \end{pmatrix} = \frac{G^2 f_\pi^2 \cos^2\theta_c}{16\pi} \quad M_\ell^3 \left(1 - \frac{M_\pi^2}{M_\ell^2}\right)^2 \int (1 \pm \vec{w}\cdot\hat{p}_\pi) \ \frac{d\Omega}{4\pi} \tag{2.5}$$

where $\hat{p}_\pi$ is the unit vector in the direction of motion of the pion. Again the invariance under CP says that the decay angular distribution of $\ell^+ \rightarrow \bar{\nu}_\ell + \pi^+$ can be obtained from that of $\ell^- \rightarrow \nu_\ell + \pi^-$ by changing the sign of $\vec{w}$. (Proof similar to Fig. 1.) Comparing Eq. (2.5) with Eq. (2.1) we notice that the x integration is missing from the latter because in the two-body decay the energy of each particle is fixed in the rest frame of ℓ: $E_\pi = (M_\ell^2 + M_\pi^2)/(2M_\ell)$ and $E_\nu = (M_\ell^2 - M_\pi^2)/(2M_\ell)$. We also notice that the sign in front of $\vec{w}\cdot\hat{p}_\pi$ is opposite to that of $\vec{w}\cdot\hat{p}_\ell$ when $x \sim 1$. The $\pm$ sign in Eq. (2.5) can be understood easily if we draw pictures similar to Fig. 2. Consider $\ell^- \rightarrow \nu_\ell + \pi^-$. Since ν_ℓ and π^- come out back to back, the component of the orbital angular momentum along the direction of ν_ℓ is zero. Now ν_ℓ has helicity $-$ hence, it likes to be emitted opposite to the direction of the spin of ℓ^-. Therefore, π^- likes to be emitted in the direction of the spin of ℓ^-.

Integrating Eq. (2.5) with respect to the solid angle, the spin dependent part vanishes. From Eq. (2.2) and (2.5), we obtain the ratio

$$\frac{2\Gamma_{\ell^- \rightarrow \nu_\ell + \bar{\nu}_e + e^-}}{\Gamma_{\ell^- \rightarrow \pi^- + \nu}} \approx \frac{M_\ell^2}{6\pi^2 f_\pi^2 \cos^2\theta_c} \approx \frac{M_\ell^2}{M_p^2}\ \frac{1}{1.04} \quad . \tag{2.6}$$

This equation shows that when the lepton mass is equal to the proton mass, the width for the pionic decay mode is roughly equal to the sum of the widths of the electronic and muonic decay modes of ℓ. If $M_\ell < M_p$, then the pionic decay mode is more important than the total leptonic decay mode (e plus μ).

C. $\underline{\ell^- \to k^- + \nu_\ell}$

We can calculate this decay rate from the known rate of $k^- \to \mu^- + \bar{\nu}_\mu$, or equivalently we may obtain this by simply replacing $\cos^2\theta_c$ and M_π in Eq. (2.6) by $\sin^2\theta_c$ and M_k respectively. f_π is equal to f_k because this is how Cabibbo[11] obtained θ_c from comparison of the decay rates of $\pi \to \mu + \nu$ and $k \to \mu + \nu$. Hence we obtain

$$\Gamma_{\ell \to k+\nu} = \Gamma_{\ell \to \pi+\nu} \tan^2\theta_c \left(1 - \frac{M_k^2}{M_\ell^2}\right)^2 \Bigg/ \left(1 - \frac{M_\pi^2}{M_\ell^2}\right)^2 , \tag{2.7}$$

where $\tan^2\theta_c \approx \frac{1}{13.7}$.

D. $\underline{\ell^- \to \rho^- + \nu_\ell}$

This decay mode can be calculated from the cross section of $e^+ + e^- \to \rho$ using CVC. CVC is equivalent to the statement that the coupling of W to ρ is obtainable from the $\gamma\rho$ coupling by replacing e in the latter by $\sqrt{2}\,g \cos\theta_c$, where

$$g^2/M_W^2 = G/\sqrt{2} .$$

The width for this decay can be calculated easily and we obtain (neglecting the ρ width)

$$\Gamma(\ell^- \to \rho^- + \nu) = \frac{g_{\rho\ell\nu}^2}{8\pi} \frac{M_\ell^3}{M_\rho^2} \left(1 - \frac{M_\rho^2}{M_\ell^2}\right)^2 \left(1 + \frac{2M_\rho^2}{M_\ell^2}\right) , \tag{2.8}$$

where

$$g_{\rho\ell\nu} = \frac{\sqrt{2}\,g^2 \cos\theta_c}{M_W^2} \quad g_{\rho\gamma} M_\rho^2 \equiv \frac{G}{\sqrt{2}} \cos\theta_c\, f_\rho \tag{2.9}$$

and

$$g_{\rho\gamma}^2 \approx 1/g_{\rho\pi\pi}^2 \approx \frac{1}{8\pi} . \tag{2.10}$$

Substituting Eqs. (2.8) and (2.9) into Eq. (2.8), we obtain

$$(\ell^- \not\to \rho^- + \nu_\ell) = \frac{G^2 M_\ell^3}{2^6 \pi^2} \cos^2\theta_c \; M_\rho^2 \left(1 - \frac{M_\rho^2}{M_e^2}\right)^2 \left(1 + \frac{2M_\rho^2}{M_e^2}\right) . \qquad (2.11)$$

In terms of the leptonic decay width we have

$$\Gamma_{\ell \to \rho + \nu} = \cos^2\theta_c \, \Gamma(\ell \to e + \nu + \bar{\nu}) \; 3\pi \frac{M_\rho^2}{M_\ell^2} \left(1 - \frac{M_\rho^2}{M_\ell^2}\right)^2 \left(1 + \frac{2M_\rho^2}{M_\ell^2}\right) . \qquad (2.12)$$

We next consider the energy-angle distributions of $\pi^\pm$ from the decays of polarized $\ell^\pm$ via

$$\ell^- \to \nu_\ell + \rho^- \qquad \rho^- \to \pi^0 + \pi^-$$

and

$$\ell^+ \to \bar{\nu}_\ell + \rho^+ \qquad \rho^+ \to \pi^0 + \pi^+ \quad .$$

Two decays are related to each other by CP invariance, hence we give the derivation for the ℓ^- decay. The width can be calculated from

$$\Gamma\left(\ell^- \to \nu_\ell + \rho^- , \; \rho^- \to \pi^- + \pi^0\right)$$

$$= \frac{1}{2M_\ell} \frac{1}{(2\pi)^5} \int \frac{d^3p_2}{2E_2} \int \frac{d^3q_1}{2\omega_1} \int \frac{d^3q_2}{2\omega_2} \; \delta^4(p_1 - p_2 - q_1 - q_2) \, |M|^2 \qquad (2.13)$$

where p_1, p_2, q_1 and q_2 are four momenta of ℓ^-, ν_ℓ, π^- and π^0 respectively; E_2, ω_1 and ω_2 are energies of ν_ℓ, π^- and π^0 respectively; and $|M|^2$ is the matrix element squared.

$$|M|^2 = g_{\ell\rho\nu}^2 \, g_{\rho\pi\pi}^2 \; \mathrm{tr}\left\{ \frac{1+\gamma_5 \not{W}_-}{2} (\not{p}_1 + M_\ell)(1+\gamma_5)\not{Q}\not{p}_2\not{Q}(1-\gamma_5)\right\} \times \left| \frac{1}{(q_1+q_2)^2 - M_\rho^2 + i\Gamma_\rho M_\rho} \right|^2 , \qquad (2.14)$$

where $g_{\rho\ell\nu}$ is given by Eq. (2.9), $g_{\rho\pi\pi}$ is given by Eq. (2.10), $Q = q_1 - q_2$, w_- is the four vector which reduces to the three dimensional polarization vector $\vec{w}$ in the rest frame of ℓ^-, and

$$\Gamma_\rho = \frac{g^2_{\rho\pi\pi}}{48\pi} M_\rho \left(1 - \frac{4M_\pi^2}{M_\rho^2}\right)^{3/2} . \tag{2.15}$$

For simplicity, let us make a narrow width approximation, i.e., we replace the Breit Wigner factor by a δ function:

$$\left| \frac{1}{(q_1+q_2)^2 - M_\rho^2 + i\Gamma_\rho M_\rho} \right|^2 \longrightarrow \frac{\pi}{\Gamma_\rho M_\rho} \delta\left((q_1+q_2)^2 - M_\rho^2\right) \tag{2.16}$$

After taking the trace we have

$$|M|^2 = \frac{4\pi g^2_{\rho\ell\nu} g^2_{\rho\pi}}{M_\rho \rho} \left[2(p_1\cdot Q)(p_2\cdot Q) - (p_1\cdot p_2)Q^2 - M_\ell \left\{ 2(w_-\cdot Q)(p_2\cdot Q) - (w_-\cdot p_2)Q^2 \right\} \right]$$
$$\delta\left((q_1+q_2)^2 - M_\rho^2\right). \tag{2.17}$$

In the rest frame of ℓ^-, we have

$$p_2\cdot Q = p_1\cdot Q = M_\ell(\omega_1 - \omega_2),$$
$$Q^2 = 4M_\pi^2 - M_\rho^2 ,$$
$$p_1\cdot p_2 = M_\ell(M_\ell - \omega_1 - \omega_2),$$
$$w_-\cdot p_2 = \vec{w}\cdot\vec{q}_1 + \vec{w}\cdot\vec{q}_2,$$

and $$w_-\cdot Q = -(\vec{w}\cdot\vec{q}_1 - \vec{w}\cdot\vec{q}_2) .$$

Since we are interested in the energy and angle distribution of q_1, we have to integrate Eq. (2.13) with respect to p_2 and q_2. We first integrate with respect to d^3p_2 with the help of the $\delta^4(p_1-p_2-q_1-q_2)$ and obtain

$$\int \frac{d^3p_2}{2E_2} \delta^4(p_1-p_2-q_1-q_2) = \delta\left((p_1-q_1-q_2)^2\right) = \delta\left(M_\ell^2 + M_\rho^2 - 2M_\ell(\omega_1+\omega_2)\right) . \tag{2.18}$$

This δ function says that ω_2 is fixed if ω_1 is fixed,and the δ function in Eq. (2.16) says that the angle between $\vec{q}_1$ and $\vec{q}_2$ are fixed. Therefore, choosing the direction of $\vec{q}_1$ as the z axis and letting $\vec{w}$ lie on the z x plane, the three dimensional integration d^3q_2 reduces to the integration with respect to the azimuthal angle ϕ. The only term in the integrand which depends upon ϕ is $\vec{w}\cdot\vec{q}_2$ and

$$\int \frac{d^3q_2}{2\omega_2}\ \vec{w}\cdot\vec{q}_2\ \delta\left((q_1+q_2)^2-M_\rho^2\right)\ \delta\left(M_\ell^2+M_\rho^2-2M_\ell(\omega_1+\omega_2)\right) = \frac{\pi}{4M_\ell q_1^3}\ (\vec{w}\cdot\vec{q}_1)(\vec{q}_1\cdot\vec{q}_2)\ . \tag{2.19}$$

After some manipulation, we obtain the energy-angle distribution of $\pi^{\pm}$ from a polarized $\ell^{\pm}$ via $\ell^{\pm}\to\nu_\ell+\rho^{\pm}$:

$$\Gamma\begin{pmatrix}\ell^-\to\nu_\ell+\rho^-\to\nu_\ell+\pi^-+\pi^0\\ \ell^+\to\bar{\nu}_\ell+\rho^+\to\bar{\nu}_\ell+\pi^++\pi^0\end{pmatrix} = \frac{3g^2_{\rho\ell\nu}}{16\pi^2M_\rho^2\left(1-\dfrac{4M_\pi^2}{M_\rho^2}\right)^{3/2}}\int_{\omega_{1\,min}}^{\omega_{1\,max}} d\omega_1\int d\Omega_1$$

$$\left[16\,M_\ell^2\left(\omega_1-\frac{M_\ell^2+M_\rho^2}{4M_\ell}\right)^2+M_\rho^2\left(1-\frac{4M_\pi^2}{M_\rho^2}\right)\left(M_\ell^2-M_\rho^2\right)\right.$$

$$\left.\pm\frac{(\vec{w}\cdot\vec{q}_1)\,\omega_1}{q_1^2}\left\{16\,M_\ell^2\left(\omega_1-\frac{M_\ell^2+M_\rho^2}{4M_\ell}\right)^2+M_\rho^2\left(1-\frac{4M_\pi^2}{M_\rho^2}\right)(3\,M_\ell^2-M_\rho^2)\right\}\right],$$

where

$$\omega_{1\,\substack{max\\min}} = \left[M_\ell^2+M_\rho^2\pm(M_\ell^2-M_\rho^2)(1-4M_\pi^2\ M_\rho^2)^{1/2}\right]\Big/(4M_\ell)\ . \tag{2.20}$$

Comparing Eq. (2.20) with Eq. (2.5), we see that the spin dependent term in two cases have the same sign, namely π^- likes to be emitted along the direction of polarization of ℓ^-, whereas π^+ likes to be emitted opposite to the direction of polarization of ℓ^+. Since the terms inside the curly bracket are positive definite, this is true independent of the energy ω_1. Equation (2.20) reduces to Eq. (2.8) after integrations with respect to energy and angle as it should.

III. WIDTH OF ℓ AND SPECTRAL FUNCTIONS OF CURRENTS

In the previous section we considered in detail the energy-angle distributions of simple decay products from a polarized $\ell^\pm$. If the mass of $\ell^\pm$ is less than 1 GeV, the consideration given so far is sufficient (except $\ell \to \nu + k^*$ (890) to be considered in this section). When the mass of $\ell^\pm$ is greater than 1 GeV, $\ell^\pm$ decays into $\nu + A_1$ (1070), $\nu + Q$ (1300) and ν + hadron continuum in addition to simple discrete states considered in the previous section. If weak vector bosons ($W^\pm$) exist and if $M_\ell > M_W$, then $\ell^\pm$ will first decay into $\nu + W^\pm$ semi-weakly rather than decay directly (weakly) into leptons and hadrons. In this section we consider the width of $\ell^\pm$ from a more systematic point of view which enables us to deal with these new problems and put the special cases discussed in the previous section in better perspective.

The width of $\ell \to$ hadrons + ν_ℓ can be written as

$$\Gamma_{\ell^- \to \text{hadrons}+\nu_\ell} = \frac{1}{2M_\ell} \frac{G^2}{2} \int \frac{d^3p_2}{2E_2} \frac{1}{(2\pi)^3} \frac{1}{2} \mathrm{tr}(\not{p}_1 + M_\ell)(1+\gamma_5)\gamma_\mu \not{p}_2 (1-\gamma_5)$$

$$\sum_f \langle 0 | J_h^\mu(0) | f \rangle \langle f | J_h^{\nu\dagger}(0) | 0 \rangle (2\pi)^4 \delta^4(p_1 - p_2 - p_f) \tag{3.1}$$

where J^μ_h is the Cabibbo current[11,5]:

$$J^\mu_h = \left[\left(F^\mu_1 + i F^\mu_2\right) - \left(F^{5\mu}_1 + i F^{5\mu}_2\right)\right] \cos \theta_c + \left[\left(F^\mu_4 + i F^\mu_5\right) - \left(F^{5\mu}_4 + i F^{5\lambda}_5\right)\right] \sin \theta_c \ . \tag{3.2}$$

The four types of currents, $F^\mu_1 + i F^\mu_2$, $F^{5\mu}_1 + i F^{5\mu}_2$, $F^\mu_4 + i F^\mu_5$ and $F^{5\mu}_4 + i F^5_5$, do not interfere with each other because the final states associated with each current have different quantum numbers,as shown in Table I.

TABLE I

	S	Q	G	J^P	I_3	Examples
$F^\mu_1 + i F^\mu_2$	0	-1	+	1^-	-1	$\rho^-, 2\pi, 4\pi, k^- + k_0$
$F^{5\mu}_1 + F^{5\mu}_2$	0	-1	-	$0^-, 1^+$	-1	$\pi^-, 3\pi, A^-_1$
$F^\mu_4 + i F^\mu_5$	-1	-1		$0^+, 1^-$	$-\frac{1}{2}$	k* (892)
$F^{5\mu}_4 + i F^{5\mu}_5$	-1	-1		$0^-, 1^+$	$-\frac{1}{2}$	k^-, Q^- (1300)

Since $F^\mu_1 + i F^\mu_2$ is conserved (CVC), the final state $|f>$ cannot be a $J = 0$ state with nonzero mass for this current.

Let us define the spectral functions:

$$\sum_f <0\left|\begin{pmatrix} F_1^{\mu}(0)+iF_2^{\mu}(0) \\ F_1^{5\mu}(0)+iF_2^{5\mu}(0) \\ F_4^{\mu}(0)+iF_5^{\mu}(0) \\ F_4^{5\mu}(0)+iF_5^{5\mu}(0) \end{pmatrix}\right|f><f\left|\begin{pmatrix} F_1^{\nu}(0)-iF_2^{\nu}(0) \\ F_1^{5\nu}(0)-iF_2^{5\nu}(0) \\ F_4^{\nu}(0)-iF_5^{\nu}(0) \\ F_4^{5\nu}(0)-iF_5^{5\nu}(0) \end{pmatrix}\right|0>(2\pi)^4\,\delta^4(q-p_f)$$

$$= (q^{\mu}q^{\nu}-q^2g^{\mu\nu})\begin{pmatrix} v_1(q^2) \\ a_1(q^2) \\ v_1^s(q^2) \\ a_1^s(q^2) \end{pmatrix} + q^{\mu}q^{\nu}\begin{pmatrix} 0 \\ a_0(q^2) \\ v_0^s(q^2) \\ a_0^s(q^2) \end{pmatrix} . \tag{3.3}$$

The spectral functions v_1, a_1, v_1^s and a_1^s come from the final states with $J = 1$, whereas a_0, v_0^s and a_0^s come from the final states with $J = 0$. $v_0 = 0$ is due to CVC. All these spectral functions are positive semidefinite (≥ 0) as can be verified by going to the rest frame of the final state, $q^{\mu} = (q^0, 0, 0, 0)$, and let $\mu = \nu = 0$ and $\mu = \nu = 1$. This form of decomposition shows explicitly that the final states with $J = 1$ is always conserved and if the current is not conserved it must decay into $J = 0$ states at some q^2 in addition to $J = 1$ states. Let us show that $J = 1$ final states contribute only to v_1, a_1 v_1^s and a_1^s and $J = 0$ final states contribute only to a_0, v_0^s and a_0^s. Since we sum over everything in $|f>$, we may simulate $|f>$ by a particle specified by its momentum and spin. Let us consider a matrix element of a current J_{μ}, which may or may not be conserved, between a vacuum and a spin 1 particle with a polarization vector ϵ and a momentum q .

This matrix element must transform like a vector and linear in ϵ. The only vectors in the problem are ϵ_μ and q_μ, hence

$$< 0 | J_\mu(0) | J = 1,\ q > = f_1 \epsilon_\mu + Bq_\mu$$

Now B must be a Lorentz scalar and linear in ϵ. The only Lorentz scalar linear in ϵ is $\epsilon \cdot q = 0$, hence $B = 0$. Thus

$$< 0 | J_\mu(0) | J = 1,\ q > = f_1 \epsilon_\mu \ . \tag{3.4}$$

After summing over the polarization, we obtain

$$\sum_{\text{spin}} < 0 | J_\mu(0) | J = 1,\ q >< J = 1, q | J_\nu^\dagger(0) | 0 > = (f_1^2/q^2)(q_\mu q_\nu - q^2 g_{\mu\nu}) \tag{3.5}$$

This shows that the spectral function associated with $J = 1$ final states must have tensor coefficients $(q^\mu q^\nu - q^2 g^{\mu\nu})$.

Similarly let us consider a matrix element of $J_\mu(0)$ between a vacuum state and a particle with spin 0. Now the only vector available in the problem is q, hence

$$< 0 | J_\mu(0) | q > = f_0 q_\mu \tag{3.6}$$

and the spectral function associated with $J = 0$ states must have a coefficient $q^\mu q^\nu$ as shown in Eq. (3.3). From Eq. (3.6), if the current is conserved, the $J = 0$ state can exist only if its mass is equal to zero. We also note that if W boson exists, its hadronic decay width can be written as

$$\Gamma_{W^- \to \text{hadrons}} = \Gamma_{W \to {s=0 \atop p=-}} + \Gamma_{W \to {s=0 \atop p=+}} + \Gamma_{W \to {s=-1 \atop p=-1}} + \Gamma_{W \to {s=-1 \atop p=+1}}$$

$$\simeq \frac{M_W^3 G}{2\sqrt{2}} \left[\left\{ v_1(M_W^2) + a_1(M_W^2) \right\} \cos^2\theta_c + \left\{ v_1^s(M_W^2) + a_1^s(M_W^2) \right\} \sin^2\theta_c \right] \tag{3.7}$$

This decay is semiweak, because it is proportional to G instead of G^2. Only the spectral functions associated with the J = 1 states contribute to this width. From Eqs. (3.1), (3.2) and (3.3) we obtain

$$\Gamma_{\ell^- \to \text{hadrons}+\nu} = \frac{G^2}{(2\pi)^3 (2M_\ell)^3} \int_0^{M_\ell^2} (M_\ell^2 - q^2)\, dq^2 \int d\Omega_2$$

$$\left(p_{1\mu} p_{2\nu} + p_{1\nu} p_{2\mu} - g_{\mu\nu}\, (p_1 \cdot p_2)\right) \sum_f < 0 \,|J_h^\mu\,(0)|\, f><f\,|J_u^{\nu +}(0)\,|\,0> \,(2\pi)^4\, \delta^4(q-p_f) \qquad (3.8)$$

$$= \frac{G^2}{(2\pi)^2 (2M_\ell)^3} \int_0^{M_\ell^2} (M_\ell^2 - q^2)^2\, dq^2$$

$$\left[\left\{(M_\ell^2+2q^2)\left(v_1(q^2)+a_1(q^2)\right) + M_\ell^2 a_0(q^2)\right\} \cos^2\theta_c\right.$$

$$\left. + \left\{(M_\ell^2+2q^2)\left(v_1^s(q^2) + a_1^s(q^2)\right) + M_\ell^2 \left(v_0^s(q^2) + a_0^s(q^2)\right)\right\} \sin^2\theta_c\right] \ . \qquad (3.9)$$

Let us first obtain various partial decay widths considered in the previous section. For $\ell^- \to \nu_\ell + \pi^-$, we let f_0 in Eq. (3.6) equal to f_π, then from Eq. (3.3) we obtain

$$a_0(q^2) = 2\pi\, f_\pi^2\, \delta(q^2 - M_\pi^2) \qquad (3.10)$$

Substituting Eq. (3.10) into Eq. (3.9), we obtain immediately Eq. (2.5).

For $\ell^- \to \nu_e + k^-$, we let f_0 in Eq. (3.6) equal to f_k, then from Eq. (3.3) we obtain

$$a_0^s(q^2) = 2\pi f_k^2\, \delta(q^2 - M_k^2) \ . \qquad (3.11)$$

Substituting Eq. (3.11) into Eq. (3.9) and remembering $f_\pi = f_k$ by definition of the Cabibbo angle, we obtain Eq. (2.7).

For $\ell^- \to \nu_e + \rho^-$, we let f_1 in Eq. (3.5) equal to $\sqrt{2}\, g_{\rho\gamma} M_\rho^2$, then from Eq. (3.3) we obtain

$$v_1(q^2) = 4\pi\, g_{\rho\gamma}^2 M_\rho^2\, \delta(q^2 - M_\rho^2) \quad . \tag{3.12}$$

Substituting Eq. (3.12) into Eq. (3.9), we obtain immediately Eq. (2.11). The fact that the constant f_1 in Eq. (3.5) is equal to $\sqrt{2}\, g_{\rho\gamma} M_\rho^2$ comes from CVC. In general CVC relates $v_1(q^2)$ to the isovector part of the total cross section for $e^+ + e^- \to$ hadrons. From

$$2 \sum_f \langle 0 | F_3^\mu(0) | f' \rangle \langle f' | F_3^\nu(0) | 0 \rangle (2\pi)^4 \delta^4(q - p_{f'})$$

$$= \sum_{f'} \langle 0 | F_1^\mu(0) + i F_2^\mu(0) | f \rangle \langle f | F_1^\nu(0) - iF_2^\nu(0) | 0 \rangle (2\pi)^4 \delta^4(q - p_f) \tag{3.13}$$

we obtain

$$v_1(q^2) = \frac{q^2 \sigma_{I=1}^{e^+ + e^-}(q^2)}{4\pi^2 \alpha^2} \quad . \tag{3.14}$$

The total cross section for $e^+ + e^- \to \rho^o \to \pi^+ + \pi^-$ is given by

$$\sigma(e^+ + e^- \to \rho) = \frac{\alpha^2 \pi}{3q^2} \left(1 - \frac{4M_\pi^2}{q^2}\right)^{3/2} \frac{M_\rho^4 g_{\rho\pi\pi}^2 g_{\gamma\rho}^2}{(M_\rho^2 - q^2)^2 + \Gamma_\rho^2 M_\rho^2} \tag{3.15}$$

where

$$\Gamma_\rho = \frac{g_{\rho\pi\pi}^2}{48\pi} \frac{q^3}{M_\rho^2} \left(1 - \frac{4M_\pi^2}{q^2}\right)^{3/2} \quad . \tag{3.16}$$

and

$$g_{\rho\pi\pi}^2 = \frac{1}{g_{\gamma\rho}^2} \approx 8\pi \tag{3.17}$$

Replacing the Breit Wigner factor by a δ function (see Eq. (2.16)), we obtain

$$\sigma(e^+ + e^- \rightarrow \rho^0) \approx 16\, \alpha^2 \pi^2\, g^2_{\gamma\rho}\ \delta(q^2 - M^2_\rho) \tag{3.18}$$

Subsituting Eq. (3.18) into Eq. (3.14), we obtain Eq. (3.12).

A. Weinberg Sum Rules and $\ell \rightarrow A_1 + \nu_\ell$ [13]

If SU 3 $\times$ SU 3 were an exact symmetry, then $v_1(q^2) = a_1(q^2) = v^s_1(q^2) = a^s_1(q^2)$ and $v_0(q^2) = a_0(q^2) = v^s_0(q^2) = a^s_0(q^2) = 0$. In this case ρ and A_1 would have the same mass and the width of $\ell \rightarrow \nu_\ell + \rho$ would be identical to that of $\ell \rightarrow \nu_\ell + A_1$. In order to do better we have to know how the symmetry is broken. Weinberg's sum rules can be used to relate these two widths. Weinberg's ρ_V, ρ_A and F_π are related to our v_1, a_1 and F_π by

$$2\pi\, \rho_V(q^2)/q^2 = v_1(q^2),$$
$$2\pi\, \rho_A(q^2)/q^2 = a_1(q^2)$$

and

$$F_\pi = f_\pi \quad .$$

In our notation Weinberg's two sum rules can be written as

$$\int_0^\infty \left[v_1(q^2) - a_1(q^2)\right] dq^2 = 2\pi\, f^2_\pi \tag{3.19}$$

and

$$\int_0^\infty q^2 \left[v_1(q^2) - a_1(q^2)\right] dq^2 = 0 \quad . \tag{3.20}$$

If v_1 and a_1 are dominated by ρ and A_1 respectively, we have

$$v_1(q^2) = 2\pi(f^2_\rho/M^2_\rho)\ \delta(q^2 - M^2_\rho) \tag{3.21}$$

and

$$a_1(q^2) = 2\pi(f^2_{A_1}/M^2_{A_1})\ \delta(q^2 - M^2_{A_1}) \quad , \tag{3.22}$$

where f_ρ and f_{A_1} are the coupling constants which appear in Eq. (3.4) for respective cases. Substituting Eqs. (3.21) and (3.22) into Eq. (3.20) we obtain

$$f_{A_1}^2 = f_\rho^2 = 2 g_{\rho\gamma}^2 M_\rho^4 \approx M_\rho^4/4\pi \quad . \tag{3.23}$$

The width of $\ell^- \to A_1^- + \nu_\ell$ can be written like Eq. (2.8) with $g_{\rho\ell\nu}$ and M_ρ replaced by $g_{A_1\ell\nu}$ and M_{A_1} respectively. Equations (3.23) and (2.9) say that $g_{\rho\ell\nu} = g_{A_1\ell\nu}$. Using the fact that $M_\rho^2 \approx M_{A_1}^2/2$, we obtain

$$\Gamma(\ell^- \to A_1^* + \nu_\ell) = \frac{G^2 M_\ell^3}{2^8 \pi^2} \cos^2\theta_c \, M_{A_1}^2 \left(1 - \frac{M_{A_1}^2}{M_\ell^2}\right)^2 \left(1 + \frac{2 M_{A_1}^2}{M_\ell^2}\right)^2 \tag{3.24}$$

The result of the colliding beam experiment indicates that $v_1(q^2)$ should behave like a constant instead of behaving like a tail of the Breit Wigner factor of ρ. Hence the ρ and A_1 dominance assumptions are very unlikely to be justifiable. Weinberg obtained the relation $M_{A_1}^2 = 2 M_\rho^2$ from Eqs. (3.19 to 3.22). This can be either accidental or that neither a_1 nor v_1 is dominated by ρ and A_1 but $v_1 - a_1$ is. If the latter is true v_1 must be equal to a_1 above the mass of A_1.

B. Das-Mathur-Okubo Sum Rules and $\ell \to k^* + \nu_\ell$, $Q + \nu_\ell$ [14]

In our notations, DMO sum rules can be written as

$$\int_0^\infty \left[v_1^s(q^2) - a_1^s(q^2)\right] dq^2 = f_k^2 \quad , \tag{3.25}$$

$$\int_0^\infty \left[v_1(q^2) - v_1^s(q^2)\right] dq^2 = 0 \tag{3.26}$$

and

$$\int_0^\infty q^2 \left[v_1^s(q^2) - a_1^s(q^2)\right] dq^2 = 0 \quad . \tag{3.27}$$

Let us assume that $v_1^s - a_1^s$ is dominated by k*(890) in v_1^s and Q(1313) in a_1^s, and that $v_1 - v_1^s$ is dominated by ρ(760) in v_1 and k*(890) in v_1^s. These assumptions mean that when using the sum rules we may let

$$v_1^s(q^2) = 2\pi(f_{k*}^2/M_{k*}^2)\,\delta(q^2 - M_{k*}^2) \tag{3.28}$$

$$a_1^s(q^2) = 2\pi(f_Q^2/M_Q^2)\;\delta(q^2 - M_Q^2) \tag{3.29}$$

$$v_1(q^2) = 2\pi(f_\rho^2/M_\rho^2)\;\delta(q^2 - M_\rho^2) \tag{3.30}$$

From Eqs. (3.26) and (3.27) we obtain

$$f_\rho^2/M_\rho^2 = f_{k*}^2/M_{k*}^2 = f_Q^2/M_{k*}^2 \quad . \tag{3.31}$$

From (3.31) and the known value of f_ρ^2, we can calculate the widths for the decays $\ell \to \nu + k^*(890)$ and $\ell \to \nu + Q(1313)$:

$$\Gamma(\ell \to \nu_\ell + k^*) = \frac{G^2 M_\ell^3 \sin\theta_c}{2^6\,\pi^2}\quad M_\rho^2 \left(1 - \frac{M_{k*}^2}{M_\ell^2}\right)^2 \left(1 + \frac{2M_{k*}^2}{M_\ell^2}\right) \quad . \tag{3.32}$$

$$\Gamma(\ell \to \nu_\ell + Q) = \frac{G^2 M_\ell^3 \sin^2\theta_c}{2^6\,\pi^2}\quad \frac{M_\rho^2 M_{k*}^2}{M_Q^2} \quad \left(1 - \frac{M_Q^2}{M_\ell^2}\right)^2 \left(1 + \frac{2M_Q^2}{M_\ell^2}\right) \quad . \tag{3.33}$$

C. <u>$\ell \to W + \nu_\ell$</u>

If weak vector bosons exist and if $M_W < M_\ell$, then ℓ will first decay into $W + \nu_\ell$ rather than decay directly into hadrons and leptons. The width of $\ell^- \to W^- + \nu_\ell$ can be obtained from the width of $\ell^- \to \rho^- + \nu_\ell$ given in Eq. (2.8) by making substitutions: $M_\rho \to M_W$ and $g_{\rho\ell\nu} \to g$.

$$\Gamma(\ell^- \to W^- + \nu_\ell) = \frac{G}{8\pi\sqrt{2}} \quad M_\ell^3 \left(1 - \frac{M_W^2}{M_\ell^2}\right)^2 \left(1 + \frac{2M_W^2}{M_\ell^2}\right) . \tag{3.34}$$

This decay is proportional to G instead of G^2, hence numerically it is much larger than the decay mechanisms we have considered so far, unless M_W and M_ℓ are almost degenerate to each other. W decays also semiweakly into leptons and hadrons. The hadronic decay modes of W can be written as Eq. (3.7) and the leptonic decay modes ($W^- \to \bar{\nu}_\ell + \ell^-$, $W^- \to \bar{\nu}_\mu + \mu^-$, or $W^- \to \bar{\nu}_\ell + \ell^-$ if $M_\ell < M_W$) can be written as

$$\Gamma(W^- \to \ell^- + \bar{\nu}_\ell) = \frac{G M_W^3}{6\sqrt{2}\,\pi} \left(1 - \frac{M_\ell^2}{M_W^2}\right)\left(1 + \frac{1}{2}\,\frac{M_\ell^2}{M_W^2}\right) \quad . \tag{3.35}$$

D. $\ell \to \nu_\ell$ + Hadron Continuum

Let us assume that either $W^\pm$ do not exist or $M_\ell < M_W$. The decay width of $\ell^\pm \to \nu_\ell$ + hadron continuum can be estimated from the results of $e^+ + e^- \to$ hadrons using Eq. (3.14). In order to do this let us make the following reasonable assumptions:

1. When q^2 is large, say $q^2 > 1\ \mathrm{GeV}^2$, the magnitudes of $v_1(q^2)$, $a_1(q^2)$, $v_1^s(q^2)$ and $a_1^s(q^2)$ corresponding to the decay $\ell \to \nu_\ell$ + hadron continuum are roughly equal to each other. This can be regarded as the basic assumption about the symmetry of currents.

2. The spectral functions for J = 0 states $a_0(q^2)$, $v_0^s(q^2)$ and $a_0^s(q^2)$ in Eq. (3.9) are negligible compared with the spectral functions for J = 1 states when q^2 is large. This is true if we accept the notion that the symmetry of currents becomes exact in the limit $q^2 \to \infty$.

3. The isoscalar part of the cross sections for $e^+ + e^- \to$ hadrons is small compared with the isovector part. For production of ρ, ϕ and ω, the ratios of these cross sections are given experimentally by

$$4\pi\, g_{\gamma\rho}^2 : 4\pi g_{\gamma\phi}^2 : 4\pi\, g_{\gamma\omega}^2 = \frac{1}{2} : \frac{1}{15} : \frac{1}{11.5} \quad .$$

Hence the isovector cross section is three times as large as the isoscalar cross section. Whether this is true for large q^2 is an interesting open question. SU3 gives the ratio of 3 to 1 for the isovector to the isoscalar cross sections. If we accept this, then Eq. (3.14) becomes

$$\lim_{q^2\to\infty} v_1(q^2) = \frac{3}{4} \frac{q^2 \sigma_{e^+ + e^-}(q^2)}{4\pi^2 \alpha^2} . \tag{3.36}$$

These assumptions say that for estimating the partial width of $\ell^- \to \nu_\ell$ + hadron continuum we may let $\theta_c = 0$, $a_0 = 0$ and $v_1 = a_1$ in Eq. (3.9), hence

$$\Gamma_{\ell^- \to \nu_\ell + \text{hadron continuum}}$$

$$= \frac{G^2}{(2\pi)^2 (2M_\ell)^3} \int_{\Lambda^2}^{M_\ell^2} dq^2 \, (M_\ell^2 - q^2)^2 \, (M_\ell^2 + 2q^2) \, \frac{3}{2} \frac{q^2 \sigma_{e^+ + e^-}(q^2)}{4\pi^2 \alpha^2} . \tag{3.37}$$

The $e^+ + e^-$ colliding beam experiment of Frascati[16] shows that in the energy range 1.6 GeV $\leq \sqrt{q^2} \leq$ 2.0 GeV, the cross section for $e^+ + e^- \to$ hadrons is $(3 \pm 0.3)\, 10^{-32}\ \text{cm}^2$ compared with the cross section for $e^+ + e^- \to \mu^+ + \mu^-$ of $2.5 \times 10^{-32}\ \text{cm}^2$ at 1.8 GeV. However it is not clear whether the observed cross section really represents the process $e^+ + e^- \to$ hadrons. For example some of the events may be due to the production of heavy leptons as considered in Section IV of this paper, or some may be due to the two photon process.[15] It is interesting to note that in the quark parton model, the cross section for $e^+ + e^- \to$ hadron is less than that of $e^+ + e^- \to \mu^+ + \mu^-$ by a factor of 2/3. Let us denote the cross section for $e^+ + e^- \to$ hadron given by the quark parton model as

$$\sigma_{\text{quark}} (e^+ + e^- \to \text{hadron}) = \frac{2}{3}\, \sigma(e^+ + e^- \to \mu^+ + \mu^-) = \frac{2}{3} \frac{4\pi}{3} \frac{\alpha^2}{q^2} , \tag{3.38}$$

and the quoted cross section from Frascati[16] as

$$\sigma_{\text{Frascati}}(e^+ + e^- \to \text{hadron}) \approx \frac{4}{3}\ \sigma(e^+ + e^- \to \mu^+ + \mu^-)\ . \tag{3.39}$$

Substituting Eq. (3.38) into Eq. (3.36) we obtain

$$\frac{\Gamma(\ell^- \to \text{hadron continuum})}{\Gamma(\ell^- \to e^- + \nu_\ell + \overline{\nu}_e)} = 2\int_{\frac{\Lambda^2}{M_\ell^2}}^{1} (1-x)^2\ (1+2x)\,dx \equiv R\left(\frac{\Lambda^2}{M_\ell^2}\right)$$

$$= \left(1 - \frac{2\Lambda^2}{M_\ell^2} + \frac{2\Lambda^6}{M_\ell^6} - \frac{\Lambda^8}{M_\ell^8}\right) \tag{3.40}$$

This result corresponds to using the quark model. If we use Eq. (3.39), the right-hand side of Eq. (3.40) would be multiplied by 2. Λ should be taken to be ~ 1 GeV. We notice that as $\Lambda^2/M_\ell^2 \to 0$, the ratio R becomes unity if quark model were used and the ratio becomes two if the experimental result of Frascati is taken at its face value.

IV. $e^+ + e^- \to \ell^+ + \ell^-$ AND DECAY CORRELATIONS

The cross section for this reaction is well known. Ignoring the mass of the electron we have in the center-of-mass system,[17]

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2}\ \beta\left[1 + \cos^2\theta + \frac{\sin^2\theta}{\gamma^2}\right] \tag{4.1}$$

where E is the energy of ℓ^+ or ℓ^-, $\beta = (E^2 - M_\ell^2)^{1/2}/E$ and $\gamma = E/M_\ell$. Integrating Eq. (4.1) with respect to the solid angle we obtain the total cross section

$$\sigma(e^+ + e^- \to \ell^+ + \ell^-) = \frac{\alpha^2\pi}{3E^2}\ \beta\left[1 + \frac{1}{2\gamma^2}\right]\ . \tag{4.2}$$

Since heavy leptons are unstable particles and their decay angular distributions depend upon their spin orientations, we have to know the probabilities of the heavy leptons coming out at different spin orientations. In order to do this let us calculate the probability of the reaction $e^+ + e^- \to \ell^+ + \ell^-$ with the spin of ℓ^- in the direction $\vec{s}$ and the spin of ℓ^+ in the direction $\vec{s}'$. $\vec{s}$ and $\vec{s}'$ are unit vectors defined in the rest frames of ℓ^- and ℓ^+ respectively. In order to calculate the spin effect covariantly, let us define two four-vectors (axial) s^- and s^+ which reduce to the three vectors $\vec{s}$ and $\vec{s}'$, respectively, in the rest frames of ℓ^- and ℓ^+. Let us choose a coordinate system where the direction of p_- in the center-of-mass system is the z axis and the direction $\vec{p}_- \times \vec{p}_1$ is the y axis, as shown in Fig. 4. In this frame the components of s^- and s^+ can be written in terms of the components of s and s' as

$$s_- = (s_-^0,\ s_-^1,\ s_-^2,\ s_-^3) = (\beta\gamma s_z, s_x, s_y, \gamma s_z)\ . \qquad (4.3)$$

$$s_+ = (-\beta\gamma s'_z, s'_x, s'_y, \gamma s'_z) \qquad (4.4)$$

Let us denote the four momenta of e^-, e^+, ℓ^- and ℓ^+ by p_1, p_2, p^- and p^+, respectively. We have then

$$\begin{aligned} p_- &= (E, 0, 0, \beta E) \\ p_+ &= (E, 0, 0, -\beta E) \\ p &\equiv (p_1 - p_2)/2 = (0, E\sin\theta, 0, E\cos\theta). \end{aligned} \qquad (4.5)$$

The projection operator for ℓ^- with momentum p_- and spin in the direction $\vec{s}$ is then

$$\frac{1+\gamma_5 \not{s}_-}{2}\ \frac{\not{p}_- + M_\ell}{2M_\ell}\ , \qquad (4.6)$$

and for ℓ^+ with momentum p_+ and spin $\vec{s}'$ is

$$\frac{1+\gamma_5 \not{s}_+}{2} \; \frac{-p_+ + M_\ell}{2M_\ell} \quad . \tag{4.7}$$

We use the following representation for γ matrices:

$$\gamma_5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \quad \gamma_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \gamma_i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix} \quad .$$

Let us show that Eq. (4.7) indeed is the required projection operator for ℓ^+ with spin $\vec{s}'$. In the rest frame of ℓ^+ we have

$$\frac{1+\gamma_5 \not{s}_+}{2} = \begin{bmatrix} \frac{1+\vec{\sigma}\cdot\vec{s}'}{2} & 0 \\ 0 & \frac{1-\vec{\sigma}\cdot\vec{s}'}{2} \end{bmatrix} \tag{4.8}$$

and

$$\frac{-\not{p} + M_\ell}{2M_\ell} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad ,$$

hence

$$\frac{1+\gamma_5 \not{s}_+}{2} \; \frac{-\not{p} + M_\ell}{2M_\ell} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1-\vec{\sigma}\cdot\vec{s}'}{2} \end{bmatrix} \quad . \tag{4.9}$$

In our representation a positron with spin in $\vec{s}'$ direction is represented by a negative energy state with spin in $-\vec{s}'$ direction, hence the projection operator is $(1 - \vec{\sigma}\cdot\vec{s}')/2$. (Notice the negative sign in front of $\vec{\sigma}$ for positrons and positive sign for electrons.)

The desired cross section can be calculated in the standard way:

$$\frac{d\sigma}{d\Omega}\ (\vec{s},\vec{s}\,') = \frac{e^4}{(2\pi)^2}\ \frac{1}{4(p_1\cdot p_2)} \int \frac{d^3p_+}{2E} \int \frac{d^3p_-}{2E}\ \delta^4\ (p_1+p_2-p_+-p_-)$$

$$\frac{tr}{4}\ (\not{p}_1+m)\gamma_\mu(\not{p}_2-m)\,\gamma_\nu\ \frac{tr}{4}\ (1+\gamma_5\not{s}_-)\,(\not{p}_-+M_\ell)\ \gamma_\mu(1+\gamma_5\not{s}_+)\,(\not{p}_+-M_\ell)\gamma_\nu \tag{4.10}$$

$$= \frac{\alpha^2}{16E^2}\ \beta\left[1+\cos^2\theta + \frac{\sin^2\theta}{\gamma^2} + s_z s'_z\left(1+\cos^2\theta - \frac{\sin^2\theta}{\gamma^2}\right)\right.$$

$$\left. + s_x s'_x\left(1+\frac{1}{\gamma^2}\right)\sin^2\theta - s_y s'_y\ \beta^2\sin^2\theta\ + (s_x s'_z + s'_x s_z)\ \frac{1}{\gamma}\ \sin 2\theta\right]\ . \tag{4.11}$$

We notice that $\vec{s}$ and $\vec{s}\,'$ occur only bilinearly. This means that if only one particle is analyzed,then no effect of polarization can be observed, but if two particles are analyzed simultaneously, their spins are correlated. The correlation is such that if $s_z = \pm 1$, then the cross section is maximum when $s'_z = \pm 1$. In other words the helicities of ℓ^+ and ℓ^- like to be opposite to each other. We also notice that the coefficients of $s_x s'_x$ and $s_y s'_y$ do not approach zero as $\gamma \to \infty$. For a massless lepton pair the terms $s_x s'_x$ and $s_y s'_y$ need not be considered and hence we have always $s_z = \pm 1$ and $s'_z = \pm 1$. In this case ℓ^+ and ℓ^- always have the opposite helicity. When the leptons have finite mass, the spin correlation is not complete even in the limit of $\gamma \to \infty$. For example, the probability for $s_z = 1$ and $s'_z = \frac{1}{\sqrt{2}}$ and $s'_x = \frac{1}{\sqrt{2}}$ is not zero even if we let $\gamma \to \infty$. Another interesting feature is the behavior near threshold. Near the threshold we have $\beta \to 0$ and $\gamma \to 1$. Hence Eq. (4.11) gives

$$\frac{d\sigma}{d\Omega}\ (\vec{s},\vec{s}\,') \underset{\text{threshold}}{\Longrightarrow} \frac{\alpha^2}{8E^2}\ \beta\left[1+(\vec{s}\cdot\hat{p})(\vec{s}\,'\cdot\hat{p})\right] \tag{4.12}$$

where $\hat{p}$ is the unit vector along the direction of the incident electron. Equation (4.12) shows that $\vec{s}$ and $\vec{s}'$ like to be both parallel or both antiparallel to the direction of the incident electron beam.

Let us try to understand Eqs. (4.11) and (4.12) using a more illuminating but clumsy method. Let us denote the electron-positron current by

$$j_\mu = \bar{v}(p_2)\gamma_\mu u(p_1) \quad . \tag{4.13}$$

Current conservation gives $2E\, j_0 = 0$, therefore $j_0 = 0$. If the mass of the electron is ignored, we have

$$p_1^\mu j_\mu = \bar{v} \not{p}_1 \; u(p_1) = 0 = E\, j_0 - p_1 j_z = -p_1 j_z ,$$

hence $j_z = 0$. Instead of j_x and j_y, it is more convenient to consider

$$j_\pm = \frac{j_x \pm i j_x}{2} \quad . \tag{4.14}$$

Writing

$$u(p_1) = (E + M)^{1/2} \begin{bmatrix} 1 \\ \\ \dfrac{\vec{\sigma}\cdot\vec{p}_1}{E+M} \end{bmatrix} \chi \tag{4.15}$$

and

$$v(p_2) = (E + M)^{1/2} \begin{bmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}_2}{E+M} \\ \\ 1 \end{bmatrix} \mathscr{H} \tag{4.16}$$

where χ is the two component spinor representing the spin state of the electron in its rest frame and $\mathscr{H}$ is the two component spinor representing the spin state of the hole. For example $\chi = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ represents a state with spin up for the electron but $\mathscr{H} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ represents a hole state with spin up hence it represents a positron with spin down. A positron with helicity + and momentum $\vec{p}_2$ satisfies

$$\sigma \cdot \hat{p}_2 \mathscr{H} = -\mathscr{H} \quad .$$

From Eqs. (4.13 to 4.16) we have

$$j_{\pm} = 2E\mathcal{U}^{+} \sigma_{\pm} X , \tag{4.17}$$

where $\sigma_{+} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\sigma_{-} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and the axis of quantization is along the direction of p_1. Taking the spin average, we notice that j_+ is nonvanishing only when the electron has negative helicity and the positron has a positive helicity, whereas j_- is nonvanishing only when the electron has a positive helicity, and the positron has a negative helicity. The numerical value of each matrix element is 2E. Equation (4.17) shows also that the total angular momentum of the electron positron system is unity and the direction of the angular momentum is parallel to the total spin $\vec{s}_1 + \vec{s}_2$ which is either parallel or antiparallel to the direction of the incident electron.

Let us denote the current of the final lepton pair by

$$J_{\mu} = \bar{u}\,(p_-)\ \gamma_{\mu}\ v(p_+) \quad .$$

Then

$$J_{\mu} j^{\mu} = -j_x J_x - j_y J_y = -2j_+ J_- - 2j_- J_+$$

because $j_0 = j_z = 0$. Now for discussion of the matrix element of J_{μ} it is more convenient to quantize the angular momentum along the direction of motion of $\vec{p}_-$ than along the direction of the incident electron. Using the coordinate system shown in Fig. 4 and denoting it by a prime, we obtain

$$J_{\pm} \equiv \frac{1}{2}\,(J_x \pm iJ_y) = \frac{1}{2}\,(J'_x \cos\theta - J'_z \sin\theta \pm i\,J'_y)$$

$$= \frac{1}{2}\,(\cos\theta \pm 1)\ J'_+ + \frac{1}{2}\,(\cos\theta \mp 1)\ J'_- - \frac{1}{2}\sin\theta\ J'_z \quad ,$$

hence

$$J_{\mu} j^{\mu} = j_+ \left[(1-\cos\theta)\ J'_+ - (1+\cos\theta) J'_- + \sin\theta\ J'_z\right]$$

$$+ j_-\left[-(1+\cos\theta)\ J'_+ + (1-\cos\theta)\ J'_- + \sin\theta\ J'_z\right] \quad . \tag{4.18}$$

In terms of two components spinors, $J'_{\pm}$ and J'_z can be written as:

$$J'_{\pm} = 2\,E\,\chi^{+}\,\sigma_{\pm}\,\mathcal{W} \quad \text{and} \quad J'_z = 2\,M_{\ell}\,\chi^{+}\,\sigma_z\,\mathcal{W}\ . \tag{4.19}$$

In our representation the states of ℓ^{-} with spin pointing in x', y' and z' directions are given respectively by

$$\chi_{x'} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\ , \quad \chi_{y'} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix} \quad \text{and} \quad \chi_{z'} = \begin{bmatrix}1\\0\end{bmatrix}\ . \tag{4.20}$$

Similarly the states of ℓ^{+} with spin pointing in x', y' and z' directions are given respectively by

$$\mathcal{W}_{x'} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}\ , \quad \mathcal{W}_{y'} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix} \quad \text{and} \quad \mathcal{W}_{z'} = \begin{bmatrix}1\\0\end{bmatrix}\ . \tag{4.21}$$

The cross section is proportional to the square of Eq. (4.18). Since the helicity states contributing to j_{+} are different from those contributing to j_{-}, the two square bracket terms do not interfere with each other. Averaging over the spin of the incident particles, we obtain

$$\frac{d\sigma}{d\Omega}(\vec{s},\vec{s}\,') = \frac{\alpha^2}{16E^2\beta}$$

$$\left[\left|\chi_s\left\{(1-\cos\theta)\ \sigma_{+} - (1+\cos\theta)\ \sigma_{-} + \frac{\sin\theta}{\gamma}\ \sigma_z\right\}\mathcal{W}_{s'}\right|^2\right.$$

$$\left. + \left|\chi_s\left\{-(1+\cos\theta)\ \sigma_{+} + (1-\cos\theta)\ \sigma_{-} + \frac{\sin\theta}{\gamma}\ \sigma_z\right\}\mathcal{W}_{s'}\right|^2\right]\ . \tag{4.22}$$

With the aid of Eqs. (4.20) and (4.21) we can verify Eq. (4.11) from Eq. (4.22). The latter derivation of the spin correlation is clumsy, but it brings out many subtle points of the problem.

Let us discuss qualitatively the experimental consequences of spin correlation. In Section II, we calculated the decay angular distribution of arbitrary polarized ℓ^- and ℓ^+. In general we may write symbolically:

$$\Gamma(\ell^- \to X) = \int d\Omega \ \mathrm{tr} \frac{1+\gamma_5 \not{w}_-}{2} \ \ldots = \int (A+B\vec{q}\cdot\vec{w}) \ d\Omega \tag{4.23}$$

and

$$\Gamma(\ell^+ \to X') = \int d\Omega' \ \mathrm{tr} \frac{1+\gamma_5 \not{w}_+}{2} \ \ldots = \int (A'+B' \ \vec{q}'\cdot\vec{w}') \ d\Omega' \ , \tag{4.24}$$

where q and q' are momenta of the decay products to be detected. Let us write symbolically the spin correlation in the production, Eq. (4.11), as

$$\frac{d\Omega}{d\Omega_\ell} = C + D_{ij} \ s_i \ s_j' \quad . \tag{4.25}$$

The combined angular distribution of the decay products $\ell^- \to X$ and $\ell^+ \to X'$ for a fixed production angle, can be written as

$$\frac{d\sigma}{d\Omega_\ell d\Omega d\Omega'} = \frac{CAA' + D_{ij} q_i q_j' BB'}{\Gamma^2_{\mathrm{total}}} \quad , \tag{4.26}$$

where Γ_{total} is the total width of $\ell^\pm$, $d\Omega_\ell$ is the solid angle for ℓ^- (or ℓ^+) in the center-of-mass system, $d\Omega$ and $d\Omega'$ are solid angles for the decay products of ℓ^- and ℓ^+ in the rest frames of ℓ^- and ℓ^+, respectively. Equation (4.26) can be derived in the following way: $\vec{w}$ represents the polarization vector of ℓ^-, and by definition each component of $\vec{w} = (w_x, w_y, w_z)$ represents

$$w_i = \frac{\text{number of } \ell^- \text{ polarized along } \hat{e}_i - \text{number of } \ell^- \text{ polarized along } -\hat{e}_i}{\text{number of } \ell^- \text{ polarized along } \hat{e}_i + \text{number of } \ell^- \text{ polarized along } -\hat{e}_i} \quad .$$

Now the number of ℓ^- having spin along the direction $\hat{e}_i$ with the polarization of ℓ^+ in a certain direction s' is proportional to $C + D_{ij}s'_j$, whereas the corresponding number of ℓ^- having spin along the direction $-\hat{e}_i$ is $C - D_{ij}s'_j$.

Hence

$$w_i = \frac{D_{ij}\, s'_j}{C} \tag{4.27}$$

and the angular distribution of the decay product of ℓ^- is proportional to

$$CA + D_{ij}\, s'_j\, q_i\, B \quad . \tag{4.28}$$

For a fixed angular distribution of the decay product of ℓ^- given by Eq. (4.28), the components of the spin of ℓ^+ are given by

$$w'_j = \frac{D_{ij} q_i\, B}{CA} \quad . \tag{4.29}$$

Substituting Eq. (4.29) into Eq. (4.24) we obtain the combined angular distribution of the decay products of ℓ^+ and ℓ^- at a fixed production angle:

$$CAA' + D_{ij}\, q_i q'_j\, BB' \quad .$$

In order to obtain the proper normalization factor, we notice that the partial decay width $\Gamma(\ell^- \rightarrow X)$ is independent of the polarization, hence

$$\Gamma(\ell^- \rightarrow X) = \int A\, d\Omega \quad \text{and} \quad \int B\, \vec{q} \cdot \vec{w}\, d\Omega = 0 \quad . \tag{4.30}$$

Similarly

$$\Gamma(\ell^- \rightarrow X') = \int A'\, d\Omega' \quad \text{and} \int B'\, \vec{q}' \cdot \vec{w}'\, d\Omega' = 0 \quad . \tag{4.31}$$

This implies

$$\int D_{ij}\, B_i\, B'_j\, d\Omega = \int D_{ij}\, B_i\, B'_j\, d\Omega' = 0 \quad . \tag{4.32}$$

and therefore integrating Eq. (4.26) we obtain

$$\frac{d\Omega}{d\Omega_\ell}\left(e^+ + e^- \to \ell^+ + \ell^- \begin{smallmatrix} \ulcorner\!\to X' \\ \\ \llcorner\!\to X \end{smallmatrix}\right) = C\,\frac{\Gamma(\ell^+\to X')\,\Gamma(\ell^-\to X)}{\Gamma^2_{\text{total}}} \quad . \tag{4.33}$$

This shows that Eq. (4.26) is indeed properly normalized because C is equal to $d\Omega/d\Omega_\ell$ summed over the polarizations of ℓ^+ and ℓ^-. Equation (4.32) shows explicitly that if the decay angular distribution of only ℓ^+ or only ℓ^- is observed, then the effects of the spin correlation vanish. This is as mentioned earlier due to the absence of the terms linear in $\vec{s}$ and $\vec{s}'$ in Eq. (4.11). The absence of linear terms in Eq. (4.11) is due to the approximation of one photon exchange.

In the absence of spin correlation we have only the CAA' term in Eq. (4.26). It is very important to notice that the existence of B and B' in Eqs. (4.24) and (4.25) is due to parity nonconservation in the decay of heavy leptons. Since the spin correlation term $D_{ij}\, q_i q_j'\, BB'$ in Eq. (4.26) is proportional to BB', we conclude that <u>if we detect in coincidence one particle from the decay products of ℓ^- and one particle from those of ℓ^+, the effects of spin correlation exist only if parity conservation is violated in the decays of both ℓ^+ and ℓ^-</u>. This is due to the fact that the polarization vector for a spin $\frac{1}{2}$ particle is an axial vector. (This would not be true if ℓ^+ and ℓ^- were spin 1 particles. In this case the correlation exists even if parity is conserved.)

In order to see the effects of the spin correlation let us compare the magnitudes of the isotropic term CAA' and the spin correlation term $D_{ij}\, q_i\, q_j'\, BB'$ in Eq. (4.26) corresponding to various combinations of decay channels.

<u>Example 1</u> $\quad \ell^- \to \nu_\ell + \bar{\nu}_\mu + \mu^-$,

$\ell^+ \to \bar{\nu}_\ell + \nu_e + e^+$ and

μ^- and e^+ are detected in coincidence.

From Eqs. (2.1) and (4.11) we have (we have ignored the mass of muon for simplicity)

$$CAA' = F\left(1+\cos^2\theta + \frac{\sin^2\theta}{\gamma^2}\right)\int_0^1 x^2 dx\,(3-2x) \int_0^1 x'^2 dx'\,(3-2x') \tag{4.34}$$

and

$$D_{ij}q_iq'_j BB' = -F\int_0^1 x^2 dx \int_0^1 x'^2 dx'\,(1-2x)(1-2x')$$

$$\times \frac{1}{qq'}\left[q_zq'_z\left(1+\cos^2\theta - \frac{\sin^2\theta}{\gamma^2}\right) + q_xq'_x\left(1+\frac{1}{\gamma^2}\right)\sin^2\theta\right.$$

$$\left. - q_yq'_y\,\beta^2\sin^2\theta + (q_xq'_z+q'_xq_z)\,\frac{1}{\gamma}\,\sin 2\theta\right] \quad , \tag{4.35}$$

where

$$F = \left(\frac{G^2M_\ell^5}{3\times 2^7\pi^4}\right)^2 \frac{\alpha^2}{16E^2}\,\beta \quad , \qquad x = q/q_{max} \quad ,$$

$$x' = q'/q'_{max} \quad \text{and} \quad q_{max} = q'_{max} = M_\ell/2 \quad .$$

We observe the following:

1. Correlation is maximum when x and x' are both near 1.
2. In the limit $\gamma \to \infty$, and both x and x' are near 1, e^+ and μ^- like to come out either both along the directions of motion of parent particles, or both opposite to the directions of motion of parent particles.
3. Near the threshold $(\gamma \to 1)$, we may write

$$CAA'+D_{ij}q_iq'_j BB' \propto (3-2x)(3-2x')-(1-2x)(1-2x')(\hat{p}\cdot\hat{q})(\hat{p}\cdot\hat{q}') \tag{4.36}$$

where $\hat{p}$ is the unit vector along the direction of the incident electron. This shows that the correlation is maximum when both x and x' are near 1. The effect of correlation vanishes when either $\hat{q}$ or $\hat{q}'$ is perpendicular to $\hat{p}$; and the maximum correlation occurs when both $\hat{q}$ and $\hat{q}'$ are in the direction of the incident beam. e^+ and μ^- like to come out in the opposite direction from each other if both x and x' are near 1 and both $\vec{q}$ and $\vec{q}'$ are in the incident beam direction.

<u>Example 2</u> $\quad \ell^- \rightarrow \nu_\ell + \pi^-$,

$\ell^+ \rightarrow \bar{\nu}_\ell + \pi^+$ $\quad$ and π^+ and π^- are detected in coincidence.

From Eqs. (2.5) and (4.11) we have

$$CAA' = F_2\left(1 + \cos^2\theta + \frac{\sin^2\theta}{\gamma^2}\right) \tag{4.37}$$

and

$$D_{ij}q_iq_j' \, BB' = F_2 \frac{1}{qq'} \,\Big|\, q_zq_z' \quad 1 + \cos^2\theta - \frac{\sin^2\theta}{\gamma^2} \quad + q_xq_x' \quad 1 + \frac{1}{\gamma^2} \quad \sin^2\theta$$

$$-q_yq_y' \quad \beta^2\sin^2\theta + (q_xq_z' + q_x'q_z)\frac{1}{\gamma} \quad \sin 2\theta \Big] \; , \tag{4.38}$$

where

$$F_2 = \frac{\Gamma^2(\ell \rightarrow \pi + \nu)}{(4\pi)^2} \quad \frac{\alpha^2}{16E^2} \quad \beta \; . \tag{4.39}$$

Near the threshold we have

$$CAA' + D_{ij}q_iq_j' \, BB' = 2F_2 \left[1 - (\hat{p}\cdot\hat{q})(\hat{p}\cdot\hat{q}')\right] \; . \tag{4.40}$$

Comparison with the results of Example 1, we see that the two cases are very similar except that in the present case $\pi^{\pm}$ have a definite momentum in the rest frames of $\ell^{\pm}$. As far as the ratio $D_{ij}q_iq_j'BB'/CAA'$ is concerned, the present case is identical to x = x' = 1 of the previous example.

Example 3 $\quad \ell^- \to \nu_\ell + \pi^-$, $\ell^+ \to \bar{\nu}_\ell + \nu_\mu + \mu^+$ and π^- and μ^+ are detected in coincidence. From Eqs. (2.1), (2.5) and (4.11) we obtain

$$CAA' = F_3 \int_0^1 (3-2x')\, x'^2 dx' \left(1 + \cos^2\theta + \frac{\sin^2\theta}{\gamma^2}\right) \tag{4.41}$$

and

$$D_{ij} q_i q_j' \, BB' = -F_3 \int_0^1 (1-2x')\, x'^2 dx' \frac{1}{qq'} \left[q_z q_z' \left(1 + \cos^2\theta - \frac{\sin^2\theta}{\gamma^2}\right) + q_x q_x' \left(1 + \frac{1}{\gamma^2}\right) \sin^2\theta - q_y q_y' \beta^2 \sin^2\theta + (q_x q_z' + q_x' q_z) \frac{1}{\gamma} \sin 2\theta \right] \tag{4.42}$$

where
$$F_3 = \frac{\Gamma(\ell \to \pi + \nu)}{4\pi} \frac{G^2 M_\ell^5}{3 \times 2^7 \pi^4} \frac{\alpha^2}{16 E^2} \beta ,$$

$x' = q'/q'_{max}$ and $q'_{max} = M_\ell/2$.

In this case the effect of spin correlation is again maximum at $x' = 1$. However, near $x' = 1$, the relative sign between CAA' and $D_{ij} q_i q_j BB'$ is opposite to the previous two examples. Hence, if π^- is emitted along the direction of motion of ℓ^-, then μ^+ likes to be emitted opposite to the direction of motion of ℓ^+ when γ is large and x' is near 1. Near the threshold π^- and μ^+ like to be emitted in the same direction and along the incident beams (e^+ or e^-) when x' is near 1.

V. SUMMARY AND CONCLUSIONS

In Table II, we give the partial and total decay rates of ℓ for various values of M_ℓ. The formulas used to calculate them are given in Sections II and III. We collect them here for easy reference. (All masses are in units of GeV.)

$$\Gamma(\ell \to \nu_\ell + \nu_e + \mu) = 3.47 \times 10^{10}\, M_\ell^5 \ \sec^{-1} .$$

$$\Gamma(\ell \to \nu_\ell + \nu_\mu + \mu) = 3.47 \times 10^{10}\, M_\ell^5 \,(1-8y+8y^3-y^4-12y^2 \ell n y) \ \sec^{-1},\ (y = M_\mu^2/M_\ell^2) .$$

$$\Gamma(\ell \to \nu_\ell + \pi) = 5.3 \times 10^{+10}\, M_\ell^3 \left(1 - \frac{M_\pi^2}{M_\ell^2}\right)^2 \ \sec^{-1}.$$

$$\Gamma(\ell \to \nu_\ell + k) = 0.46 \times 10^{+10}\, M_\ell^3 \left(1 - \frac{M_k^2}{M_\ell^2}\right)^2 \ \sec^{-1}.$$

$$\Gamma(\ell \to \rho + \nu_\ell) = 18 \times 10^{10}\, M_\ell^3 \left(1 - \frac{M_\rho^2}{M_\ell^2}\right)^2 \left(1 + \frac{2M_\rho^2}{M_\ell^2}\right) \ \sec^{-1} .$$

$$\Gamma(\ell \to k^* + \nu_\ell) = 1.29 \times 10^{10}\, M_\ell^3 \left(1 - \frac{M_{k^*}^2}{M_\ell^2}\right)^2 \left(1 + \frac{2M_{k^*}^2}{M_\ell^2}\right) \ \sec^{-1} .$$

$$\Gamma(\ell \to A_1 + \nu_\ell) = 7.2 \times 10^{10}\, M_\ell^3 \left(1 - \frac{M_{A_1}^2}{M_\ell^2}\right)^2 \left(1 + \frac{2M_{A_1}^2}{M_\ell^2}\right) \ \sec^{-1} .$$

$$\Gamma(\ell \to Q + \nu_\ell) = 0.614 \times 10^{10}\, M_\ell^3 \left(1 - \frac{M_Q^2}{M_\ell^2}\right)^2 \left(1 + \frac{2M_Q^2}{M_\ell^2}\right) \ \sec^{-1}.$$

$$\Gamma(\ell \to \nu_\ell + \text{hadron continuum}) = 3.47 \times 10^{10}\, M_\ell^5 \left(1 - \frac{2\Lambda^2}{M_\ell^2} + \frac{2\Lambda^6}{M_\ell^6} - \frac{\Lambda^8}{M_\ell^8}\right) \ \sec^{-1} .$$

For construction of Table II, we have used the following numerical values: $G = 1.02 \times 10^{-5}/M_p^2$, $M_p = .938$, $M_\mu = .106$, $M_\pi = .14$, $M_k = .495$, $M_\rho = .765$, $M_{k^*} = .892$, $M_{A_1} = 1.070$, $M_Q = 1.3$, $\Lambda = 1$, $\sin^2\theta_c = .068$, $\cos^2\theta_c = .932$, and $\tan^2\theta_c = .073$. We have also computed the decay lengths in the vacuum corresponding to $E_\ell = 5$ GeV and $E_\ell = 50$ GeV. We make the following comments and observations:

1. In the photo pair production $\gamma + z \to \ell^+ + \ell^- + z^*$, if the decay length is greater than 1 cm, then ℓ^+ and ℓ^- can be identified visually in a streamer chamber.[18]

Hence, at SLAC energy ($E_\ell \leq 5$ GeV) we see from Table II that the identification of $\ell^\pm$ up to $M_\ell = 1$ GeV is possible. Another scheme, suggested by M. Davier,[19] is to aim a spectrometer in the decay region and detect the decay products. If the production region and the decay region is well separated (>1 cm), $\ell^\pm$ can be identified. At NAL energies, one can identify the production of $\ell^\pm$ if $M_\ell < 1.3$ GeV using these two methods.

2. In Table II, the first four decay modes (e, μ, π and k) depend only on the validity of the current-current interaction hypothesis of weak interaction. The decay $\ell \to \rho + \nu$ depends on CVC besides the current-current hypothesis. Since we have made a narrow width approximation for the ρ decay the numerical value near ρ threshold is not reliable. This can be improved easily if we use the experimental cross section for $e^+ + e^- \to \rho^o \to \pi^+ + \pi^-$ in Eqs. (3.14) and Eq. (3.9). The last three modes of decay depend upon the assumptions of SU3 symmetry and the asymptotic behavior of the spectral functions. They are probably correct to within a factor of two except near the threshold where again the assumption of narrow widths was used. If the mass M_ℓ happens to be near one of these resonances, one should restore the Breit Wigner factor in the spectral function Eq. (3.9) instead of approximating it by a δ function (see Eq. (2.16)).

3. For the partial width $\Gamma(\ell \to \nu +$ hadron continuum) we have used the expression obtained from the quark model, Eq. (3.38), instead of the quoted experimental value of the Frascati experiment, Eq. (3.39).

If both heavy leptons and weak vector bosons exist and $M_\ell > M_W$, then the decay mode $\ell \to W + \nu_\ell$ will completely dominate the widths given in Table II. For example, if $M_W = 2$ GeV and $M_\ell = 3$ GeV, then from Eq. (3.34), we obtain $\Gamma(\ell^- \to W^- + \nu_\ell)/\hbar = 2.6 \times 10^{18}\ \mathrm{sec}^{-1}$ which is much larger than the total weak decay rate of $3.4 \times 10^{13}\ \mathrm{sec}^{-1}$ given in Table II.

W bosons decay rapidly into $\nu_e + e$, $\nu_\mu + \mu$ and hadrons. The leptonic decay widths can be calculated by using Eq. (3.35). The hadronic decay width can be estimated by using Eq. (3.7). Assuming $v_1(M_W^2) = a_1(M_W^2) = v_1^s(M_W^2) = a_1^s(M_W^2)$ and using Eqs. (3.7), (3.36) and (3.38) we obtain

$$\Gamma(W \rightarrow \text{hadrons}) = \Gamma(W \rightarrow e + \nu_e) = \frac{M_W^3 G}{6\sqrt{2}\pi} \quad .$$

This is the result of the quark model. If the experimental result from Frascati is taken at its face value, we obtain

$$\Gamma(W \rightarrow \text{hadrons}) = 2\,\Gamma(W \rightarrow e + \nu_e) \quad .$$

In the experiment $e^+ + e^- \rightarrow \ell^+ + \ell^-$, the first thing to try is to look for $\mu\pi$ coincidence. The energy and angle of π and μ are correlated,as shown in Eqs. (4.41) and (4.42). The effects of correlations, together with the branching ratios into various channels, can be used to confirm the existence of $\ell^\pm$. The work,as presented in this paper, is not complete because in the colliding beam experiment one probably detects only the decay products, hence the production angle of $\ell^\pm$ has to be integrated out. Also, the energy-angle distributions of the decay products have to be given in the overall center-of-mass system instead of the rest frames of ℓ^+ and ℓ^-. Both of these can be done easily by a computer using the technique similar to the one used in the calculation of $e^+ + e^- \rightarrow W^+ + W^- \rightarrow \mu^+ + \nu_\mu + e^- + \bar{\nu}_e$ by A. C. Hearn and the author[20] many years ago.

In the experiment $\nu_\ell + z \rightarrow \ell + z^*$, the heavy lepton is polarized, and we expect the angular distribution of the decay product to be quite different from that of an unpolarized ℓ. The calculation of this process is being performed.

In the experiment $\gamma + z \rightarrow \ell^+ + \ell^- + z^*$, the polarizations of ℓ^+ and ℓ^- have similar correlations to the process $e^+ + e^- \rightarrow \ell^+ + \ell^-$. The detail of these correlations have never been investigated.

VI. ACKNOWLEDGEMENTS

The author wishes to thank Drs. J. D. Bjorken, M. Davier, A. Odian, F. Villa and D. R. Yennie for useful conversations. He also wishes to thank Dr. J. J. Sakurai for a preprint of his work which partially overlaps with the contents of our sections II and III.

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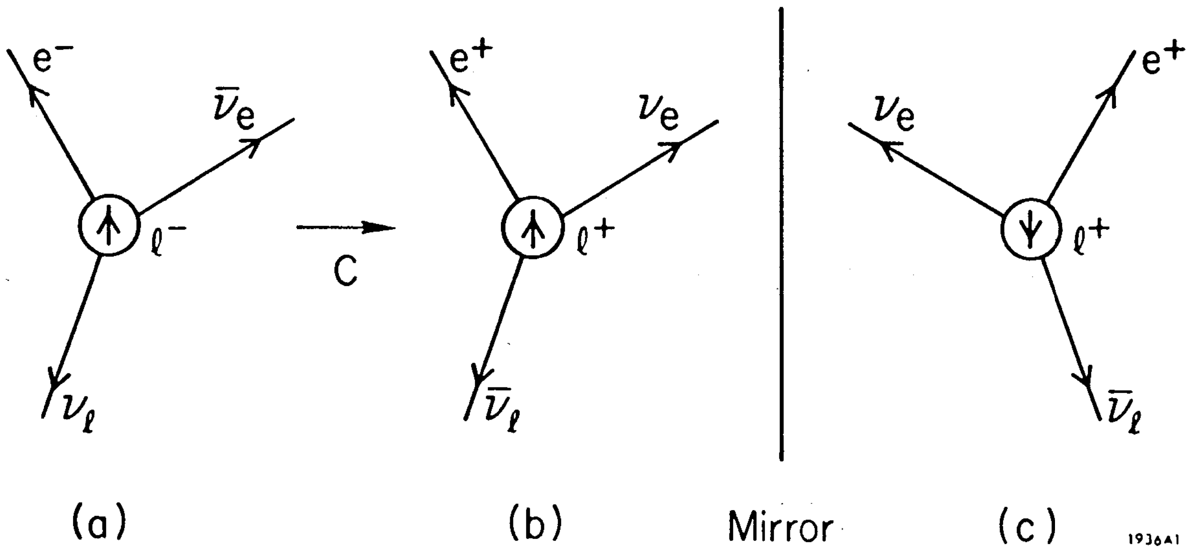
TABLE II

Decay rate (10^{10} sec^{-1}) = ($\Gamma/\hbar$) = $1/\tau$

M_ℓ (GeV)		0.6	0.8	0.938	1.2	1.8	3.0	6.0
Decay Mode								
ℓ	$\nu_\ell + \nu_e + e$	0.266	1.12	2.46	8.5	64.6	831	26600
	$\nu_\ell + \nu_\mu + \mu$	0.2	0.96	2.21	7.97	63	823	26533
	$\pi + \nu_\ell$	1.02	2.57	4.17	9.0	30	143	1145
	$k + \nu_\ell$	0.0092	0.09	0.2	0.55	2.3	11.7	98
	$\rho + \nu_\ell$	0	0.21	3.8	19	96	486	3900
	$k^* + \nu_\ell$	0	0	0.03	0.96	6.3	33	280
	$A_1 + \nu_\ell$	0	0	0	0.6	33.7	364	1550
	$Q + \nu_\ell$	0	0	0	0	0.17	15.2	133
	ν + hadron ℓ continuum	0	0	0	0.5	27	737	25900
ν + hadrons		1.03	2.87	8.2	29.6	195	1790	33006
Total Rate		1.5	3.95	12.9	46.1	323	3444	85539
Decay length in cm at E_ℓ = 5 GeV		16.5	4.7	1.2	0.26	0.024	--	--
Decay length in cm at E_ℓ = 50 GeV		167	48	12.2	2.7	0.257	0.0145	--

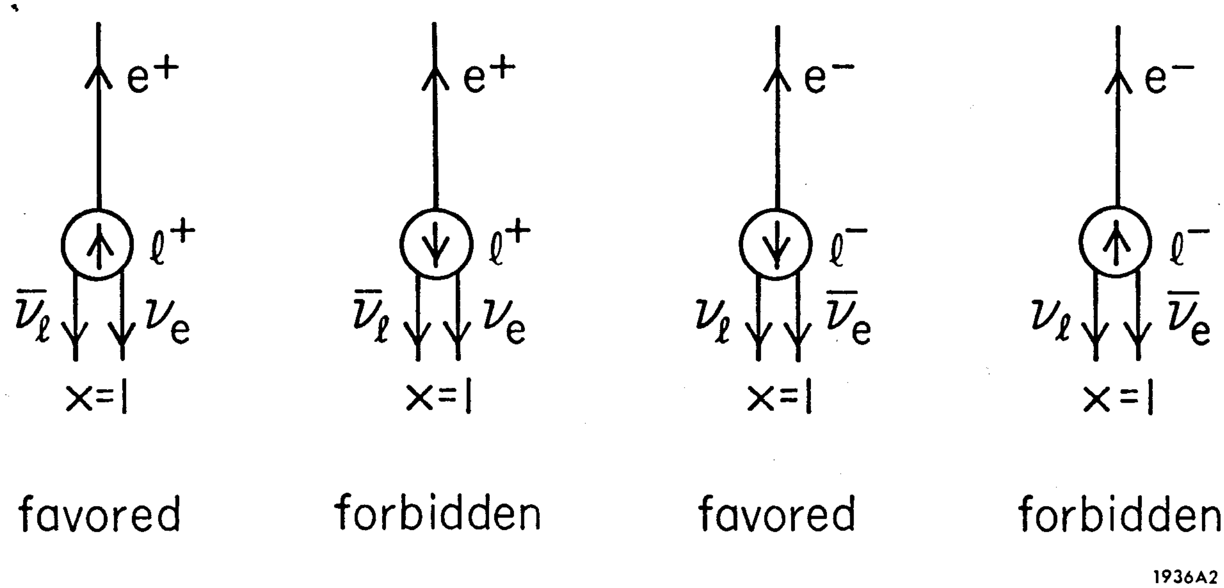
FIGURE CAPTIONS

1. (a) is an arbitrary energy-angle distribution of decay products of a polarized ℓ^-. (b) is a charge conjugate of (a) and is physically unrealizable because e^+, ν_e and $\bar{\nu}_\ell$ have wrong helicities. (c) is a mirror image of (b) and is physically realizable. Since the decay is invariant under CP, the probability of (c) is equal to the probability of (a). These figures show that the decay energy angle distribution of a polarized ℓ^+ can be obtained from that of a polarized ℓ^- by changing the sign of the polarization vector.
2. Both neutrinos must come out in the opposite direction to the direction of the electron when the electron has the maximum allowed energy. Neutrino and antineutrino have opposite helicities, therefore the z component of their total angular momentum must be zero. e^+ has a positive helicity and e^- has a negative helicity when their mass can be ignored. Thus e^+ likes to be emitted in the direction of the spin of ℓ^+ whereas e^- likes to be emitted opposite to the direction of the spin of ℓ^- when x is near 1.
3. These two diagrams show why the decay $\ell^- \rightarrow \pi^- \nu_\ell$ can be calculated from the knowledge of the decay $\pi^- \rightarrow \mu^- \bar{\nu}_\mu$.
4. Coordinate system and notations used in the calculation of the cross section of $e^+ + e^- \rightarrow \ell^+ + \ell^-$.



e^-
$\bar{\nu}_e$
ℓ^-
ν_ℓ
C
e^+
ν_e
ℓ^+
$\bar{\nu}_\ell$
ν_e
e^+
ℓ^+
$\bar{\nu}_\ell$
(a)
(b)
Mirror
(c)
1936A1

Fig. 1


e+
ℓ+
$\bar{\nu}_\ell$
ν_e
x=1
favored
e+
ℓ+
$\bar{\nu}_\ell$
ν_e
x=1
forbidden
e−
ℓ−
ν_ℓ
$\bar{\nu}_e$
x=1
favored
e−
ℓ−
ν_ℓ
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forbidden
1936A2

Fig. 2

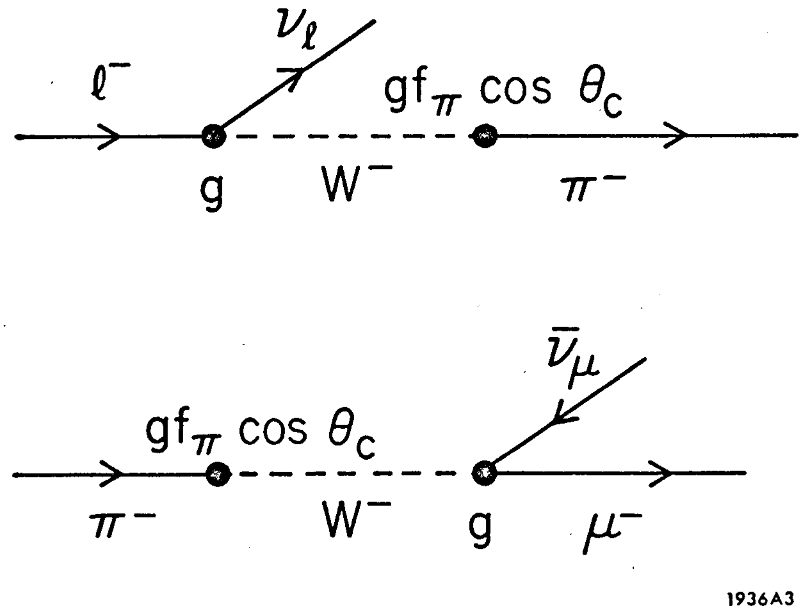
ν_ℓ
ℓ^-
$gf_\pi \cos\theta_c$
g
W^-
π^-
$\bar{\nu}_\mu$
$gf_\pi \cos\theta_c$
π^-
W^-
g
μ^-
1936A3

Fig. 3

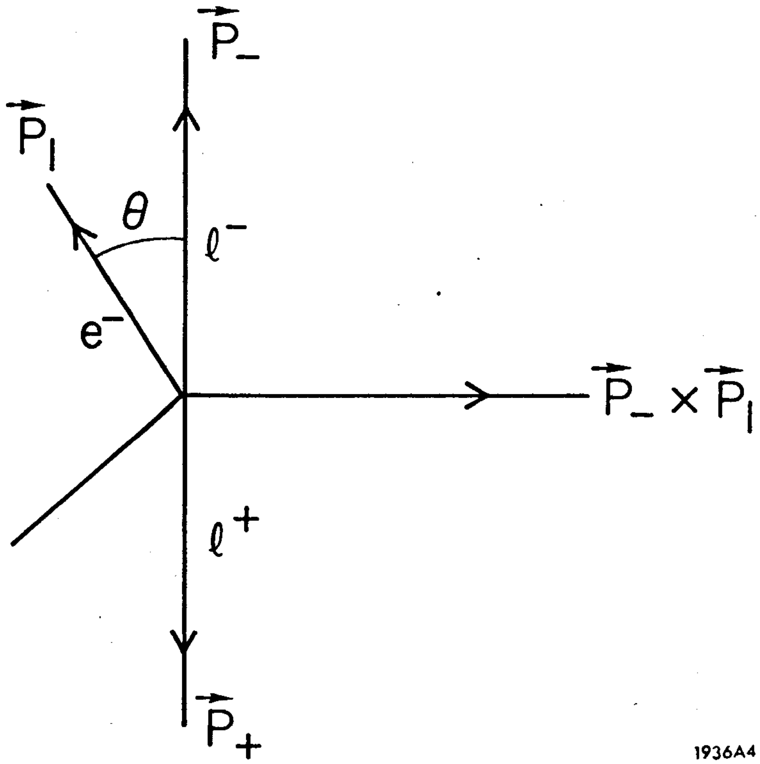

Fig. 4