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CONCENTRIC SPHERICAL CAVITIES AND LIMITS ON THE PHOTON REST MASS*

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ABSTRACT

The concentric spherical cavity is used as a vehicle for study of the complete breakdown of the "relativistic particle in a box" formula for the photon rest mass effect. The results are applied to a determination of the limit on the photon's rest mass which can be inferred from the Schumann resonances.

We consider a cavity consisting of the space between two concentric conducting spheres and study the dependence of its resonant frequencies upon an assumed photon rest mass for various values of the ratio of inner to outer radius. Our interest arises on the one hand from its relevance to a suggestion of H. Kendall, that the Schumann resonances in the cavity formed by the earth and the ionosphere[2] be used to set a limit in the photon's rest mass; and on the other, from its utility as a vehicle for exploring in detail the breakdown of the "relativistic particle in a box" formula for the photon rest mass effect.

For the case of a Klein-Gordon particle whose wave function is required to vanish at the boundary of a closed cavity, it is easy to show that the dependence of the energy upon the rest mass of the particle is given by the relation

$$k'^2 = k_0^2 + \kappa^2 \tag{1}$$

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for every mode of the cavity. In Eq. (1), $\hbar k_0 c$ is the energy of a zero rest mass particle for a particular mode and $\hbar k'c$ the energy of the corresponding mode for a particle with rest mass $\hbar\kappa/c$. We shall refer to Eq. (1) as the "Relativistic Particle in a Box" (RPB) formula. While it has been generally assumed[3] that the RPB formula would also hold for photons in a perfectly conducting cavity, it has recently been shown[4,5] that it is not generally valid. For purposes of discussion it is convenient to parameterize a corrected form of Eq. (1), valid for $(k'^2-k_0^2)/k_0^2 \ll 1$, by defining an in general complex "mass sensitivity coefficient", g, via the equation

$$k'^2 = k_0^2 + g\kappa^2 \tag{2}$$

The case of the spherical cavity was briefly discussed in Ref. 4, where it was pointed out that the corrections to Eq. (1) either vanish or are very small. Thus $g \approx 1$ for that case. The situation for concentric spheres can be expected to be different, however. This circumstance is due to the existence of a class of modes ("special" modes) for which k_0 remains finite when the inner (a) radius approaches the outer radius (b).[6] Since this behavior has no counterpart for the RPB, there is no reason to expect any validity for the RPB formula. Indeed we expect on the basis of the discussion in Ref. 4 that g will vanish as $a \to b$. By studying the dependence of g upon the ratio a/b one can observe the full breakdown of the RPB formula in a situation subject to exact analytic solution. We assume exp $ik'ct$ time dependence for all fields, use spherical coordinates (v, θ, ϕ), and consider only ϕ independent modes.[7]

It was shown in Ref. 4 that magnetic multipole modes for a spherical cavity satisfy Eq. (1). By the same argument this continues to hold for the concentric sphere case. For all of these modes $k_0 \to \infty$ as $a \to b$ so that all are regular.

Turning to the electric multipole case, we assume perfect conductivity for $r < a$ and $r > b$.[8] Then with $k'^2 = k^2 + \kappa^2$, we have

$$\left.\begin{aligned} V &= V_1 = -i\alpha_1 j_n(k'r)\, P_n(\cos\theta) \\ \underset{\sim}{A} &= \underset{\sim}{A}_1 = \frac{i}{k'}\underset{\sim}{\nabla} V_1 \end{aligned}\right\} \quad r < a \tag{3}$$

$$\left.\begin{aligned} V &= V_2 = -i\left[\alpha_\ell j_n(kr) + \beta_\ell y_n(kr)\right] P_n(\cos\theta) \\ \underset{\sim}{A}_T &= \frac{1}{k}\underset{\sim}{\nabla}\times\hat{\phi}\left[\alpha_t j_n(kr) + \beta_t y_n(kr)\right]\frac{dP_n(\cos\theta)}{d\theta} \\ \underset{\sim}{A} &= \underset{\sim}{A}_2 = \underset{\sim}{A}_T + i\,\frac{k'}{k^2}\underset{\sim}{\nabla} V_2 \end{aligned}\right\} \quad a < r < b \tag{4}$$

$$\left.\begin{aligned} V &= V_3 = -i\,\alpha_2 h_n^{(2)}(k'r)\, P_n(\cos\theta) \\ \underset{\sim}{A} &= \underset{\sim}{A}_3 = \frac{i}{k'}\underset{\sim}{\nabla} V_3 \end{aligned}\right\} \quad r > b \tag{5}$$

In Eqs. (3), (4), (5), j_n, y_n, $h_n^{(2)}$ are the usual spherical Bessel functions,[9] and n is a positive interger. The six constants $(\alpha_1, \alpha_2, \alpha_\ell, \beta_\ell, \alpha_t, \beta_t)$ are to be determined by the six boundary matching conditions $V_1=V_2$, $\underset{\sim}{A}_1=\underset{\sim}{A}_2$ at $r=a$, and $V_2=V_3$, $\underset{\sim}{A}_2=\underset{\sim}{A}_3$ at $r=b$. The six resultant linear homogeneous equations have nontrivial solutions only if the determinant, Δ, of the coefficients vanish. The condition $\Delta=0$ thus determines the eigenvalues k.[10]

At $\kappa=0$ the eigenvalue condition reduces to the familiar relation

$$\omega_n(k_0a)\, z_n(k_0b) - z_n(k_0a)\, \omega_n(k_0b) = 0 \tag{6}$$

where

$$\begin{aligned} \omega_n(x) &\equiv \frac{d}{dx}(x\, j_n(x)) \\ z_n(x) &\equiv \frac{d}{dx}(x\, y_n(x)) \end{aligned} \tag{7}$$

The lowest root of Eq. (6) is, for each value of n, special. Indeed, k_0 <u>decreases</u> as $a \to b$, and the limiting value as $a \to b$ is given by[2]

$$k_0\frac{(a+b)}{2} = \sqrt{n(n+1)} \tag{8}$$

The behavior of $k_0 b$ as a function of a/b is shown for the n=1 case in Fig. 1. The special modes are essentially TEM modes propagating in the narrow gap between the spheres. The remaining roots of Eq. (6) are all regular.

At $\kappa \neq 0$ the eigenvalue condition is too complicated to be usefully written out explicitly. It is, however, clearly different from Eq. (6) and hence k differs from k_0. It then follows that the RPB formula is violated in every case. To study this violation we assume κ^2 small and define g (a/b) by the relation

$$g\left(\frac{a}{b}\right) = 1 + \lim_{\kappa^2 \to 0} \frac{k_0^2 - k^2}{\kappa^2} \quad . \tag{9}$$

The quantities, g, k_0, and k of course depends upon the mode which is being scrutinized. A lengthy formula for g has been derived (see appendix). It has the following limiting forms:

$$\begin{aligned} g(0) &= 1 + \frac{2n(n+1)\,(k_0 b)\, j_n(k_0 b)}{(k_0 b)^2 - n(n+1)} \left(y_n(k_0 b) + i\, j_n(k_0 b)\right) \\ &= .9856 + .298\, i \quad ; \quad \text{for the first } n=1 \text{ eigenvalue} \end{aligned} \tag{10}$$

For the special modes in the case a ~ b we have the linear approximation

$$g\left(\frac{a}{b}\right) = \bar{g}(n) \left(1 - \frac{a}{b}\right) \tag{11}$$

where

$$\bar{g}(n) = \nu^3\, j_n'(\nu) \left[y_n'(\nu) + i\, j_n'(\nu)\right] \tag{12}$$

with $\nu^2 = n(n+1)$. The full behavior of g for the special n=1 mode is shown in Fig. 1 and values of the constants $\bar{g}(n)$ are tabulated in Table 1. Evidently g does indeed vanish for each special mode as anticipated. For the regular modes $g \to 1$ as $a \to b$.

TABLE 1

Slope of the mass sensitivity coefficient at a/b = 1 for the first five modes.

Mode Number n	Re $\bar{g}$ (n)	Im $\bar{g}$ (n)
1	.4362	.0689
2	.7204	.1734
3	.9550	.2777
4	1.1623	.3778
5	1.3515	.4734

We now turn to an exploratory consideration of the bearing which our results may have open the possibility of setting a limit on the photon rest mass by means of the Schumann resonances. The observed Schumann resonances are approximately twenty percent lower in frequency than the values predicted by the infinite conductivity formula and the Q's are of the order of five. It is evident that the finite conductivity of the ionosphere has an important effect. There are additional complications arising from the effect of the earth's magnetic field upon the conductivity, and the asymmetry introduced by the diurnal variation in ionospheric properties. A reliable assessment of the effect of a photon rest mass could be obtained by repeating the elaborate analyses which has been performed with a rest mass included. We shall, however, confine ourselves here to a crude approximation which consists of applying Eq. (2) using for g the value determined in the infinite conductivity limit and using for the ionospheric height a nominal 70 kilometers. We thus obtain Re g = .00486 for the first mode. We note that the correction to the RPB formula degrades the efficacy of this means of setting a rest mass limit by a factor 14.

The results of measurements of the Schumann resonances by a number of observers have been summarized by Madden and Thompson.[12] Their results are given in the first two lines of Table 2. For any assumed value of the rest mass parameter, κ, one can use Eq. (2) and Table 1 to determine what the resonant frequencies, f_0, quality factors, Q_0 would then be in the absence of a photon rest mass. It is these values which should then be compared with the predictions of a theoretical treatment of the earth ionospheric system based upon Maxwell's equations. The third and fourth lines show $f-f_0$ and $\frac{1}{Q}-\frac{1}{Q_0}$ expressed in terms of $\delta f_1 = f-f_0$ for the first mode. The last two lines show f_0 and Q_0 for an assumed $\delta f_1 = 1$ Hz. Finally we note that Eqs. (2) and (11) imply

$$\frac{1}{\kappa} = \lambda\!\!\!^{-}_p = \frac{\hbar}{m_p c} = \frac{8.3 \times 10^7 \text{ cm}}{\sqrt{\delta f_1}} \tag{13}$$

On the basis of the theoretical analyses of Ref. 1 we judge (to a precision appropriate to our crude analysis) that $\delta f_1 > 1$ Hz is inconsistent with existing knowledge of the ionosphere[12] and hence that $\lambda\!\!\!^{-}_p > 8.3 \times 10^7$ cm. This limit (which might, for the present, be halved, to be more conservative about δf_1 and to take account of uncertainty in g) is a factor 100 poorer than the current best limit, which is based upon geomagnetic measurements.[13] On the other hand, it is about twenty times better than any other determination in which wave propagation properties play a role.[14]

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APPENDIX

For the interested reader we exhibit the complete expression for g. To keep the expressions to a reasonable length we define a number of auxiliary quantities.

$$u = k_0 a \,, \qquad v = k_0 b \qquad \text{with } u \text{ and } v \text{ related by}$$

$$\omega_n(u)\, z_n(v) - \omega_n(v)\, z_n(u) = 0$$

$$A_n(u,v) \equiv j_n(u)\, y_n(v) - j_n(v)\, y_n(u)$$

$$B_n(u,v) \equiv j_n(u)\, y_n'(v) - y_n(u)\, j_n'(v)$$

$$C_n(u,v) \equiv j_n(u)\, \omega_n(v) - y_n(u)\, z_n(v)$$

$$D_n(u,v) \equiv B_n(u,v)\, j_n'(u)/j_n(u) - B_n(v,u)\, h_n^{(2)\,'}(v)/h_n^{(2)}(v)$$

$$N_n(u,v) \equiv D_n(u,v) + \frac{A_n(u,v) - C_n(u,v) + C_n(v,u)}{uv}\left[u\,\frac{j_n'(u)}{j_n(u)} + v\,\frac{h_n^{(2)\,'}(v)}{h_n(v)} - \frac{C_n(u,v) - C_n(v,u)}{A_n(u,v)}\right]$$

$$M_n(u,v) \equiv D_n(u,v) - \frac{A_n(u,v) - C_n(u,v) + C_n(v,u)}{uv} - A_n(u,v)\,\frac{j_n'(u)\, h_n^{(2)\,'}(v)}{j_n(u)\, h_n^{(2)}(v)}$$

and finally

$$g-1 = \frac{2n(n+1)\, A_n(u,v)\, N_n(u,v)}{M_n(u,v)\left[(u^2 - n(n+1))\, C_n(u,v) - (v^2 - n(n+1))\, C_n(v,u)\right]}$$

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2. W. D. Schumann, Z. Naturforschung 7A, 149-154 (1952a).

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4. N. M. Kroll, Phys. Rev. Letters 26, 1395 (1971).

5. A. S. Goldhaber and M. M. Nieto, Phys. Rev. Letters 26, 1390 (1971); E. Williams and D. Park, Phys. Rev. Letters 26, 1393 (1971).

6. For convenience, we think of b as fixed. For the remaining modes ("regular" modes), $k \rightarrow \infty$ as $a \rightarrow b$. All of the RPB modes are "regular" in this sense.

7. On account of the well known "m" degeneracy of spherical symmetry, all resonant frequencies are found in this way.

8. The behavior of the potentials in a perfect conductor as well as the appropriate boundary conditions are discussed in Ref. 4.

9. Our notation follows that of Handbook of Mathematical Functions (U. S. Dept. of Commerce, 1964), Section 10.1.

10. The eigenvalue conditions determine both electric multipoles, corresponding to $\kappa=0$ modes; and longitudinal modes, which have no $\kappa=0$ counterpart. Longitudinal modes are, to a good approximation, gauge transformations and hence couple exceedingly weakly to charge-current distributions. Because they pass through the conductor freely one anticipates $\mathrm{Im}\, k \approx \frac{1}{b}$ for the eigenvalues. Furthermore, as $\kappa \rightarrow 0$ these eigenvalues must become infinite, and preliminary investigation suggests that $\mathrm{Im}\, k \approx \frac{1}{b} \ln(1/\kappa b)$ is typical. It is clear from these comments that the longitudinal modes have no bearing on the principal subject of this paper.

11. T. Madden and W. Thompson, Reviews of Geophysics 3, 211-254 (1965).

12. This judgement is based upon the values listed in Tables 7 and 8 and upon the discussion of the WKB and SID models on pages 232-234 (Ref. 11). We note that models such as the WKB which reduce the frequency of the lowest modes also reduce Q_0, while the rest mass effect increases Q_0.

13. A. S. Goldhaber and M. M. Nieto, Phys. Rev. Letters 21, 568 (1968).

14. G. Feinberg, Science 166, 879 (1969).

TABLE 2

Summary of information for assessing the limit on the photon rest mass which can be inferred from the Schumann resonances.

Mode Number n	1	2	3	4
Measured frequency f (Hz)	8.0	14.0	20.0	26.5
Measured Q	4	5	5	6
$\delta f = f - f_0$	δf_1	$.96\ \delta f_1$	$.88\ \delta f_1$	$.82\ \delta f_1$
$Q^{-1} - Q_0^{-1}$	$.042\ \delta f_1$	$.030\ \delta f_1$	$.028\ \delta f_1$	$.022\ \delta f_1$
f_0 *	7.0	13.0	19.1	25.7
Q_0 *	4.8	5.9	5.8	6.9

*Assuming $\delta f_1 = 1$ Hz.

FIGURE CAPTION

1. The mass sensitivity coefficient and resonant frequency of the lowest frequency mode of a concentric spherical cavity as a function of the ratio of radii (a/b).

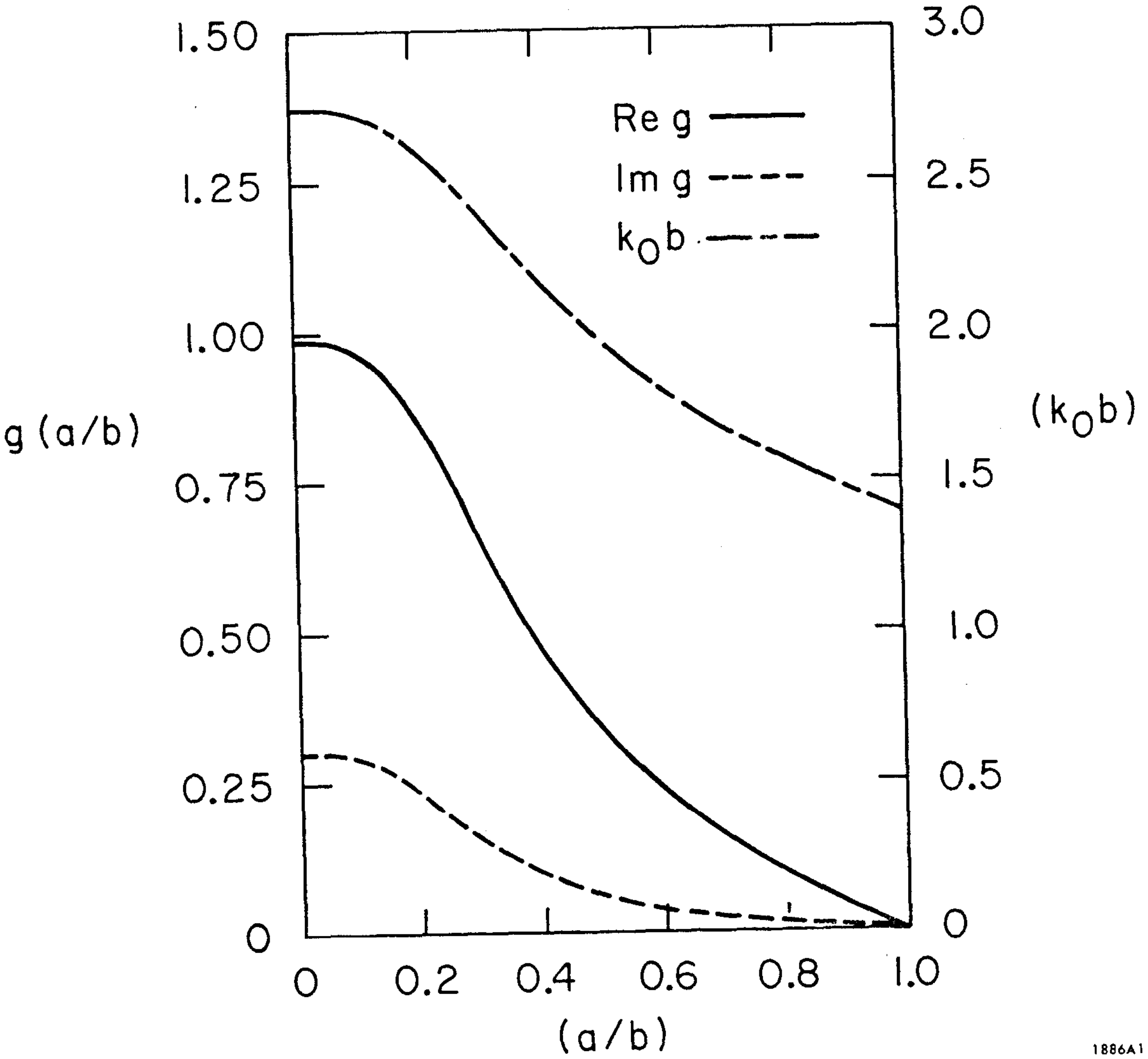
1.50
1.25
1.00
0.75
0.50
0.25
0
g (a/b)
3.0
2.5
2.0
1.5
1.0
0.5
0
(k_0b)
Re g
Im g
k_0b
0
0.2
0.4
0.6
0.8
1.0
(a/b)
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Fig. 1