(TH) and (EXP)

SLAC-PUB-895
April 1971

THEORETICAL INTERPRETATION OF A RECENT EXPERIMENTAL INVESTIGATION OF THE PHOTON REST MASS*

Norman M. Kroll†

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

ABSTRACT

It is shown that both small dimensional extremely low frequency resonant circuits, and long coaxial cables are unsuited to the setting of significant limits on the photon rest mass.

(Submitted to Phys. Rev. Letters.)

*Work supported by the U. S. Atomic Energy Commission.

†On temporary leave from the Department of Physics, University of California, San Diego, La Jolla, California.

P. A. Franken and G. W. Ampulski[1] have recently proposed that an upper limit on the photon rest mass be determined by investigation of the properties of low frequency resonant circuits. Furthermore, they have reported a measurement based upon their proposal which, even in its preliminary form provides, according to their interpretation, a factor 30 improvement over the previous limit. The authors emphasize the speculative character of their theoretical interpretation, and it is the purpose of this note to show that their reservations in this matter are, unfortunately, well founded. It is our view that their experiment yields no significant information on the photon rest mass.

Goldhaber and Nieto[2] have emphasized the uniqueness of the Proca equations for describing modifications to Maxwell's equations implied by a photon rest mass. With the assumption of harmonic time dependence exp ik'ct these may be written

$$\nabla^2 \underset{\sim}{A} + k^2 \underset{\sim}{A} = -\underset{\sim}{j}/c \quad (1) ; \qquad \nabla^2 V + k^2 V = -\rho \quad (2)$$

$$\nabla \cdot A + ik'V = 0 \quad (3) ; \qquad \nabla \cdot j + ik'\rho = 0 \quad (4)$$

$$\underset{\sim}{E} \equiv -\underset{\sim}{\nabla} V - ik'\underset{\sim}{A} \quad (5) ; \qquad \underset{\sim}{B} \equiv \underset{\sim}{\nabla} \times \underset{\sim}{A} \quad (6)$$

$$\nabla \times \underset{\sim}{E} = -ik'\underset{\sim}{B} \quad (7) ; \qquad k^2 \equiv k'^2 - \kappa^2 \equiv -q^2 \quad (8)$$

Equations (1) and (2) can be integrated to yield

$$\underset{\sim}{A}(r) = \frac{1}{4\pi c} \int \frac{\underset{\sim}{j}(r')\, e^{-q|r-r'|}}{|r-r'|}\, dr' \qquad (9)$$

$$V(r) = \frac{1}{4\pi} \int \frac{\rho(r')\, e^{-q|r-r'|}}{|r-r'|}\, dr' \qquad (10)$$

Equations (9, 10) can be used to derive the quasi-static conventional lumped parameter or electric circuit treatment of electromagnetism. In the case of fine wire inductances, the distribution of the quasi-steady current is known and Eq. (9) allows a direct calculation of the self inductance via the usual definition: Flux linked $\equiv \oint \underset{\sim}{A} \cdot d\underset{\sim}{\ell} = cLi$. In the case of $\kappa = 0$ electromagnetism one sets q=0 and asserts that the effect of time dependence is small provided qR is small, where R is some size parameter of the system. This condition sets the limit of the quasi-static approximation. For $\kappa \neq 0$ in the static limit,[3] we have $q=\kappa$ but the same remark applies. Thus $L(\kappa) \approx L(q) \approx L(0)$ to order κR or qR.

For the case of capacitance it is simplest if we confine our attention to a symmetric pair of conductors in the form of a parallel plate capacitor with very small plate separation and carrying charge Q and -Q respectively. In this case we argue that the charge distribution will be essentially uniform for small qR or κR just as it is in the case of $\kappa=0$ electromagnetism. From $Q=C(V_1-V_2)$ we conclude the $C(\kappa) \approx C(q) \approx C(0)$ again to order κR or qR.

We now imagine the inductance and capacitor connected together, apply current conservation Eq. (4), and Faraday's law Eq. (7), to obtain

$$-L(q)\ \ddot{Q} = \frac{Q}{C(q)} \tag{11}$$

whence

$$\omega'^2(q) = \frac{1}{L(q)\ C(q)} \tag{12}$$

We emphasize that the κ dependent corrections to ω' are of order κR or $\frac{\kappa}{k'}\ \kappa R$ accordingly as $k' \ll \kappa$ or $k' \gg \kappa$ respectively.[4]

The theoretical conjecture of Ref. 1 is based upon an analogy with rectangular cavity resonators, for which it is claimed that the relation

$$k'^2 = k_0^2 + \kappa^2 \tag{13}$$

holds rigorously. Here k_0c is the resonant frequency for massless photons. In order to clarify the status of Eq. (13) and its relation to the quasi-static situation, we have carried out a study of the theory of transmission lines, wave guides, and cavities for the Proca equation. We shall find that, while Eq. (13) holds for some modes of some cavities together with an analogue for some modes of some waveguides, it has no general validity and indeed fails for all modes of the rectangular cavity case. Another particularly conspicuous and significant failure occurs in the case of the TEM modes of multiconductor transmission lines (such as coaxial cables). Propagation in such modes occurs at phase velocity c regardless of the magnitude of κ^2.

It is useful to begin with an incorrect derivation of Eq. (13). Inside the cavity $j=\rho=0$. Equations (2) and (3) are satisfied by V=0, $\nabla \cdot A=0$. The vanishing of $\underset{\sim}{E}\times\hat{n}$ on the boundaries then requires $\underset{\sim}{A}\times\hat{n}$ vanish on the boundaries. Formulated in this way the problem of determining the eigenvalues, k^2, of Eq. (1) appears to be independent of κ^2, so that k^2 has precisely the values it has for conventional, zero mass, electromagnetic theory. Equation (13) is an immediate consequence.[5] That this derivation is incorrect becomes apparent on noting that while the charges and currents vanish inside the cavity, there are, nevertheless, charges and currents in the cavity walls. Equation (10) then tells us that V cannot vanish. In the Proca theory the potentials as well as the fields are physical quantities, and it is necessary to construct solutions throughout space which are consistent with Eqs. (9, 10).

To proceed, it is necessary to properly formulate the boundary value problem. As in the case of $\kappa=0$ electromagnetism, ρ and $\underset{\sim}{j}$ will be no more singular than surface distributions. Equations (9, 10) then imply that V and $\underset{\sim}{A}$ are <u>everywhere</u> continuous. Discontinuities in the derivatives of V and $\underset{\sim}{A}$ occur at surface distributions of charge and current.

We next discuss solutions in a medium of conductivity σ. Equations (1), (2), (3) and (4) are supplemented by

$$\underset{\sim}{j} = \sigma \underset{\sim}{E} = -\sigma \underset{\sim}{\nabla} V - ik'\sigma \underset{\sim}{A} \tag{14}$$

Writing

$$\underset{\sim}{A} = \underset{\sim}{A}_T + \underset{\sim}{A}_\ell \tag{15}$$

with $\nabla \cdot \underset{\sim}{A}_T = 0$ and

$$\underset{\sim}{A}_\ell = \frac{ik'}{k_\ell^2} \underset{\sim}{\nabla} V \tag{16}$$

$$k_\ell^2 = k'^2 - \frac{ik'c\kappa^2}{\sigma + ik'c} \tag{17}$$

we find that (1), (2) and (3) are satisfied if

$$\nabla^2 \underset{\sim}{A}_T + \left(k^2 - \frac{i\sigma k'}{c}\right) \underset{\sim}{A}_T = 0 \tag{18}$$

$$\nabla^2 V + k_\ell^2 V = 0 \tag{19}$$

In the limit $\sigma \to \infty$, $k_\ell^2 = k'^2$ and $A_T = 0$. Thus we have

$$\underset{\sim}{A} = \underset{\sim}{A}_\ell = \frac{i}{k'} \underset{\sim}{\nabla} V \tag{20}$$

$$\nabla^2 V + k'^2 V = 0 \tag{21}$$

Thus the potentials propagate unattenuated at the speed of light through a perfect conductor! Equation (20) implies via (5) and (6) that E and B both vanish. Nevertheless these potentials have physical significance. They carry energy and, on emergence from the conductor, will generate electromagnetic fields.

For the discussion of cavities and waveguides the appropriate boundary conditions are: V, $\underset{\sim}{A}$ continuous at boundary, Eqs. (20) and (21) inside the conductor ($\sigma = \infty$ case); or V, $\underset{\sim}{A}$, and derivatives of $\underset{\sim}{A}$ continuous at boundary, Eqs. (15), (16), (17), (18), (19) inside the conductor (σ finite).[7] Outgoing wave conditions are imposed at infinity. To avoid a multitude of boundaries we shall, at times, assume a uniform conducting medium extending to infinity.[8] For the most part we take $\sigma = \infty$.

Wave Guides

We assume a closed cylindrical boundary oriented along the z axis. The TM and TEM modes can be discussed in general. For both cases we may write

$$\underset{\sim}{A} = \frac{k'}{h} \psi(x, y)\, e^{-ihz}\, \hat{z} \tag{22}$$

$$V = \psi(x,y)\, e^{-ihz} \tag{23}$$

Equation (1), (2), and (3) are satisfied provided

$$\nabla^2 \psi + k_c^2 \psi = 0 \tag{24}$$

$$k_c^2 = k^2 - h^2 \tag{25}$$

The TM modes are obtained by requiring ψ to vanish on the boundary. $\underline{A}$=V=0 outside obviously satisfies the boundary conditions. This condition, together with (24) and (25) are identical to the $\kappa=0$ TM waveguide conditions. Hence Eq. (13') holds, viz

$$k'^2 = k_{c0}^2 + h^2 + \kappa^2 \tag{13'}$$

Two (or more) conductor systems support TEM modes. One may imagine a two wire system or a coaxial cable. For definiteness consider a two wire system. If $\psi=0$ on both conductors then we are back to the TM case. If $\psi\neq 0$, then $V\neq 0$ on at least one conductor. Since $\underline{A}$ is parallel to the z axis, Eq. (20) tells us that ψ is constant inside the conductor. But then Eq. (21) requires h=k' and hence $k_c^2=-\kappa^2$. As a check we note that the equation

$$E_z = \left(-\underline{\nabla}\psi - i\,\frac{k_c^2+\kappa^2}{h}\,\psi\hat{z}\right) e^{-ihz} \tag{26}$$

also requires $k_c^2=-\kappa^2$ if ψ is constant rather than zero on a boundary. The equation $\nabla^2\psi=\kappa^2\psi$ with ψ given by different constant values on two boundaries always has solutions. We see that TEM modes violate Eq. (13') and indeed propagate with velocity c independent of κ^2!

In order to become convinced that this result was not associated with an "unphysical" $\sigma=\infty$ limit we have also examined the finite conductivity solution for the case of plane parallel geometry. The propagation now includes attenuation, but is still found to be independent of κ^2. (In contrast to the $\sigma=\infty$ case this result is not exact. It depends upon the assumption that both κ and the attenuation constant are small compared to the inverse lateral dimension.)

It is useful to recall that transmission lines are often discussed in the context of inductance and capacitance per unit length. It is easy to see, using the explicit solutions obtained, that corrections to these quantities are of order κR and appear in such a way as to precisely cancel in the LC product which determines the propagation velocity.

The TE modes can not be discussed with generality. However, they typically violate Eq. (13'). As an example, we have studied the case of a circular cross section. One finds that the pure TE character is lost. The eigenvalue condition is found to be

$$J'(k_c a) = -\frac{n^2}{(k_c a)^2} \frac{\kappa^2 J_n(k_c a)}{\kappa^2 + k_c^2} \frac{k'^2 H_n^{(2)}(k_c a)\, J_n(k_c a)}{k^2 H_n^{(2)'}(k_c a)\, J_n(k_c a) - k'^2 H_2^{(2)}(k_c a) J_n'(k_c a)} \tag{27}$$

where n is the angular mode number. For n=0 one gets the $\kappa^2=0$ equation and hence Eq. (13') holds. Otherwise it fails but the corrections are small. We find for the TE_{11} mode, $\delta k_c^2 = (-.15 + .44i)\kappa^2$. Note that this result implies attenuation, corresponding to radiation through the perfect conductor!

<u>Cavity Resonators</u>

In $\kappa^2=0$ electromagnetism one obtains exact solutions of cavity problems by terminating waveguides with conductors. This procedure is no longer exact for $\kappa^2 \neq 0$, although the presumption $h = \frac{2n}{L}$ is probably an excellent approximation, especially for large n.[9]

In order to study a problem capable of exact solution, we have considered the case of a spherical cavity. Because of the m degeneracy of spheres it is sufficient to consider the m=0 modes, (i.e., invariant under rotations about the z axis). For the magnetic multipoles one may take V=0, $\underset{\sim}{A} = \hat{\phi}\,\psi(r, \theta)$. The condition $\underset{\sim}{A}=V=0$ outside together with $\psi(a, \theta)=0$ satisfies all continuity conditions. Again the situation is identical to that at $\kappa^2=0$ and Eq. (13) holds. For the electric multipoles an equation analogous to Eq. (27) has been derived. Hence Eq. (13) fails but the corrections to it are very small. For these modes the cavity radiates through the perfect conductor, and analogous to Eq. (27), the eigenvalue is complex, yielding a finite cavity Q.[10]

The fact that Eqs. (13) and (13') fail for TE and electric multipole modes is in satisfactory agreement with our quasi-static treatment, as it is these modes which "deform" into quasi-static modes as the guide or cavity is deformed into a reentrant shape.

We conclude with the general comment that photon mass effects appear to be of the order κR or $\kappa c\tau$, where R or $c\tau$ is the shortest length which is relevant to the "apprehension" or control of the photon. This result is somewhat prejudicial to table top experiments of less than exquisite sensitivity.

REFERENCES

1. P. A. Franken and G. W. Ampulski, Phys. Rev. Letters 26, 115 (1971).
2. See A. S. Goldhaber and M. M. Nieto, "Terrestrial and extraterrestrial limits on the photon mass" (to be published).
3. We note that Eq. (4) of Ref. 1 would seem to imply for zero resistance $\omega' > \omega_c^2$ (their notation), and hence nonexistance of static solutions. On the other hand the interpretation of the geomagnetic measurements is based upon static solutions.
4. We have omitted in our discussion, the fact that for $k' \neq 0$, $\rho \neq 0$ in the inductance and $\underset{\sim}{j} \neq 0$ in the capacitor. The error is of order $k'R$.
5. Since it is obviously possible to have a reentrant cavity with a capacitative gap of infinitesimal width and a resonant wavelength arbitrarily large compared to the spatial dimensions of the cavity, the above argument is in contradiction to the quasi-static argument.
6. Compare with M. E. Gertsenshtein and L. G. Solovei, Zh. ETF Pis. Red. 9, 137 (1969) (translation: JETP Letters 9, 79 (1969)). Our result agrees with theirs only in the limit $\sigma \ll k'c$. For typical conductors the opposite inequality always holds. While this difference has no effect on their conclusions, it is crucial for us. Their result implies V, A_ℓ vanish for $\sigma \to \infty$. We would be unable to satisfy our boundary conditions if this were actually the case. (Their Eq. (5) for $\underset{\sim}{E}$ is incorrect.)
7. For σ finite, $\underset{\sim}{j}$ must be finite. Hence there are no surface currents. On the other hand, for $\underset{\sim}{j} \cdot \hat{n} \neq 0$ there will be surface charges. Hence $\hat{n} \cdot \nabla V$ may be discontinuous. Not all of the specified boundary conditions are independent when one takes the equations of motion and the Lorentz condition into account.

8. The following "thin perfect conductor" boundary conditions also avoid a multitude of surfaces. Free space equations of motion are assumed on both sides of the boundary. Only V, A are required to be continuous ($\hat{n}\cdot\nabla(\hat{n}\cdot A)$ is then automatically continuous). In addition, one requires $\hat{n}\times E$ to vanish at the boundary. All of the "thick" perfect conductor problems discussed in the text, have also been treated as thin conductor problems. The results for the interior fields and eigenvalues are always similar, and to first order in κ^2, identical.

9. Combining this comment with our discussion of waveguides we see that for rectangular cavity modes with mode indices (ℓ, m, n), Eq. (13) is inexact, but probably an excellent approximation for $(\ell, m, n)\neq 0$. If, however, any of these indices vanish the frequency is essentially independent of κ^2.

10. H. Kendall (private communication) has suggested that the fact that the earth-ionosphere resonance has been observed sets some sort of limit on κ^2. It is probable that Eq. (13) has substantial corrections for the lowest mode of two concentric spheres but not so large as to eliminate the possibility of a useful determination. The evaluation of the effect, especially with realistic conductivities is straight forward but more tedious than we were willing to undertake.