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ACCURACY OF IBM'S SUBROUTINE GAUSS*

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ABSTRACT

Subroutine GAUSS is an IBM routine which computes a normally distributed random number with given mean and standard deviation. This routine is based upon an application of the central limit theorem and the use of the random number generator. In this note we will give a short explanation of how this routine works.

Since we have found that for Monte Carlo studies of tracking algorithms and propagation of errors that IBM's GAUSS is not quite sufficient, the main point of this note is to provide explicit bounds for the accuracy of GAUSS. The modification required in IBM's routine is trivial. Since the proof and development of the ideas necessary to establish the bounds are deep and computational, we will only indicate the development of the ideas and state the result without proof.

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I. THE CENTRAL LIMIT THEOREM (CLT)

The CLT states that the sum of independently identically distributed (IID) random variables tends to a normal distribution. The precise statement of the CLT is as follows:

Let $X_1, X_2, \ldots$ be a sequence of RV's which are IID. Let $E(X_i) = \mu$ and $\mathrm{Var}\,(X_i) = \sigma^2$, $\mu < \infty$, $\sigma^2 < \infty$. Let $S_n = \sum_{i=1}^{n} X_i$. Then for each Z,

$$\lim_{n \to \infty} P\left(\frac{S_n - n\mu}{\sqrt{n}} \leq Z\right) = \Phi(Z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{Z} \exp(-u^2/2)\, du.$$

The proof for a large class of distributions can be found in [1]. For an indication of the proof see [2]. For generalizations see [3].

The idea of the proof is to employ the linearity of characteristic functions, and the unique inversion property of characteristic functions.

II. APPLICATION OF CLT IN SUBROUTINE GAUSS

As is often the case in the application of limit theorems one merely applies the limit theorem with finite n. The subroutine uses $X_1, \ldots, X_n$ generated by the random number generator i.e.:

$$X_i = \begin{cases} 1, & \text{if } X \in [0,1] \\ 0, & \text{otherwise} \end{cases} \qquad i=1,n$$

Then, $E(X_i) = 1/2$, $E(X_i^2) = 1/3 \Rightarrow \mathrm{Var}(X_i) = 1/12$. Then applying CLT for this case we get:

$$Y = \frac{\sum_{i=1}^{n} X_i - \frac{n}{2}}{\sqrt{\frac{n}{12}}}, \qquad Y \to \Phi \text{ as } n \to \infty .$$

For computation reasons only IBM takes n=12 to obtain $Y = \sum_{i=1}^{12} X_i - 6$. Then the user mean, μ, and user standard deviation, σ, are introduced as follows:

$$Y' = Y\sigma + \mu$$

to obtain the desired normal distribution about μ with SD=σ.

III. HOW WELL DOES THE CENTRAL LIMIT WORK?

In the application of limit theorems what is really needed is an explicit bound for the remainder. Obtaining such bounds is often a much harder problem, since bounds demand detailed understanding of the process involved in the limit. That is to say limit theorems are really vague and general and appeal to those working in generalities. However, if one is working with actual computations remainders are needed.

<u>Theorem</u> If the random variables $X_1, \ldots, X_n, \ldots$ are IID such that $E(X_i) = \mu$, $Var(X_i) = \sigma^2 < \infty$, $E(X_i^3) = \beta < \infty$. Then if we put,

$$F_n(x) = P\left\{\frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}} \leq x\right\}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{u^2}{2}\right) du$$

Then,

$$\left|F_n(x) - \Phi(x)\right| \leq \frac{C\beta}{\sqrt{n}} \qquad C = 1.88$$

This result is known as the Berry-Essen Theorem ([4], [5]).

The proof of this theorem depends upon using Chebyshev-Hermite expansion of $F_n(x) - \Phi(x)$, and improving Lyapunov's Theorem:

$$\left|F_n(x) - \Phi(x)\right| < \frac{C\beta \log(n)}{\sqrt{n}}$$

IV. APPLICATION OF THE BERRY-ESSEN THEOREM TO IBM'S GAUSS

$\beta = \int_0^1 X^3 dx = .25$, $C = 1.88$ and for IBM's routine $n=12$. Therefore,

$$\left| F_n(x) - \Phi(x) \right| \leq \frac{(1.88)(.25)}{\sqrt{12}} = 0.14$$

Thus it is clear at the values of x where the upper bound is attained n=12 IBM is not sufficient. To alter IBM's routine merely use the Berry-Essen Theorem to compute adequate n and normalize properly according to the formula in II.

V. OTHER METHODS OF COMPUTING NORMAL DEVIATES

Several other methods for computing normal deviates have been studied [6], [7]. One of the most interesting of these is the direct approach of Box and Muller. Briefly this method is as follows: Let X_1, X_2 by independent random variables uniformly distributed on 0, 1]. Then it can be shown that if,

$$Y_1 = (-2 \log X_1)^{1/2} \text{ Cos } 2\pi X_2$$

$$Y_2 = (-2 \log X_1)^{1/2} \text{ Sin } 2\pi X_2$$

Then, Y_1 and Y_2 are independent normal random variables with zero mean and unit variance.

The advantage of this method is that the computation is exact. However, it does require more core and computing time than the method based upon the sum of random variables (i.e., approximately 30% more time per deviate based upon summing 12 random variables). Consequently if it turns out that n is moderate for any particular application for,

$$\left| F_n(x) - \Phi(x) \right| < \frac{C\beta}{\sqrt{n}}$$

to be within the desired error bounds, then it seems worthwhile to use the sum of random variables approach for generating normal deviates.

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