SLAC-PUB-854
January 1971

RELATIVISTIC EXTENSION OF THE ELECTROMAGNETIC CURRENT FOR COMPOSITE SYSTEMS*

Francis E. Close**
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

and

Hugh Osborn
Department of Natural Philosophy
Glasgow University, Glasgow, Scotland

ABSTRACT

We formulate a general method for evaluating relativistic and binding energy corrections to matrix elements of the electromagnetic current for multiparticle composite systems. Application is made to the calculation of the g-factor for bound electrons and compared with recent experiments.

(Submitted to Phys. Letters B)

---

* Work supported in part by the U. S. Atomic Energy Commission.

** NATO postdoctoral fellow 1970-72. Fellowship administered by the Science Research Council, London.

The present theoretical difficulties of handling composite systems when relativistic and strong binding corrections are significant is well illustrated by the problems of constructing dynamical information from the quark model [1], especially for baryons when three quarks are involved. Particularly interesting here is resonance photoproduction where the nonrelativistic quark model has some success [2], so that an estimate of corrections is desirable. In an attempt to overcome such problems, in this Letter the lowest order relativistic and binding energy[1] corrections to the electromagnetic current are obtained from the direct application of Lorentz invariance, rather than a reduction of Dirac type equations as has been customary.

Working in terms of the generators of the Poincaré group, the Hamiltonian and boosts[2] for a collection of free particles have the canonical forms [3]

$$H_0 = \sum_i E_i, \qquad \underset{\sim}{K}_0 = \sum_i \underset{\sim}{K}_i$$

$$E_i = \left(m_i^2 + \underset{\sim}{p}_i^2\right)^{1/2}, \qquad \underset{\sim}{K}_i = -\frac{1}{2}(\underset{\sim}{r}_i E_i + E_i \underset{\sim}{r}_i) - \frac{\underset{\sim}{p}_1 \times \underset{\sim}{\sigma}_i}{4(E_i + m_i)} \tag{1}$$

which reduce to the usual nonrelativistic limit.

The free particle currents can be regarded phenomenologically as the solution of the current conservation, four vector equations

$$i\left[E_i, \rho_i(\underset{\sim}{x})\right] = -\underset{\sim}{\nabla} \cdot \underset{\sim}{j}_i(\underset{\sim}{x}) \tag{2a}$$

$$i\left[\underset{\sim}{K}_i, \rho_i(\underset{\sim}{x})\right] = \underset{\sim}{j}_i(\underset{\sim}{x}) - \underset{\sim}{x} i\left[E_i, \rho_i(\underset{\sim}{x})\right] \tag{2b}$$

$$i\left[\underset{\sim}{K}_i, \underset{\sim}{j}_i(\underset{\sim}{x})\right] = -\underset{\sim}{x} i\left[E_i, \underset{\sim}{j}_i(\underset{\sim}{x})\right] + \frac{1}{2} \rho_i(\underset{\sim}{x}) \tag{2c}$$

Expanding in powers of v^2/c^2, to first order beyond the leading terms, Eqs. (2a, b, c) give[3]

$$\rho_i(\underset{\sim}{x}) = e_i\left(1+\frac{1}{6}\langle r_i^2\rangle\nabla^2\right)\delta(\underset{\sim}{x}-\underset{\sim}{r}_i) + \frac{1}{2m_i}\left(2\mu_i - \frac{e_i}{2m_i}\right)\underset{\sim}{\sigma}_i\times\underset{\sim}{p}_i\cdot\underset{\sim}{\nabla}_x\delta(\underset{\sim}{x}-\underset{\sim}{r}_i)$$

$$\underset{\sim}{j}_i(\underset{\sim}{x}) = \frac{e_i}{2m_i}\left\{\underset{\sim}{p}_i,\delta(\underset{\sim}{x}-\underset{\sim}{r}_i)\right\} - \frac{e_i}{8m_i^3}\left\{\underset{\sim}{p}_i^2,\left\{\underset{\sim}{p}_i,\delta(\underset{\sim}{x}-\underset{\sim}{r}_i)\right\}\right\} + \frac{1}{2m_i}\left(\frac{e_i}{6}\langle r_i^2\rangle - \frac{\mu_i}{4m_i}\right)\left\{\underset{\sim}{p}_i,\nabla^2\delta(\underset{\sim}{x}-\underset{\sim}{r}_i)\right\}$$

$$+\frac{1}{4m_i^2}\left(\mu_i-\frac{e_i}{2m_i}\right)\left\{\underset{\sim}{p}_i,\underset{\sim}{\sigma}_i\times\underset{\sim}{p}_i\cdot\underset{\sim}{\nabla}_x\delta(\underset{\sim}{x}-\underset{\sim}{r}_i)\right\} - \mu_i\underset{\sim}{\sigma}_i\times\underset{\sim}{\nabla}_x\left(1+G'_{M,i}(0)\nabla^2\right)\delta(\underset{\sim}{x}-\underset{\sim}{r}_i)$$

$$+\frac{\mu_i}{4m_i^2}\left\{\underset{\sim}{p}_i^2,\underset{\sim}{\sigma}_i\times\underset{\sim}{\nabla}_x\delta(\underset{\sim}{x}-\underset{\sim}{r}_i)\right\} + \frac{\mu_i}{8m_i^3}\left\{\underset{\sim}{p}_i\cdot\underset{\sim}{\nabla}_x,\underset{\sim}{\nabla}_x\delta(\underset{\sim}{x}-\underset{\sim}{r}_i) + \left\{\underset{\sim}{\sigma}_i\times\underset{\sim}{p}_i,\delta(\underset{\sim}{x}-\underset{\sim}{r}_i)\right\}\right\} \qquad (3)$$

If the charge, current density $(Q(\underset{\sim}{x}), \underset{\sim}{J}(\underset{\sim}{x}))$ of a system of particles is to be a conserved four vector, then the equations

$$\begin{aligned} i[H, Q(\underset{\sim}{x})] &= -\underset{\sim}{\nabla}\cdot\underset{\sim}{J}(\underset{\sim}{x}) \\ i[\underset{\sim}{K}, Q(\underset{\sim}{x})] &= \underset{\sim}{J}(\underset{\sim}{x}) + \underset{\sim}{x}\nabla\cdot\underset{\sim}{J}(\underset{\sim}{x}) \\ i[\underset{\sim}{K}, \underset{\sim}{J}(\underset{\sim}{x})] &= -ix[H, \underset{\sim}{J}(\underset{\sim}{x})] + \underset{\sim}{1}\,Q(\underset{\sim}{x}) \end{aligned} \qquad (4)$$

must be satisfied. If the system is a collection of free particles whose charge, current densities $\rho_i(x)$, $j_i(x)$ satisfy Eqs. (2), then the Hamiltonian and boost for such a system will be $H = H_0 = \sum_i E_i$, $\underset{\sim}{K} = \underset{\sim}{K}_0 = \sum_i \underset{\sim}{K}_i$ and so the Eqs. (4) will be satisfied by

$$Q(x) = \sum_i \rho_i(\underset{\sim}{x}), \quad \underset{\sim}{J}(\underset{\sim}{x}) = \sum_i \underset{\sim}{j}_i(\underset{\sim}{x}). \qquad (5)$$

In general there is interaction present between the constituents of the system. The interaction between particles i, j is assumed to be a local potential $V_{ij}(|r_i - r_j|)$. The interaction terms H', $\underset{\sim}{K}'$, depending upon the potential V, to be added to $H_0, \underset{\sim}{K}_0$ are then required to satisfy [4]

$$i[\underset{\sim}{K}', \underset{\sim}{P}_0] = \underset{\sim}{1}\,H'; [\underset{\sim}{K}', H_0] + [\underset{\sim}{K}_0, H'] = 0; \underset{\sim}{K}'\times\underset{\sim}{K}_0 + \underset{\sim}{K}_0\times\underset{\sim}{K}' = 0 \qquad (6)$$

so as to preserve the Poincaré algebra to first order in V. Equation (6) uniquely (up to a unitary transformation) determines $H', \underset{\sim}{K}'$, extended to include v^2/c^2 corrections, for either a scalar or a vector potential [5]. In the latter case we have [4]

$$H' = \int d\underset{\sim}{x}\, d\underset{\sim}{x}'\, \mathcal{H}(\underset{\sim}{x}, \underset{\sim}{x}') \qquad \underset{\sim}{K}' = -\int d\underset{\sim}{x}\, d\underset{\sim}{x}'\, \frac{1}{2}(\underset{\sim}{x}+\underset{\sim}{x}')\, \mathcal{H}(\underset{\sim}{x}, \underset{\sim}{x}')$$

$$\mathcal{H}(\underset{\sim}{x}, \underset{\sim}{x}') = \sum_{i>j} \left\{ \bar{\rho}_i(\underset{\sim}{x})\bar{\rho}_j(\underset{\sim}{x}') - \bar{\underset{\sim}{j}}_i(\underset{\sim}{x}) \cdot \bar{\underset{\sim}{j}}_j(\underset{\sim}{x}') \right\} V_{ij}(|\underset{\sim}{x} - \underset{\sim}{x}'|)$$

$$+ \frac{1}{2} \sum_{i>j} \underset{\sim}{\nabla} \cdot \bar{\underset{\sim}{j}}_i(\underset{\sim}{x})\, \underset{\sim}{\nabla} \cdot \bar{\underset{\sim}{j}}_j(\underset{\sim}{x}') \int_{|x-x'|}^{\infty} dr\, r\, V_{ij}(r) \tag{7}$$

The $\bar{\rho}_i, \bar{\underset{\sim}{j}}_i$ are the internal currents between which the vector potential acts. Their leading terms are[4]

$$\bar{\rho}_i(\underset{\sim}{x}) = \delta(\underset{\sim}{x} - \underset{\sim}{r}_i), \quad \bar{\underset{\sim}{j}}_i(\underset{\sim}{x}) = \frac{1}{2m_i} \left\{ \underset{\sim}{p}_i, \delta(\underset{\sim}{x} - \underset{\sim}{r}_i) \right\} - \frac{1}{2m_i} (1+\bar{\kappa}_i) \underset{\sim}{\nabla}_i \times \underset{\sim}{\sigma}_x \delta(\underset{\sim}{x} - \underset{\sim}{r}_i) \tag{8}$$

so that nonrelativistically H' goes over to a sum of two body local potentials. In the particular case of a two body system when V is just the normal Coulomb interaction, the familiar two particle electromagnetic interaction Hamiltonian is obtained [4]. Verification of Eqs. (6) relies on the current conservation, four vector equations for $\bar{\rho}_i, \bar{\underset{\sim}{j}}_i$ with the neglect of $i[E_i, \bar{\underset{\sim}{j}}_i(\underset{\sim}{x})]$ in accordance with the v^2/c^2 approximations.

For systems with interparticle interactions the Eqs. (4) are no longer satisfied by (5). For these cases the solution of (4) will be

$$Q(\underset{\sim}{x}) = \sum_i \rho_i(\underset{\sim}{x}) + \delta\rho_i(\underset{\sim}{x})$$

$$\underset{\sim}{J}(\underset{\sim}{x}) = \sum_i \underset{\sim}{j}_i(\underset{\sim}{x}) + \delta\underset{\sim}{j}_i(\underset{\sim}{x}) \tag{9}$$

The $\rho_i(\underset{\sim}{x}), \underset{\sim}{j}_i(\underset{\sim}{x})$ satisfy the Eqs. (2) and so, to first order in V, the $\delta\rho_i(\underset{\sim}{x})$, $\delta\underset{\sim}{j}_i(\underset{\sim}{x})$, must satisfy

$$i[H_0, \delta\rho_i(\underset{\sim}{x})] + \underset{\sim}{\nabla} \cdot \delta\underset{\sim}{j}_i(\underset{\sim}{x}) = -i[H', \rho_i(\underset{\sim}{x})]$$

$$i[\underset{\sim}{K}_0, \delta\rho_i(x)] - x\underset{\sim}{\nabla} \cdot \delta\underset{\sim}{j}_i(x) - \delta\underset{\sim}{j}_i(x) = -i[\underset{\sim}{K}', \rho_i(\underset{\sim}{x})]$$

$$i[\underset{\sim}{K}_0, \delta\underset{\sim}{j}_i(\underset{\sim}{x})] + \underset{\sim}{x}i[H_0, \delta\underset{\sim}{j}_i(\underset{\sim}{x})] - 1\delta\rho_i(\underset{\sim}{x}) = -i[\underset{\sim}{K}', \underset{\sim}{j}_i(\underset{\sim}{x})] - \underset{\sim}{x}i[H', \underset{\sim}{j}_i(\underset{\sim}{x})] \tag{10}$$

The right hand sides of (10) may be evaluated with the current commutators which follow from (3),

$$[\rho_i(\underset{\sim}{x}),\ \bar{\rho}_j(\underset{\sim}{x}')] = i\delta_{ij}\frac{1}{m_i}\left(\mu_i + \frac{\rho_i\bar{\kappa}_i}{2m_i}\right)\nabla_{\underset{\sim}{x}}\delta(\underset{\sim}{x}-\underset{\sim}{x}')\cdot\underset{\sim}{\sigma}_i\times\nabla_{\underset{\sim}{x}'}\bar{\rho}_i(\underset{\sim}{x}')$$

$$[\rho_i(\underset{\sim}{x}),\bar{j}_j(\underset{\sim}{x}')] = -i\delta_{ij}\frac{e_i}{m_i}\nabla_{\underset{\sim}{x}}\left\{\delta(\underset{\sim}{x}-\underset{\sim}{x}')\,\rho_i(\underset{\sim}{x})\right\}$$

$$[j_i(\underset{\sim}{x}),\bar{\rho}_j(\underset{\sim}{x}')] = -i\delta_{ij}\frac{e_i}{m_i}\rho_i(\underset{\sim}{x})\,\nabla_{\underset{\sim}{x}}\delta(\underset{\sim}{x}-\underset{\sim}{x}'). \tag{11}$$

The solution of Eqs. (10) is not unique [5], but a minimal solution which corresponds to that given by reduction of the Dirac equation is

$$\delta\underset{\sim}{j}_i(\underset{\sim}{x}) = -\frac{1}{m_i}\left(\mu_i + \frac{e_i\bar{\kappa}_i}{2m_i}\right)\rho_i(\underset{\sim}{x})\,\underset{\sim}{\sigma}_i\times\underset{\sim}{E}_i(\underset{\sim}{x}) - \frac{e_i}{m_i}\rho_i(\underset{\sim}{x})\,\underset{\sim}{A}_i(\underset{\sim}{x}),$$

$$\delta\rho_i(x)\sim\frac{1}{m^3}, \qquad \underset{\sim}{E}_i(\underset{\sim}{x}) = -\nabla\int dx'\sum_{j\neq i}V_{ij}(|\underset{\sim}{x}-\underset{\sim}{x}'|)\rho_j(\underset{\sim}{x}'),$$

$$\underset{\sim}{A}_i(\underset{\sim}{x}) = \frac{1}{2}\int dx'\sum_{j\neq i}\left\{\frac{1}{\underset{\sim}{x}-\underset{\sim}{x}'}V_{ij}(|\underset{\sim}{x}-\underset{\sim}{x}'|) - (\underset{\sim}{x}-\underset{\sim}{x}')\nabla_{\underset{\sim}{x}}V_{ij}(|\underset{\sim}{x}-\underset{\sim}{x}'|)\right\}\underset{\sim}{j}_j(\underset{\sim}{x}'). \tag{12}$$

Also in this v^2/c^2, first order in V, approximation the c.m. decomposition of the single particles' dynamical variables $\underset{\sim}{r}_i,\underset{\sim}{p}_i,\underset{\sim}{\sigma}_i$ is modified [3,4,6]. The additional terms are necessary essentially to take care of the Wigner spin rotations for the composite system, both for the individual particle spins and orbital angular momentum. In the c.m. frame $\underset{\sim}{P}=0$, $\underset{\sim}{p}_i\longrightarrow\underset{\sim}{k}_i$ with $\sum_j\underset{\sim}{k}_i=0$ where $\underset{\sim}{k}_i,\underset{\sim}{\sigma}_i$ and $\underset{\sim}{r}_{ij}=\underset{\sim}{r}_i-\underset{\sim}{r}_j$ are the internal variables. With $\underset{\sim}{P}=0$

$$\underset{\sim}{r}_i = \underset{\sim}{R}+\Delta\underset{\sim}{r}_i - \frac{1}{2\mathcal{M}}\sum_j\frac{1}{2}\left\{\Delta\underset{\sim}{r}_j,\ \frac{\underset{\sim}{k}_j^2}{m_j}+\sum_{k\neq j}V_{jk}(|\underset{\sim}{r}_{jk}|)\right\} + \frac{1}{4\mathcal{M}}\sum_j\frac{1}{m_j}\underset{\sim}{\sigma}_j\times\underset{\sim}{k}_j,$$

$$\Delta\underset{\sim}{r}_i = \sum_j\frac{m_j}{\mathcal{M}}\underset{\sim}{r}_{ij}, \qquad \mathcal{M}=\sum_i m_i \tag{13}$$

One application of our results is for the calculation of magnetic moments of composite systems. The magnetic moment operator is

$$\underset{\sim}{\mu} = \frac{1}{2}\int d\underset{\sim}{x}\,\underset{\sim}{x}\times\underset{\sim}{J}(\underset{\sim}{x}),$$

in the c.m. frame $P = 0$. This may be explicitly evaluated in terms of (3), (9), (12) and (13). For a two particle system, employing a unitary transformation to simplify $\underset{\sim}{\mu}$ evaluated between eigenstates of the mass operator (=H for $\underset{\sim}{P} = 0$),[5] then for particle 1,

$$\underset{\sim}{\mu}_1 = \left(1 - \frac{\underset{\sim}{k}^2}{2m_1^2}\right)\left(\frac{e_1}{2m_1}\frac{m_2}{\mathcal{M}}\,\underset{\sim}{\ell} + \mu_1\underset{\sim}{\sigma}_1\right) + \frac{e_1}{2\mathcal{M}^2}\,V(|\underset{\sim}{r}|)\underset{\sim}{\ell} + e_1\,\frac{m_1 - m_2}{4m_1^2 m_2 \mathcal{M}}\,\underset{\sim}{k}^2\underset{\sim}{\ell}$$
$$- \frac{1}{2m_1^2}\left(\mu_1 + \frac{e_1}{4m_1}(2\bar{\kappa}_1 - 1)\right)\underset{\sim}{k}\times(\underset{\sim}{k}\times\underset{\sim}{\sigma}_1) - \frac{e_1}{4m_1^2 m_2}(1+\bar{\kappa}_2)\,\underset{\sim}{k}\times(\underset{\sim}{k}\times\underset{\sim}{\sigma}_2)$$
$$+ \frac{e_1}{8m_1\mathcal{M}}\,\underset{\sim}{k}\times\left\{\underset{\sim}{k}\times\left(\frac{\underset{\sim}{\sigma}_1}{m_1} - \frac{\underset{\sim}{\sigma}_2}{m_2}\right)\right\}, \qquad \underset{\sim}{\ell} = \underset{\sim}{r}\times\underset{\sim}{k} \tag{14}$$

This agrees with previous results as far as spin dependent terms [7]. For a hydrogenic atom with an electron in a (1S) state,

$$g(1S) = g_e\left[1 - \frac{1}{3}(Z\alpha)^2\left\{\frac{M^2}{(M+m_e)^2}\left(1 - \frac{3\alpha}{4\pi}\right) + Z\,\frac{m_e^2}{(M+m_e)^2} + \frac{Mm_e}{2(M+m_e)^3}(M+Zm_e)\left(1 - \frac{\alpha}{2\pi}\right)\right\}\right] \tag{15}$$

neglecting terms of $O(\alpha^4)$, which include v^4/c^4 effects, and using $\kappa_e = \alpha/2\pi$.[6] For the bound nucleus the corresponding g factor is

$$g_{\substack{\text{bound}\\\text{nucleus}}} = g_{\text{nucleus}}\left\{1 - \frac{1}{3}Z\alpha^2\left[\frac{M^2}{(m_e+M)^2} - \frac{Z\,m_e^2}{2(m_e+M)^2}\right]\right\} - Z\alpha^2\left\{\frac{Z\,m_e^2}{(m_e+M)^2} + \frac{Mm_e}{3(m_e+M)^3}(Zm_e+M)\right\} \tag{16}$$

to the same accuracy.[7] These results are in agreement with recently published results [8].

The method presented here neglects radiative corrections but as shown by Grotch [8] these are unimportant for the present accuracy except insofar as the electron has an anomalous moment, which has been included. Numerically from (15) the ratio of the bound electron g factors for hydrogen and deuterium is

$$\frac{g_H(1S)}{g_D(1S)} = 1 + \alpha^2\,\frac{m_e}{4M_p} - \alpha^2\,\frac{m_e}{8M_p^2}(2M_p - M_D) - \alpha^2\,\frac{3m_e^2}{4M_p^2} - \frac{5}{24\pi}\,\alpha^3\,\frac{m_e}{M_p}$$
$$= 1 + 7.25\times 10^{-9} \tag{17}$$

whereas two experiments [9] give $1 + (7.2 \pm 3.0) \times 10^{-9}$ and $1 + (9.4 \pm 1.4) \times 10^{-9}$ in good agreement.

Thus the methods described above provide a consistent framework for the calculations of relativistic corrections to the electromagnetic multipole moments of composite systems of arbitrary numbers of particles.

One of us (F. E. C.) wishes to thank S. J. Brodsky for comments and discussions, and H. Grotch for communications.

FOOTNOTES

1. Thus the present results cannot be immediately applied to the naive quark model of very massive quarks and strong binding. It is hoped to extend this approach to include all orders in the nonrelativistic binding potential V but only first order in $V \times v^2/c^2$.

2. Commutation relations involving the momentum $\underset{\sim}{P}_0 = \sum_i \underset{\sim}{p}_i$, and the angular momentum $\underset{\sim}{J}_0 = \sum_i (\underset{\sim}{r}_i \times \underset{\sim}{p}_i + 1/2\, \underset{\sim}{\sigma}_i)$ can be satisfied trivially and so will not be discussed here.

3. The sole arbitrariness in the solution of Eq. (2) lies in the overall normalization and the relative magnitudes of sets of terms which are independent. Then to order v^2/c^2 this arbitrariness is exhibited by the numbers $e_i, \mu_i, \langle r_i^2 \rangle$, $G'_{M,i}(0)$ in (3). If the ρ, j at (3) are to be thought of as electromagnetic current densities for the free particles then the above arbitrary parameters are fixed by requiring that the current so constructed gives the values of the charge (ρ_i), magnetic moment (μ_i), etc., of the free particles described. Hence $\langle r_i^2 \rangle = 6 \left\{ (1/8m_i^2) + G'_{E,i}(0) \right\}$ is the mean square radius, normalizing $G_{E,i}(0) = G_{M,i}(0) = 1$; $G'_E(0)$, $G'_m(0) > 0$ for the proton.

4. The $\bar{\rho}, \bar{j}$ are free particle currents satisfying Eq. (2), and normalized as at (8). The spin dependent term in $\underset{\sim}{j}$ is independent of the spin independent terms and hence its arbitrary magnitude is reflected in the parameter κ.

5. This also leads to the disappearance of the R dependence.

6. For Coulomb binding $V(r) = -Z\alpha/r$, $\bar{\kappa} = \kappa$, $g_e = 2\left[1 + \frac{\alpha}{2\pi} - 0.328\,(\alpha^2/\pi^2)\right]$ where the α^2 term should not be affected by the binding except to $O(\alpha^4)$.

7. This method verifies Ref. (15) for arbitrary nuclear spin, with the usual definition $\mu_{\text{nucleus}} = (Ze/2M)\, g_{\text{nucleus}} S_{\text{nucleus}}$.

REFERENCES

[1] See H. J. Lipkin, Proceedings of the Lund International Conference on Elementary Particles (1969).

[2] L. A. Copley, G. Karl and E. Obryk, Phys. Letters 29B, 117 (1969), and Nucl. Phys. B13, 303 (1969). R. L. Walker, Proceedings of the 4th International Symposium on Electron and Photon Interactions at High Energies, Liverpool (1969).

[3] H. Osborn, Phys. Rev. 176, 1514 (1968); which contains further references. We are specializing to spin 1/2. The general spin case can be treated without further complication.

[4] F. E. Close and H. Osborn, Phys. Rev. D2, 2127 (1970). The momentum and angular momentum are not affected by the interaction, $\underset{\sim}{P} = \underset{\sim}{P}_0 \cdot \underset{\sim}{J} = \underset{\sim}{J}_0$.

[5] The scalar case and other aspects of the present work will be discussed in more detail elsewhere.

[6] F. E. Close and L. A. Copley, Nucl. Phys. B19, 477 (1970).

[7] R. Faustov, Nuovo Cimento 69A, 37 (1970). The orbital terms in this reference are fallacious in that the interaction dependence must vanish for a central vector potential (corresponding to $m_2 \longrightarrow \infty$) (R.G. Sachs, Phys. Rev. 72, 91 (1947)).

[8] R. Faustov, Phys. Letters B422 (1970); H. Grotch, Phys. Rev. A2, 1605 (1970); H. Grotch, to be published in Proceedings of the International Conference on Precision Measurement and Fundamental Constants; R. A. Hegstrom, ibid.

[9] D. J. Larson, P. A. Valberg and N. F. Ramsey, Phys. Rev. Letters 23, 1369 (1969); W. M. Hughes and H. G. Robinson, Phys. Rev. Letters 23, 1209 (1969).