SLAC-PUB-853
January 1971

THE EIGENVALUES AND EIGENVECTORS OF THE INTERACTION TERM IN LOCAL QUANTUM FIELD THEORY*

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ABSTRACT

A basic feature of nonrelativistic quantum mechanics is the existence of one representation (the momentum representation) which diagonalizes the free part of the Hamiltonian, a second representation (the position representation) which diagonalizes the interaction part of the Hamiltonian, and a unitary transformation (Fourier transform) which connects these two representations. In local Lagrangian field theory the free particle representation which diagonalizes the free part of the Hamiltonian is well known, but the representation which diagonalizes the interaction part of the Hamiltonian has not been systematically studied. In what follows, this representation is explicitly constructed and it is shown that there is no unitary transformation connecting it with the free particle representation. In fact this representation space is not even a Hilbert space in the sense that it seems impossible to define a meaningful norm.

(Submitted to Phys. Rev.)

*Work supported by the U. S. Atomic Energy Commission.

INTRODUCTION

The most fundamental structures in nonrelativistic quantum mechanics are the two basic representation spaces, position space and momentum space, and the operation of Fourier transformation which connects them. Although one may abstract and consider arbitrary complete sets of commuting observables and their eigenvectors and eigenvalues, in practical physical problems one solves for the eigenvectors and eigenvalues of a Hamiltonian whose free part is diagonal in the momentum representation and whose interaction part is diagonal (or nearly so) in the position representation. It is this feature of physical Hamiltonians which gives the position, and momentum representations their paramount importance. Most often one solves problems in position space, the representation which diagonalizes the interaction.

When one turns to quantum field theory, the situation is radically altered. One invariably works in the representation which diagonalizes the free particle part of the Hamiltonian and one is completely unfamiliar with the representation which diagonalizes the interaction part. Let us attempt to remedy this situation.

Consider the case of a local Lagrangian field theory with a scalar boson field coupled to a spin 1/2 fermion field through a simple scalar coupling. The physically more interesting cases of quantum electrodynamics (in the transverse gauge), psuedoscalar meson theory, etc., require only straightforward generalizations of the methods used to solve the simple scalar coupling case. The Hamiltonian is given by,

$$H = H_0 + H_I \ , \tag{1}$$

$$H_0 = \int d^3\underset{\sim}{x} \ :\psi^\dagger(-i\underset{\sim}{\alpha}\cdot\underset{\sim}{\nabla} + \beta m)\psi: + \frac{1}{2}\int d^3\underset{\sim}{x} \ :\left\{\left(\frac{\partial\phi}{\partial t}\right)^2 + (\underset{\sim}{\nabla}\phi)^2 + \mu^2\phi^2\right\}: \ , \tag{1a}$$

and

$$H_I = g\int d^3x\, j(x)\, \phi(x), \tag{1b}$$

where

$$j(x) = \frac{1}{2}\left[\bar{\psi}(x),\ \psi(x)\right] .$$

To start with we take the usual representation of the commutation (anticommutation) relations at t = 0, given by

$$\psi(x) = \sum_{\alpha=1}^{2}\int d\zeta\left[u_\alpha(p)\, e^{ip\cdot x}\, b_\alpha(p) + v_\alpha(p)\, e^{-ip\cdot x}\, c_\alpha^\dagger(p)\right] \tag{2a}$$

$$\phi(x) = \int d\zeta\left[e^{ip\cdot x}a(p) + e^{-ip\cdot x}a^\dagger(p)\right]. \tag{2b}$$

Here $u_\alpha(p)$ and $v_\alpha(p)$ are the free positive and negative energy spinors,

$$\left[a(p),\ a^\dagger(p')\right] = \delta(\zeta - \zeta'), \tag{3a}$$

$$\left[b_\alpha(p),\ b_\beta^\dagger(p')\right]_+ = \left[c_\alpha(p),\ c_\beta^\dagger(p')\right]_+ = \delta_{\alpha\beta}\,\delta(\zeta - \zeta'), \tag{3b}$$

and all other commutators (anticommutators) = 0,

$$d\zeta = \frac{d^3p}{(2\pi)^3\ 2E_p}\ , \tag{3c}$$

and

$$\delta(\zeta - \zeta') = (2\pi)^3\ 2E_p\ \delta^3(p - p'). \tag{3d}$$

Our problem is to find an alternate representation which diagonalizes H_I at t = 0. Therefore, for the rest of this paper we consider the time to be fixed at t = 0. Further if we can find a representation in which $\phi(x)$ is diagonal for every x, and $j(x)$ is diagonal for every x, then H_I will be diagonal in this representation. This is certainly feasible since $\phi(x)$ and $j(x)$ are Hermitian operators, and since

$$\left[\phi(x),\ \phi(x')\right] = \left[j(x),\ j(x')\right] = 0.$$

PART I

THE DIAGONALIZATION OF $\phi(\underset{\sim}{x})$

To find the eigenvectors and eigenvalues of $\phi(\underset{\sim}{x})$ it is easiest to go over to the Fourier transform field operators given by,

$$\phi(\underset{\sim}{p}) = 2E_p \int d^3\underset{\sim}{x}\, e^{-i\underset{\sim}{p} \cdot \underset{\sim}{x}} \phi(\underset{\sim}{x})$$
$$= a(\underset{\sim}{p}) + a^{\dagger}(-\underset{\sim}{p}). \tag{4}$$

Clearly since $[\phi(\underset{\sim}{x}),\ \phi(\underset{\sim}{x}')] = 0$, we have

$$[\phi(\underset{\sim}{p}),\ \phi(\underset{\sim}{p}')] = 0$$

and in particular, since $\phi(-\underset{\sim}{p}) = \phi(\underset{\sim}{p})^{\dagger}$, we have

$$[\phi(\underset{\sim}{p}),\ \phi(\underset{\sim}{p}')^{\dagger}] = 0$$

so $\phi(\underset{\sim}{p})$ is a normal operator and may be diagonalized. $\phi(\underset{\sim}{p})$ couples bosons of momentum $\underset{\sim}{p}$ only with bosons of momentum $-\underset{\sim}{p}$, which is to be expected since the eigenvalues of $\phi(\underset{\sim}{x})$ must be real. Let the free particle vacuum be denoted by $|\phi\rangle$, $\Longrightarrow a(\underset{\sim}{p})|\phi\rangle = 0$ for all $\underset{\sim}{p}$. Then consider the operator

$$Z = \exp\left[-\frac{1}{2}\int d\zeta\, a^{\dagger}(\underset{\sim}{p})\, a^{\dagger}(-\underset{\sim}{p})\right]. \tag{5}$$

Clearly,

$$[a^{\dagger}(-\underset{\sim}{p}), Z)] = 0,$$

and

$$[a(\underset{\sim}{p}),\ Z] = -\, a^{\dagger}(-\underset{\sim}{p})\, Z\ ,$$

so

$$\phi(\underset{\sim}{p})\, Z|\phi\rangle = [a(\underset{\sim}{p}) + a^{\dagger}(-\underset{\sim}{p})]\, Z|\phi\rangle$$
$$= Z[a(\underset{\sim}{p}) - a^{\dagger}(-\underset{\sim}{p}) + a^{\dagger}(-\underset{\sim}{p})]\,|\phi\rangle = 0,$$

Therefore $Z|\phi\rangle = |0\rangle$ is the eigenvector of $\phi(\underset{\sim}{p})$ whose eigenvalue is zero for all $\underset{\sim}{p}$, i.e.,

$$\phi(\underset{\sim}{x})|0\rangle = 0 \text{ for all } \underset{\sim}{x}\ .$$

Consider further the operator

$$W(g(\underset{\sim}{p})) = \exp\left[\int d\zeta\, g(\underset{\sim}{p})\, a^{\dagger}(\underset{\sim}{p})\right], \tag{6}$$

where $g(\underset{\sim}{p})$ is an arbitrary C number function of $\underset{\sim}{p}$ which satisfies

$$g^{*}(\underset{\sim}{p}) = g(-\underset{\sim}{p}) . \tag{7}$$

Clearly we have

$$[a^{\dagger}(-\underset{\sim}{p}),\ W] = 0$$

and

$$[a(\underset{\sim}{p}),\ W] = g(\underset{\sim}{p})\, W,$$

so that

$$\begin{aligned}\phi(\underset{\sim}{p})\, WZ|\phi\rangle &= g(\underset{\sim}{p})\, WZ|\phi\rangle + W\phi(\underset{\sim}{p})\, Z|\phi\rangle \\ &= g(\underset{\sim}{p})\, WZ|\phi\rangle .\end{aligned}$$

Since $a^{\dagger}(\underset{\sim}{p})$ commutes with $a^{\dagger}(\underset{\sim}{p}')\, a^{\dagger}(-\underset{\sim}{p}')$ we can write

$$U(g(\underset{\sim}{p})) = W(g(\underset{\sim}{p}))Z = \exp\left[\int d\zeta \left\{g(\underset{\sim}{p})\, a^{\dagger}(\underset{\sim}{p}) - (1/2)\, a^{\dagger}(\underset{\sim}{p})\, a^{\dagger}(-\underset{\sim}{p})\right\}\right], \tag{8}$$

and we have

$$|g(\underset{\sim}{p})\rangle = U(g(\underset{\sim}{p}))|\phi\rangle \tag{9}$$

is the eigenvector of $\phi(\underset{\sim}{x})$ for all $\underset{\sim}{x}$. For all $\underset{\sim}{p}$ we have

$$\phi(\underset{\sim}{p})|g(\underset{\sim}{p})\rangle = g(\underset{\sim}{p})|g(\underset{\sim}{p})\rangle , \tag{10}$$

and for all $\underset{\sim}{x}$

$$\phi(\underset{\sim}{x})|g(\underset{\sim}{p})\rangle = g(\underset{\sim}{x})|g(\underset{\sim}{p})\rangle , \tag{11}$$

where

$$g(\underset{\sim}{x}) = \int d\zeta\, g(\underset{\sim}{p})\, e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}$$

and by Eq. (7), $g(\underset{\sim}{x})$ is real.[1]

Now let us try and fix the normalizations so as to give us a complete <u>ortho-normal</u> set of states, and thereby a unitary transformation. Toward this end let us evaluate the scalar product $\langle f(\underset{\sim}{p})|g(\underset{\sim}{p})\rangle$ for two arbitrary functions $f(\underset{\sim}{p})$ and $g(\underset{\sim}{p})$ which satisfy Eq. (7). To facilitate the calculation we insert a convergence factor by letting

$$a^{\dagger}(\underset{\sim}{p}) \longrightarrow t\, a^{\dagger}(\underset{\sim}{p})$$

in

$$U(g(\underset{\sim}{p})),$$

where

$$0 < t < 1.$$

Thus we have

$$|g(\underset{\sim}{p})\rangle_t = U_t(g(\underset{\sim}{p}))|\phi\rangle \ , \tag{11}$$

where

$$U_t(g(\underset{\sim}{p})) = \exp\left[\int d\zeta \left\{t\, g(\underset{\sim}{p})\, a^{\dagger}(\underset{\sim}{p}) - \left(\frac{t^2}{2}\right) a^{\dagger}(\underset{\sim}{p})\, a^{\dagger}(-\underset{\sim}{p})\right\}\right] \ . \tag{11a}$$

Then at the end of the calculation we shall take the limit as $t \longrightarrow 1$. We define the state $|0\rangle_t$ by

$$|0\rangle_t = U_t(0)|\phi\rangle \ , \tag{12}$$

and we then have,

$${}_t\langle f(\underset{\sim}{p})|g(\underset{\sim}{p})\rangle_t = {}_t\langle 0|e^A e^B|0\rangle_t$$

where

$$A = t\int d\zeta\, f^*(\underset{\sim}{p})\, a(\underset{\sim}{p}),$$

and

$$B = t\int d\zeta\, g(\underset{\sim}{p})\, a^{\dagger}(\underset{\sim}{p}).$$

Now consider the operator

$$\beta(\underset{\sim}{p}) = a(\underset{\sim}{p}) + t^2 a^{\dagger}(-\underset{\sim}{p}). \tag{13}$$

It has the basic properties,

$$\beta(\underset{\sim}{p})|0\rangle_t = 0 = {}_t\langle 0|\beta^{\dagger}(\underset{\sim}{p}) \ , \tag{13a}$$

$$[\beta(\underset{\sim}{p}),\ \beta(\underset{\sim}{p}')] = 0 \ , \tag{13b}$$

$$[\beta(\underset{\sim}{p}),\ \beta^{\dagger}(\underset{\sim}{p}')] = (1 - t^4)\ \delta(\zeta - \zeta') \ . \tag{13c}$$

Solving (13) for a(p) we find

$$a(\underset{\sim}{p}) = \frac{\beta(\underset{\sim}{p}) - t^2 \beta^\dagger(-\underset{\sim}{p})}{(1 - t^4)} \tag{14}$$

and substituting this into the expressions for A and B we find,

$$A = \left(\frac{t}{1-t^4}\right) \int d\zeta\, f^*(\underset{\sim}{p})\, [\beta(\underset{\sim}{p}) - t^2 \beta^\dagger(-\underset{\sim}{p})] \tag{15a}$$

and

$$B = \left(\frac{t}{1-t^4}\right) \int d\zeta\, g(\underset{\sim}{p})\, [\beta^\dagger(\underset{\sim}{p}) - t^2 \beta(-\underset{\sim}{p})] \; . \tag{15b}$$

Let us try to use relation (13a) to our advantage. We shall use the operator identity

$$e^{(B_1+B_2)} = e^{B_1} e^{B_2} e^{-1/2[B_1, B_2]} \tag{16}$$

where B_1 and B_2 are operators whose commutator is a C number. Letting B_1 be the $\beta^\dagger(\underset{\sim}{p})$ part of B, and B_2 be the $\beta(-\underset{\sim}{p})$ part of B, we find using (16), (15b), (13c), (13a), and (7)

$$e^B|0\rangle_t = \exp\left\{\frac{-t^4}{2(1-t^4)} \int d\zeta\, |g(\underset{\sim}{p})|^2\right\} \times \exp\left\{\frac{t}{(1-t^4)} \int d\zeta g(\underset{\sim}{p}) \beta^\dagger(\underset{\sim}{p})\right\} |0\rangle_t \, . \tag{17a}$$

Similarly, we find

$${}_t\langle 0|e^A = {}_t\langle 0| \exp\left\{\frac{t}{(1-t^4)} \int d\zeta\, f^*(\underset{\sim}{p}) \beta(\underset{\sim}{p})\right\} \times \exp\left\{\frac{-t^4}{2(1-t^4)} \int d\zeta |f(\underset{\sim}{p})|^2\right\} . \tag{17b}$$

Applying another operator identity

$$e^{A_1} e^{B_1} = e^{B_1} e^{A_1} e^{+[A_1, B_1]}$$

where $[A_1, B_1]$ is a C number, and

$$A_1 = \frac{t}{(1-t^4)} \int d\zeta\, f^*(\underset{\sim}{p}) \beta(\underset{\sim}{p})$$

$$B_1 = \frac{t}{(1-t^4)} \int d\zeta\, g(\underset{\sim}{p})\, \beta^\dagger(\underset{\sim}{p}),$$

we find using (17a), (17b), (7), (13a), and (13c)

$$ {}_t\langle 0|e^A e^B|0\rangle_t = {}_t\langle f(\underset{\sim}{p})|g(\underset{\sim}{p})\rangle_t = {}_t\langle 0|0\rangle_t \exp\left\{\frac{-t^4}{2(1-t^4)}\int d\zeta |f(\underset{\sim}{p}) - g(\underset{\sim}{p})|^2\right\} $$

$$ \times \exp\left\{\left(\frac{t^2}{1+t^2}\right)\int d\zeta\ f^*(\underset{\sim}{p})\ g(\underset{\sim}{p})\right\} . \tag{18} $$

It remains only to evaluate ${}_t\langle 0|0\rangle_t$. We have,

$$ {}_t\langle 0|0\rangle_t = \langle\phi|\exp\left\{\frac{-t^2}{2}\int d\zeta\ a(\underset{\sim}{p})\ a(-\underset{\sim}{p})\right\}\exp\left\{\frac{-t^2}{2}\int d\zeta\ a^\dagger(\underset{\sim}{p})\ a^\dagger(-\underset{\sim}{p})\right\}|\phi\rangle $$

$$ = \sum_{n=0}^{\infty}\frac{\left(\frac{-t^2}{2}\right)^{2n}}{(n!)^2}\langle\phi|[\int d\zeta\ a(\underset{\sim}{p})\ a(-\underset{\sim}{p})]^n \times [\int d\zeta\ a^\dagger(\underset{\sim}{p})\ a^\dagger(-\underset{\sim}{p})]^n|\phi\rangle . \tag{19} $$

The expression

$$ \langle\phi|[\int aa]^n\ [\int a^\dagger a^\dagger]^n|\phi\rangle = K_n $$

may be evaluated by induction as follows. Take one operator $a(-\underset{\sim}{p})$ from the left and carry it through to the right picking up the commutators along the way. We obtain,

$$ K_n = 2n\langle\phi|[\int aa]^{n-1}\int d\zeta a(\underset{\sim}{p})\ a^\dagger(\underset{\sim}{p})[\int a^\dagger a^\dagger]^{n-1}|\phi\rangle . $$

Now take the operator $a(\underset{\sim}{p})$ and carry it through to the right obtaining,

$$ K_n = \left\{2n\ C(\infty^2) + 4n(n-1)\right\}K_{n-1} $$

$$ = 2n\left\{C(\infty^2) + 2(n-1)\right\}K_{n-1}, \tag{20} $$

where $C(\infty^2)$ is the doubly infinite constant given by

$$ C(\infty^2) = \int d\zeta\delta(\zeta-\zeta) = \iint\frac{d^3\underset{\sim}{p}\,d^3\underset{\sim}{x}}{(2\pi)^3} . \tag{21} $$

Since $K_0 = 1$, we have

$$ K_n = 2^n n!\ C(\infty^2)\ [C(\infty^2)+2] ---- [C(\infty^2)+2(n-1)] $$

and so

$$ {}_t\langle 0|0\rangle_t = \sum_{n=0}^{\infty}\frac{(t^4)^n}{n!}\times\frac{C(\infty^2)}{2}\left[\frac{C(\infty^2)}{2}+1\right] ---- \left[\frac{C(\infty^2)}{2}+(n-1)\right] $$

$$ = (1-t^4)^{(-C(\infty^2)/2)} \tag{22} $$

by the binomial theorem. Combining (18) and (22) we have

$$\langle f(p)|g(p)\rangle = \lim_{t\to 1}\left\{\frac{1}{(1-t^4)^{[C(\infty^2)/2]}}\right\}\times \exp\left\{\frac{-t^4}{2(1-t^4)}\int d\zeta |f(\underset{\sim}{p}) - g(\underset{\sim}{p})|^2\right\}$$

$$\times \exp\left\{\left(\frac{t^2}{1+t^2}\right)\int d\zeta\ f^*(\underset{\sim}{p})\ g(\underset{\sim}{p})\right\} \quad . \tag{23}$$

First we note that unless $f(\underset{\sim}{p}) = g(\underset{\sim}{p})$, the first exponential goes rapidly to zero as $t \to 1$. Since we are interested only in the limit as $t\to 1$ we may rewrite (23) as

$$\langle f(\underset{\sim}{p})|g(\underset{\sim}{p})\rangle = \exp\left\{\frac{1}{2}\int d\zeta |f(\underset{\sim}{p})|^2\right\} \times \lim_{t\to 1} \frac{\exp\left\{\frac{-1}{2(1-t^4)}\int d\zeta\, |f(\underset{\sim}{p}) - g(\underset{\sim}{p})|^2\right\}}{(1-t^4)^{[C(\infty^2)/2]}} \quad . \tag{24}$$

It is a satisfactory feature of (24) that unless

$$\int d\zeta |f(\underset{\sim}{p}) - g(\underset{\sim}{p})|^2 = 0, \tag{25}$$

the numerator goes rapidly to zero as $t \to 1$, for this means that in some sense the orthogonality of eigenvectors with different eigenvalues holds true. However, the appearance of the factor

$$(1-t^4)^{[C(\infty^2)/2]}$$

in the denominator means that it is impossible to orthonormalize the states $|g(p)\rangle$ and hence <u>there exists no unitary transformation connecting the free particle states with the states $|g(p)\rangle$</u>.

Can we make any better intuitive sense of (24)? First observe that since

$$\int_V \frac{d^3\underset{\sim}{x}\, d^3\underset{\sim}{p}}{(2\pi)^3} = 1 ,$$

implies that V is a phase space volume element for a single state for one boson we may heuristically write

$$(1-t^4)^{C(\infty^2)/2} = \prod_S (1-t^4) \tag{26}$$

where $\prod_S$ is the product over any half of the possible states for one boson.
Similarly using relation (7) we may heuristically write

$$\exp\left\{\frac{1}{2}\int d\zeta |f(\underset{\sim}{p})|^2\right\} = \prod_S \exp\{|f(\underset{\sim}{p})|^2\} \ , \tag{27}$$

and

$$\exp\left\{\frac{-1}{2(1-t^4)}\int d\zeta |f(\underset{\sim}{p}) - g(\underset{\sim}{p})|^2\right\} = \prod_S \exp\left\{\frac{-|f(\underset{\sim}{p}) - g(\underset{\sim}{p})|^2}{(1-t^4)}\right\}, \tag{28}$$

where $\prod_S$ is the product over a particular half of the possible states for one boson, namely a set of half the values of $\underset{\sim}{p}$ chosen so that if $\underset{\sim}{p}$ is within the set, then $-\underset{\sim}{p}$ is not within the set. For example we could take the set of all $\underset{\sim}{p}$ such that either $p_z > 0$, or if $p_z = 0$ then $p_y > 0$, or if $p_z = 0$ and $p_y = 0$ then $p_x \geq 0$. Now a simple calculation shows that

$$\lim_{t \to 1}\left\{\frac{1}{(1-t^4)} \exp\left(\frac{-|f-g|^2}{(1-t^4)}\right)\right\} = \pi\, \delta(\mathrm{Re} f - \mathrm{Re} g)\, \delta(\mathrm{Im} f - \mathrm{Im} g). \tag{29}$$

Therefore combining (26), (27), (28), and (29) we have heuristically

$$\langle f(\underset{\sim}{p}) | g(\underset{\sim}{p})\rangle \prod_S \frac{\exp\{-|f(\underset{\sim}{p})|^2\}}{\pi} = \prod_S \delta(\mathrm{Re} f(\underset{\sim}{p}) - \mathrm{Re} g(\underset{\sim}{p}))\ \delta(\mathrm{Im} f(\underset{\sim}{p}) - \mathrm{Im} g(\underset{\sim}{p})).$$

Expression (30) is just the answer we intuitively expect and in fact is just the exact answer one obtains when the set S contains only a finite number of distinct possible states, i.e., if we put all our bosons in a finite box and only consider those states for which $|\underset{\sim}{p}| <$ some p_{max}. Hence we can say that the expression (24) is trying to describe a continuously infinite product of delta functions, but since such an infinite product is mathematically horribly ill defined (in fact the dual motion of integration over the manifold of all possible functions $g(\underset{\sim}{p})$ has never been successfully defined),[2] the mathematics just breaks down and produces infinite multiplicative constants as in (24).

PART II

THE DIAGONALIZATION OF $j(\underset{\sim}{x})$

First we note that by (1b) and (3b),

$$j(\underset{\sim}{x}) = \frac{1}{2}\left[\bar{\psi}(\underset{\sim}{x}), \psi(x)\right] = \bar{\psi}(\underset{\sim}{x})\,\psi(\underset{\sim}{x}) + \text{infinite constant.}$$

Since we are interested in the problem of diagonalizing $j(\underset{\sim}{x})$, the infinite constant may be ignored and we shall hereafter take

$$j(\underset{\sim}{x}) = \bar{\psi}(\underset{\sim}{x})\,\psi(\underset{\sim}{x}) \tag{31}$$

as the operator to be diagonalized.

Secondly, we reintroduce explicit awareness of the infinite Dirac sea of filled negative energy states by setting

$$\left.\begin{aligned} u_{\alpha+2}(\underset{\sim}{p}) &= v_{\alpha}(-\underset{\sim}{p}) \\ b_{\alpha+2}(\underset{\sim}{p}) &= c^{\dagger}_{\alpha}(-\underset{\sim}{p}) \end{aligned}\right\} \quad (\alpha = 1,2) \tag{32}$$

Substituting (32) in (2a) we find

$$\psi(\underset{\sim}{x}) = \sum_{\alpha=1}^{4} \int d\zeta \;\; u_{\alpha}(\underset{\sim}{p})\, e^{ip\cdot x}\, b_{\alpha}(\underset{\sim}{p}). \tag{33}$$

Thirdly we note that the Dirac vacuum, $|\phi\rangle$, for free fields is defined by the condition that for all $\underset{\sim}{p}$

$$b_{\alpha}(\underset{\sim}{p})|\phi\rangle = 0 \qquad (\alpha = 1,2),$$

and

$$b^{\dagger}_{\alpha}(\underset{\sim}{p})|\phi\rangle = 0 \qquad (\alpha = 3,4). \tag{34}$$

The b (and $b^{\dagger}$) operators destroy and create fermions in eigenstates of the free Hamiltonian. Clearly if we wish to diagonalize $j(\underset{\sim}{x})$ we shall need instead operators which create and destroy particles localized at a given position $\underset{\sim}{x}$. Therefore consider the operator

$$\eta_i(\underset{\sim}{p}) = \left(\frac{1}{2E_p}\right) \sum_{\alpha=1}^{4} u_{\alpha i}(\underset{\sim}{p})\; b_{\alpha}(\underset{\sim}{p}), \tag{35}$$

where $u_{\alpha i}(\underset{\sim}{p})$ is the i'th component $(i = 1,2,3,4)$ of the spinor $u_\alpha(\underset{\sim}{p})$. Define the spinor e_α by

$$e_{\alpha i} = \delta_{\alpha i} \tag{36}$$

From (33), (35), and (36) it follows that

$$\psi(\underset{\sim}{x}) = \sum_{\alpha=1}^{4} \int \frac{d^3 \underset{\sim}{p}}{(2\pi)^3} \; e_\alpha \, e^{i\underset{\sim}{p} \cdot \underset{\sim}{x}} \eta_\alpha(\underset{\sim}{p}), \tag{37}$$

and

$$\left[\eta_\alpha(\underset{\sim}{p}),\ \eta_\beta^\dagger(\underset{\sim}{p}')\right]_+ = \delta_{\alpha\beta} (2\pi)^3 \, \delta^3(\underset{\sim}{p} - \underset{\sim}{p}'). \tag{38}$$

Next consider the operator

$$\xi_\alpha(\underset{\sim}{x}) = \int \frac{d^3 \underset{\sim}{p}}{(2\pi)^3} \; e^{i\underset{\sim}{p} \cdot \underset{\sim}{x}} \, \eta_\alpha(\underset{\sim}{p}) \; . \tag{39}$$

Substituting (39) in (37) and (38) we find

$$\psi(\underset{\sim}{x}) = \sum_{\alpha=1}^{4} e_\alpha \, \xi_\alpha(\underset{\sim}{x}) \; , \tag{40}$$

and

$$\left[\xi_\alpha(\underset{\sim}{x}),\ \xi_\beta^\dagger(\underset{\sim}{x}')\right]_+ = \delta_{\alpha\beta} \, \delta^3(\underset{\sim}{x} - \underset{\sim}{x}'). \tag{41}$$

The ξ (and $\xi^\dagger$) operators are the desired operators which destroy and create particles localized at a given position. To construct an eigenvector of $j(\underset{\sim}{x})$ we simply define it by the condition (compare with (34)) that for all $\underset{\sim}{x}$

$$\xi_\alpha(\underset{\sim}{x})|j\rangle = 0 \qquad\qquad (\alpha = 1,2)$$

and

$$\xi_\alpha^\dagger(\underset{\sim}{x})|j\rangle = 0 \qquad\qquad (\alpha = 3,4). \tag{42}$$

If we use the standard representation where

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

we find

$$j(\underset{\sim}{x})|j\rangle = \psi^\dagger(\underset{\sim}{x})\,\gamma_0\psi(\underset{\sim}{x})|j\rangle = \sum_{\alpha=1}^{2}\left\{\xi^\dagger_\alpha(\underset{\sim}{x})\,\xi_\alpha(\underset{\sim}{x}) - \xi^\dagger_{\alpha+2}(\underset{\sim}{x})\,\xi_{\alpha+2}(\underset{\sim}{x})\right\}|j\rangle \ ,$$

which by (41) and (42)

$$= - C(\infty)\,|\,j\,\rangle \ ,$$

where $C(\infty)$ is an infinite constant given by

$$C(\infty) = \delta^3(0) = \int \frac{d^3\underset{\sim}{p}}{(2\pi)^3} \ .$$

Hence $|\,j\rangle$ is an eigenvector of $j(\underset{\sim}{x})$.

To obtain all the eigenvectors of $j(\underset{\sim}{x})$ consider an arbitrary function $f(\alpha, \underset{\sim}{x})$ whose range consists of the integers 0 and 1. Define the state $|f(\alpha, \underset{\sim}{x})\rangle$ by the condition that for all α and $\underset{\sim}{x}$,

$$\xi_\alpha(\underset{\sim}{x})|f\rangle = 0, \ \text{ if } f(\alpha, \underset{\sim}{x}) = 0$$

and

$$\xi^\dagger_\alpha(\underset{\sim}{x})|f\rangle = 0, \quad \text{if } f(\alpha, \underset{\sim}{x}) = 1. \tag{43}$$

Since the states $(\alpha, \underset{\sim}{x})$ are a complete orthonormal set for one particle we see by analogy to the case with any finite number of particles, that corresponding to the set of all distinct possible $f(\alpha, \underset{\sim}{x})$ we have a complete set of orthogonal basis states for all possible numbers of fermions. Furthermore,

$$j(\underset{\sim}{x})|f\rangle = \bar{\psi}(\underset{\sim}{x})\psi(\underset{\sim}{x})|f\rangle = \sum_{\alpha=1}^{2}\left\{\xi^\dagger_\alpha(\underset{\sim}{x})\,\xi_\alpha(\underset{\sim}{x}) - \xi^\dagger_{\alpha+2}(\underset{\sim}{x})\,\xi_{\alpha+2}(\underset{\sim}{x})\right\}|f\rangle$$

$$= C(\infty)\sum_{\alpha=1}^{2}\left\{f(\alpha, \underset{\sim}{x}) - f(\alpha + 2, \underset{\sim}{x})\right\}|f\rangle \ , \tag{44}$$

where if

$$f(\alpha, \underset{\sim}{x}) = 0$$

then

$$C(\infty)\, f(\alpha, \underset{\sim}{x}) = 0.$$

So $|f\rangle$ is an eigenvector of $j(\underset{\sim}{x})$ whose (infinite) eigenvalue is

$$C(\infty) \sum_{\alpha=1}^{2} \left\{ f(\alpha, \underset{\sim}{x}) - f(\alpha + 2, \underset{\sim}{x}) \right\} .$$

Therefore we have found the complete orthogonal set of eigenvectors and eigenvalues of $j(\underset{\sim}{x})$.[3] The infinite constant $C(\infty)$ appears for the following reason: Consider the case of only one particle localized at position $\underset{\sim}{x}'$ and spinor index α'. The corresponding state $|f_1\rangle$ has

$$f_1(\alpha, \underset{\sim}{x}) = \begin{cases} 1 & \text{if } \alpha = \alpha' \text{ and } \underset{\sim}{x} = \underset{\sim}{x}' \\ 0 & \text{otherwise} \end{cases} .$$

Then the eigenvalue of $j(\underset{\sim}{x})$ is

$$C(\infty) \sum_{\alpha=1}^{2} \left\{ f_1(\alpha, \underset{\sim}{x}) - f_1(\alpha + 2, \underset{\sim}{x}) \right\} = \begin{cases} \rho(\alpha') C(\infty) & \text{if } \underset{\sim}{x} = \underset{\sim}{x}' \\ 0 & \text{otherwise} \end{cases}$$

where

$$\rho(1) = \rho(2) = -\rho(3) = -\rho(4) = +1 .$$

However, $C(\infty) = \delta^3(0)$, so the eigenvalue of $j(\underset{\sim}{x})$ is just the expected result,

$$\rho(\alpha')\, \delta^3(\underset{\sim}{x} - \underset{\sim}{x}') .$$

Alternately consider the state $|f_2\rangle$ where

$$f_2(\alpha, \underset{\sim}{x}) = 1 \text{ if } \alpha = 1$$

and $\underset{\sim}{x}$ is within some finite volume V, and $f_2 = 0$ otherwise. Then the eigenvalue of $j(\underset{\sim}{x})$ is

$$C(\infty) \sum_{\alpha=1}^{2} \left\{ f_2(\alpha, \underset{\sim}{x}) - f_2(\alpha + 2, \underset{\sim}{x}) \right\} = \theta(\underset{\sim}{x})\, C(\infty),$$

where

$$\theta(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \underset{\sim}{x} \in V \\ 0 & \text{otherwise} \end{cases}$$

Thus in this case the eigenvalue is infinite for all $\underset{\sim}{x}$ within V, which simply reflects the infinite density of particles within V.

It is interesting to note that in a completely analogous way we can construct the set of all eigenvectors and eigenvalues of H_0, the free particle Hamiltonian. We simply consider all possible functions $g(\alpha, \underset{\sim}{p})$ whose range is 0 and 1, and define the corresponding state by the condition

$$\begin{aligned} b_\alpha(\underset{\sim}{p}) \mid g \rangle &= 0, \quad \text{if } g(\alpha, \underset{\sim}{p}) = 0 \\ b_\alpha^\dagger(\underset{\sim}{p}) \mid g \rangle &= 0, \quad \text{if } g(\alpha, \underset{\sim}{p}) = 1. \end{aligned} \tag{45}$$

Then we have

$$H_0 | g \rangle = \sum_{\alpha=1}^{4} \int d\zeta \rho(\alpha) E_p b_\alpha^\dagger(\underset{\sim}{p})\, b_\alpha(\underset{\sim}{p}) | g \rangle = \sum_{\alpha=1}^{4} \int \frac{d^3\underset{\sim}{x}\, d^3\underset{\sim}{p}}{(2\pi)^3} \rho(\alpha) E_p g(\alpha, \underset{\sim}{p}) | g \rangle \tag{46}$$

So $|g\rangle$ is an eigenvector of H_0. Just as before, the states $|g(\alpha, p)\rangle$ for all possible distinct functions g, constitute a complete orthogonal basis set for all possible numbers of fermions, but which diagonalizes H_0. Note that the conventional Hilbert space in which one does quantum field theory is an infinitesimally small subset of this complete orthogonal basis set. The conventional Hilbert space corresponds to the subset of those functions $g(\alpha, \underset{\sim}{p})$ such that for all $\underset{\sim}{p}$,

$$g(\alpha, \underset{\sim}{p}) = \begin{cases} 0 & \text{if } \alpha = 1, 2 \\ 1 & \text{if } \alpha = 3, 4 \end{cases} ,$$

<u>except</u> for a <u>finite</u> number of discrete values, α_i and p_i $(i = 1, 2, \ldots, n)$, where $g(\alpha_i, \underset{\sim}{p}_i)$ is arbitrary (but must be 0 or 1).

Returning to the eigenvectors of $j(\underset{\sim}{x})$ we may ask if there exists a unitary transformation from the complete orthogonal basis states $|g(\alpha, \underset{\sim}{p})\rangle$ which diagonalize H_0 to the complete orthogonal basis states $|f(\alpha, \underset{\sim}{x})\rangle$ which diagonalize $j(\underset{\sim}{x})$ and hence H_I. First we observe that, just as in the boson case, the set of basis states is so infinite that it seems impossible to define the notion of integration over all possible states, and therefore it is impossible to define a normalization for the basis states. So, there exists no unitary transformation connecting the $|g\rangle$ states with the $|f\rangle$ states. If we try to write any given $|f\rangle$ state as a superposition of $|g\rangle$ states this problem of the basis set being too infinite immediately appears. To see this, write the given $|f\rangle$ state in the form

$$|f(\alpha, \underset{\sim}{x})\rangle = \left[\prod_f \xi^\dagger_\alpha(\underset{\sim}{x})\right] |0\rangle \ , \tag{47}$$

where $|0\rangle$ is the no particle state defined by

$$\xi_\alpha(\underset{\sim}{x})|0\rangle = 0$$

for all α and $\underset{\sim}{x}$, and by $\prod_f$ is meant the produce over all α and $\underset{\sim}{x}$ such that $f(\alpha, \underset{\sim}{x}) = 1$. Now from Eqs. (35) and (39) it follows that

$$\xi_\alpha(\underset{\sim}{x}) = \sum_{\beta=1}^{4} \int d\zeta \, S_{\alpha\beta}(\underset{\sim}{x}, p) \; b_\beta(\underset{\sim}{p}), \tag{48}$$

where

$$S_{i\alpha}(\underset{\sim}{x}, \underset{\sim}{p}) = u_{\alpha i}(\underset{\sim}{p}) \, e^{i\underset{\sim}{p} \cdot \underset{\sim}{x}} \ .$$

Substituting this in (47) we find

$$|f\rangle = \prod_f \left[\sum_{\beta=1}^{4} \int d\zeta \; b^\dagger_\beta(\underset{\sim}{p}) \; S^\dagger_{\beta\alpha}(\underset{\sim}{p}, \underset{\sim}{x})\right] |0\rangle \ . \tag{49}$$

If there are only a finite number, N, of particles in the state $|f\rangle$ we have from (49)

$$|f\rangle = \prod_{i=1}^{N}\left[\sum_{\beta=1}^{4}\int d\zeta\, b_{\beta}^{\dagger}(\mathbf{p})\, S_{\beta\alpha_i}^{\dagger}(\mathbf{p},\mathbf{x}_i)\right]|0\rangle$$

$$= \sum_{\beta_1,\ldots,\beta_N=1}^{4}\int d\zeta_1\ldots d\zeta_N \left\{\prod_{i=1}^{N} S_{\beta_i\alpha_i}^{\dagger}(\mathbf{p}_i,\mathbf{x}_i)\right\}\times\left\{\prod_{i=1}^{N} b_{\beta_i}^{\dagger}(\mathbf{p}_i)\right\}|0\rangle. \quad (50)$$

Since the expression

$$\prod_{i=1}^{N} b_{\beta_i}^{\dagger}(\mathbf{p}_i)\,|0\rangle$$

is just a $|g\rangle$ basis state, Eq. (50) gives $|f\rangle$ as the required superposition of g states. However if, as is generally true, the state $|f\rangle$ contains a continuously infinite number of particles, then the

$$\left[\prod_{f} S_{\beta\alpha}^{\dagger}(\mathbf{p},\mathbf{x})\right]$$

will in general be either zero or infinite, and there will be a continuously infinite number of summations and integrals to do, which is also ill defined. Therefore, in general, it is impossible to write a given $|f\rangle$ state as a superposition of $|g\rangle$ states.

CONCLUSION

We have shown that a basic element of the nonrelativistic quantum theory is absent in quantum field theory. We are still able to construct explicit representations which diagonalize either the free part or the interaction part of the Hamiltonian in quantum field theory, just as in nonrelativistic quantum theory, but there is no unitary transformation connecting the two representations.

In nonrelativistic quantum mechanics all the Hermitian operators which occur in the theory can be diagonalized, and the various representations which diagonalize them are connected by unitary transformations. This feature is fundamental to both calculation and physical interpretation of the theory. The fact that this is no longer true in quantum field theory, as we have amply demonstrated, is a serious difficulty which has yet to be understood and overcome.

REFERENCES AND FOOTNOTES

1. Note that these states are fundamentally different from the usual Glauber coherent states, which are of the form $W(g(\underset{\sim}{p}))|\phi\rangle$.

2. For a discussion of this problem see F. Coester and R. Haag, Phys. Rev. 117, 1137-45 (1960).

3. States somewhat like the boson and fermion states constructed here have been used before by Schiff and others. They, however, all work on a lattice space and to my knowledge the continuum case has never been treated before. For examples of this lattice space treatment see: L. I. Schiff, Phys. Rev. 92, 766 (1953); D. H. Holland, Phys. Rev. 98, 788 (1955).