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Fourier Descriptors for Plane Closed Curves*

by

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Abstract

A method for the analysis and synthesis of closed curves in the plane is developed using the Fourier Descriptors of Cosgriff [1]. A curve is represented parametrically as a function of arc length by the accumulated change in direction of the curve since the starting point. This function is expanded in a Fourier Series and the coefficients are arranged in the amplitude/phase-angle form. It is shown that the amplitudes are pure form-invariants as well as are certain simple functions of phase-angles. Rotational and axial symmetry are related directly to simple properties of the Fourier Descriptors. An analysis of shape similarity or symmetry can be based on these relationships; also closed symmetric curves can be synthesized from almost arbitrary Fourier Descriptors. It is established that the Fourier Series expansion is optimal and unique with respect to obtaining coefficients insensitive to starting point. Several examples are provided to indicate the usefulness of Fourier Descriptors as features for shape discrimination and a number of interesting symmetric curves are generated by computer and plotted out.

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INDEX TERMS

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Introduction

The problem of discriminating planar shapes is one of the most familiar and fundamental problems in pattern recognition. It entails the assignment of an unknown shape to one of several classes of shapes based on a finite set of measurements (features) made on the shape. The digital representation of the shape usually is initially a digital picture [14] — that is, a rectangular matrix whose elements are from some finite range (0, B) of nonnegative integers. When B = 1 we have a black/white digital picture and each connected subset of elements having the value 1 (black points) constitutes a two-dimensional shape. Figure 5a exhibits such a shape. There are many different ways to obtain numerical features from digital shapes but we find features based on the boundary of the shape to be most compelling intuitively. There is a substantial body of psychological and psychophysical evidence suggesting the importance of edges in visual scene analysis. We cite in particular the work of Attneave and Arnoult [15], Attneave [16], Uhr [17], Lettvin et al. [18], and Hubel and Wiesel [19].

The transformation from binary matrix to polygonal boundary can be effected by the method of Ledley [20] or Zahn [8], [9]. When B > 1 (so-called grey-scale pictures) more sophisticated methods are sometimes required to extract boundary curves but whatever method is employed the theory developed in this paper is appropriate. The topic is analysis and synthesis of closed curves in the plane and parts of the theory could be applied to nonclosed curves. Although the boundary of a shape has as many separate closed curves as its degree of connectivity we discuss single curves only. Full comparison of multiply connected shapes should involve analysis of all boundary curves but the treatment of each curve will be similar.

At this point our problem is reduced to one of extracting a finite set of numerical features from a closed curve, features which will tend to separate the shapes of different classes relative to the intraclass dispersion. For this purpose, we shall use the Fourier Descriptors, first suggested by Cosgriff [1]. A starting point on the boundary is selected and a function $\theta(\ell)$ is defined which measures angular direction of the curve as a function of arc length. After appropriate normalization this periodic function is expanded in a Fourier series and the coefficients of a truncated expansion are used as shape features called Fourier Descriptors. Higher order terms represent changes in direction of the curve over very small arc lengths and their elimination will probably reduce noise and serve to accentuate lower order terms which contain more macroscopic information on the shape.

The report by Cosgriff [1] was followed by a sequence of reports by Fritzsche [2], Raudseps [3], Borel [4], and Brill [5], [6], [7]. Material from these reports tends to support the idea that look-alike shapes are usually near each other in a space of Fourier Descriptor features endowed with the euclidean metric. The last two papers in the sequence Brill [6], [7] are especially noteworthy because they report on the results of recognition experiments which had remarkable success considering the context in which the recognition was performed.

In this paper we define a normalized cumulative angular function $\phi^*(t)$ for a simple closed curve γ and expand ϕ^* in a Fourier Series to obtain descriptive coefficients which we call Fourier Descriptors, although ϕ^* is a slightly different function than was used in the earlier work [1-7]. There follows a formula for reconstructing a curve from its Fourier Descriptors and the explicit formulas for calculating the Fourier Descriptors of a polygonal curve. Next we state several important relationships between the geometry of shapes and algebraic properties of the Fourier Descriptors. Using these results we are able to define form-invariant measures based on the phase angle Fourier Descriptors, that is functions of these phase angles which are not dependent on the starting point used to generate the function ϕ^*. Also using the basic relationships, we suggest methods for shape similarity analysis and the testing of symmetry with several concrete examples. A short discussion of Fourier Descriptors based on the curvature function of γ shows that they differ only slightly from those based on ϕ^*. Next we consider the synthetic generation of curves from arbitrary Fourier Descriptors and establish some simple sufficient conditions for such curves to be closed. It is then proved that the Fourier Series expansion is the only expansion for which the coefficients are largely independent of starting point. Several examples are shown to indicate the usefulness of Fourier Descriptors as features for shape discrimination and the theoretical material on curve reconstruction is used to generate some interesting curves with axial as well as rotational symmetry.

Fourier Descriptors of a Curve and the Reconstruction Theorem

We assume γ is a clockwise-oriented simple closed curve with parametric representation $(x(\ell),y(\ell))=Z(\ell)$ where ℓ is arc length and $0 \leq \ell \leq L$. Denote the angular direction of γ at point ℓ by the function $\theta(\ell)$ assuming that γ is smooth in the sense of Buck [21] and let $\delta_0=\theta(0)$ be the absolute angular direction at the starting point $Z(0)$. We now define the cumulative angular function $\phi(\ell)$ as the net amount of angular bend between starting point and point ℓ (see Fig. 1a). With this definition $\phi(0)=0$ and $\phi(\ell) + \delta_0$ is identical to $\phi(\ell)$ except for a possible multiple of 2π. If the curve γ winds in a spiral then $|\phi(\ell)|$ achieves values larger than 2π. Figure 2b shows $\phi(\ell)$ for the shape depicted in Fig. 2a.

It is not hard to see that $\phi(L)=-2\pi$ because all smooth simple closed curves with clockwise orientation have a net angular bend of -2π. As a result, $\phi(L)$ does not convey any shape information. The domain of definition $[0, L]$ of $\phi(\ell)$ simply contains absolute size information and we would like to normalize to the interval $[0, 2\pi]$ which is standard for periodic functions. Hence we define a normalized variant $\phi^*(t)$ (see Fig. 2c) whose domain is $[0, 2\pi]$ and such that $\phi^*(0) = \phi^*(2\pi) = 0$. The formal definition is

$$\phi^*(t) = \phi\left(\frac{Lt}{2\pi}\right) + t \tag{1}$$

and ϕ^* is invariant under translations, rotations and changes of perimeter L. As intuitive justification for the definition of ϕ^* we note that $\phi^* \equiv 0$ for a circle which is in some sense the most <u>shapeless</u> closed curve! Viewed in this light the function $\phi^*(t)$ measures the way in which the shape in question differs from a circular shape.

What we have done so far is to map all plane simple closed curves with starting point into the class of periodic functions on $[0, 2\pi]$ in such a way that all curves of identical shape and starting point go into the same function ϕ^*. Two plane curves have <u>identical shape</u> if they differ only by a combination of translation, rotation and change in size.

We now expand ϕ^* as a Fourier series

$$\phi^*(t) = \mu_0 + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt) \tag{2}$$

In polar form the expansion is

$$\phi^*(t) = \mu_0 + \sum_{k=1}^{\infty} A_k \cos(kt - \alpha_k) \tag{3}$$

where (A_k, α_k) are polar coordinates of (a_k, b_k). These numbers A_k and α_k are the Fourier Descriptors for curve γ and are known respectively as the kth harmonic amplitude and phase angle.

Having developed this description $\left\{A_k, \alpha_k\right\}_1^\infty$ it is of interest to know how easily the curve might be reconstructed from these numbers and how well a truncated description $\left\{A_k, \alpha_k\right\}_1^N$ would reconstruct the curve. The theorem which follows answers the first question and a later section provides some insight into the second. The first question reduces quickly to one of reconstructing γ easily from its angle-versus-length function $\theta(\ell)$.

Reconstruction Theorem

If curve γ is described by $\theta(\ell)$ and starting point $Z(0)$ the position of point $Z(\ell)$ can be gotten from the expression

$$Z(\ell) = Z(0) + \int_0^\ell e^{i\theta(\lambda)} d\lambda \tag{4}$$

which is equivalent to

$$x(\ell) = x(0) + \int_0^\ell \cos\theta(\lambda)\, d\lambda; \quad y(\ell) = y(0) + \int_0^\ell \sin\theta(\lambda)\, d\lambda$$

Proof:

By definition $\theta(\ell)$ measures the direction of the velocity vector $\gamma'(\ell)=(x'(\ell), y'(\ell))$ which is tangent to γ at ℓ (see Fig. 1b). Whenever a curve is parametrized by its own arc length the speed $|\gamma'(\ell)|$ is always 1 (see Buck [22]) and we obtain immediately $x'(\ell)=\cos\theta(\ell)$ and $y'(\ell)=\sin\theta(\ell)$. The result follows by substituting these values into the fundamental therem of integral calculus

$$x(\ell) - x(0) = \int_0^\ell x'(\lambda)\, d\lambda \quad \text{and} \quad y(\ell) - y(0) = \int_0^\ell y'(\lambda)\, d\lambda$$

Corollary

Any function $\theta^*(\ell) \equiv \theta(\ell)$ and mod 2π works as well in the above reconstruction formula.

Using the reconstruction theorem and a truncated Fourier series expansion of ϕ^* we shall give a practical formula for curve reconstruction based on Fourier Descriptors $\left\{A_k, \alpha_k\right\}_1^N$. Invoking the corollary we can use $\theta^*(\ell) = \phi(\ell) + \delta_0$ in place of $\theta(\ell)$; then using Eqs. (4) and (1), the transformation of variable $\pi = \frac{Lt}{2\pi}$ and the truncated version of Eq. (3) we obtain the formula

$$Z(\ell) = Z(0) + \frac{L}{2\pi}\int_0^{\frac{2\pi\ell}{L}} e^{i\left[-t + \delta_0 + \mu_0 + \sum_{k=1}^{N} A_k \cos(kt - \alpha_k)\right]} dt \quad (5)$$

If we desire a curve of similar shape but starting point Z^*, initial direction δ^*, and total arc length L^* we use the same formula but with these values instead of $Z(0)$, δ_0 and L.

Calculating Fourier Descriptors from a Polygonal Curve

In this section we shall derive formulas for the Fourier coefficients $\left\{a_k, b_k\right\}$ and μ_0 when γ is a polygonal curve. We assume the curve γ has m vertices $V_0, \ldots, V_{m-1}$ and that the edge (V_{i-1}, V_i) has length $\Delta\ell_i$. The change in angular direction at vertex V_i is $\Delta\phi_i$ and $L = \sum_{i=1}^{m} \Delta\ell_i$. With these definitions (see Fig. 3) it is not hard to verify that

$$\phi(\ell) = \sum_{i=1}^{k} \Delta\phi_i \quad \text{for} \quad \sum_{i=1}^{k} \Delta\ell_i \leq \ell < \sum_{i=1}^{k+1} \Delta\ell_i$$

and

$$\phi(\ell) = 0 \quad \text{for} \quad 0 \leq \ell < \Delta\ell_1.$$

Expanding ϕ^* we get $\phi^*(t) = \mu_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$ where

$$\mu_0 = \frac{1}{2\pi}\int_0^{2\pi} \phi^*(t)\, dt$$

and

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \phi^*(t) \cos nt \, dt \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} \phi^*(t) \sin nt \, dt$$

Remembering that $\phi^*(t) = \phi\left(\frac{Lt}{2\pi}\right) + t$ and changing variable $\lambda = \frac{Lt}{2\pi}$ we get

$$\mu_0 = \frac{1}{L}\int_0^L \phi(\lambda) \, d\lambda + \pi$$

$$a_n = \frac{2}{L}\int_0^L \left(\phi(\lambda) + \frac{2\pi\lambda}{L}\right) \cos \frac{2\pi n\lambda}{L} \, d\lambda$$

$$b_n = \frac{2}{L}\int_0^L \left(\phi(\lambda) + \frac{2\pi\lambda}{L}\right) \sin \frac{2\pi n\lambda}{L} \, d\lambda$$

Exploiting the fact that $\phi(\ell)$ is a step function we obtain (after reasonably straightforward but somewhat lengthy calculations [3])

$$\mu_0 = -\pi - \frac{1}{L}\sum_{k=1}^{m} \ell_k \Delta\phi_k \, , \tag{6}$$

$$a_n = \frac{-1}{n\pi}\sum_{k=1}^{m} \Delta\phi_k \sin\frac{2\pi n \ell_k}{L} \, , \tag{7}$$

and

$$b_n = \frac{1}{n\pi}\sum_{k=1}^{m} \Delta\phi_k \cos\frac{2\pi n \ell_k}{L} \tag{8}$$

where

$$\ell_k = \sum_{i=1}^{k} \Delta\ell_i$$

The final forms of the expressions for a_n, b_n are especially appealing because of their similarity and also because $\Delta\phi_k$ represents the angular change (bend) in the curve's direction at the kth polygonal vertex, and ℓ_k is the arc length from starting vertex to kth vertex.

It is clear from these expressions alone that the Fourier coefficients (a_n, b_n) contain no information relating to absolute position or rotational orientation of the curve.

In the amplitude/phase angle form of the Fourier series

$$\phi^*(t) = \mu_0 + \sum_{n=1}^{\infty} A_n \cos(nt - \alpha_n)$$

the (A_n, α_n) are polar coordinates for the point (a_n, b_n).

A_n is called the nth harmonic amplitude and α_n the nth harmonic phase angle. Of course when $A_n = 0$ the nth term vanishes and α_n is undefined.

Relationships Between Fourier Descriptors and Geometry of Shapes

A number of useful relationships link algebraic properites of the Fourier Descriptors of a simple closed curve with geometric properties of the shape bounded by that curve. Probably the most powerful justification for the use of FDs (Fourier Descriptors) is the invariance of harmonic amplitudes $\{A_n\}$ and certain simple functions of pairs of phase angles $\{\alpha_n\}$ under translations, rotations, changes in size, and shifts in the starting point. It could be said that the $\{A_n\}$ are pure shape features although pure shape information resides in the $\{\alpha_n\}$ also. Intuitively speaking, if we cut from cardboard a two-dimensional shape bounded by a simple closed curve, lay it flat on either side in any position and orientation, and select an arbitrary starting point -- then respective values of $\{A_n\}$ will be the same each time. When an algorithm computes the FDs for a polygonal curve the information about position, orientation and size is so to speak factored out in the values Z_0, δ_0 and L and is therefore still available for discrimination tests. Mirror images and shifts of the starting point can be detected by an appropriate examination of the phase angles $\left\{ \alpha_n \right\}$ as described below.

We now describe informally several important algebraic-geometric properites of the Fourier Descriptors of a simple closed curve. They are stated formally and proved as Theorems 1, 2, 3 in the appendix. First we remind the reader that two curves which differ only in position, orientation and size (i.e. Z_0, δ_0, L) but have analogous starting points are transformed into the same waveform $\phi^*(t)$ and hence have identical Fourier Descriptors $\left\{ A_n, \alpha_n \right\}$. If γ and γ' represent the same curve with different starting points Z_0 and Z_0' then the FDs $\left\{ A_n, \alpha_n \right\}$ and $\left\{ A_n', \alpha_n' \right\}$ for γ and γ' satisfy

$$A_n' = A_n \qquad \alpha_n' = \alpha_n + n\Delta\alpha \qquad \Delta\alpha = \frac{-2\pi\Delta\ell}{L} \tag{9}$$

where $\Delta\ell$ is the clockwise arclength along the curve from Z_0 to Z'_0,

Two curves γ and γ' which are reflections of one another (mirror images) but have identical starting points satisfy

$$A_n' = A_n \qquad \alpha_n' + \alpha_n \equiv \pi \pmod{2\pi} \tag{10}$$

It should be understood that the relations involving α_n, α'_n hold only when α_n, α_n' exist (i.e. A_n, $A_n' \neq 0$).

In addition to relationships which exist between two different curves we can investigate the affect on FDs of axial and rotational symmetries of a single curve.

We find that a curve γ with axial symmetry satisfies

$$2\alpha_n = \pi - n\Delta\alpha \tag{11}$$

After a shift of starting point by $\beta = \Delta\alpha/2$ the resulting curve γ' has its starting point on an axis of symmetry and the $\{\alpha'_n\}$ have the following simple property:

$$\alpha'_n \equiv \pi/2 \pmod{\pi}. \tag{11a}$$

Turning our attention to rotational symmetry we find that a curve having k-fold rotational symmetry (i.e. goes into itself under a rotation of $2\pi/k$) has zero harmonic amplitudes for all indices which are not integral multiples of k, that is

$$A_n = 0 \quad \text{for} \quad n \not\equiv 0 \pmod{k}. \tag{12}$$

The general form of the propositions above was to give necessary conditions on the Fourier Descriptors for certain geometric properties to exist. They are sufficient conditions as well so that we obtain a one-one correspondence between a class of geometric properties of two-dimensional shapes and a class of algebraic properties of the Fourier Descriptors extracted from the boundary curves of those shapes. These statements are proved in the appendix.

In practice we shall want to infer geometric properties from the properties of a truncated sequence of Fourier Descriptors $\{A_n, \alpha_n\}_1^N$ with the knowledge that similarity of shapes is likely to be only approximate. The experience we have gathered using FDs suggests that shapes which "look similar" to people are near each other in the space of low-order FDs. This assumes that the phase angles have been normalized so different starting points don't affect the comparison.

The omission of μ_0 from this discussion warrants some comment. Because $\phi^*(0) = 0$ we find that

$$\mu_0 = - \sum_{n=1}^{\infty} a_n$$

and therefore μ_0 carries information related to the particular starting point used. This makes it undesirable as a shape discrimination feature.

<u>Form-Invariant Measures Based on Phase Angles</u>

Since the phase angles $\{\alpha_n\}$ contain shape information it is of interest to have measures based on $\{\alpha_n\}$ which are not affected by change of starting point. In other words, we desire form-invariant measures which are functions of $\{\alpha_n\}$. Furthermore, because the harmonic amplitudes $\{A_n\}$ do not discriminate reflected images we would hope the measures based on $\{\alpha_n\}$ would discriminate such reflections.

Define $F_{kj}[\gamma] = j^* \alpha_k - k^* \alpha_j$ where $j^* = j/\gcd(j,k)$, $k^* = k/\gcd(j,k)$ and gcd denotes greatest common divisor. In other words, k^*/j^* is k/j reduced to lowest terms.

Now if $\gamma = \{A_n, \alpha_n\}$ and $\gamma' = \{A'_n, \alpha'_n\}$ is γ with a shifted starting point then $\alpha'_n = \alpha_n + n\Delta\alpha$ and we get

$$\begin{aligned}
F_{kj}[\gamma] - F_{kj}[\gamma'] &= j^* \alpha_k - k^* \alpha_j - (j^* \alpha'_k - k^* \alpha'_j) \\
&= j^* \alpha_k - k^* \alpha_j - j^*(\alpha_k + k\Delta\alpha) + k^*(\alpha_j + j\Delta\alpha) \\
&= (k^* j - j^* k)\Delta\alpha \\
&= \frac{(kj - jk)}{\gcd(j,k)} \cdot \Delta\alpha \\
&= 0
\end{aligned}$$

Hence

$$F_{kj}[\gamma] \equiv F_{kj}[\gamma'] \pmod{2\pi} \text{ when } \gamma' = \text{shift}(\gamma) \qquad (13)$$

On the other hand, if $\gamma' = \{A'_n, \alpha'_n\}$ is a reflected and shifted version of γ then $\alpha'_n = (\pi - \alpha_n) + n\Delta\alpha$ and we obtain

$$F_{kj}[\gamma] + F_{kj}[\gamma'] = j^*\alpha_k - k^*\alpha_j + j^*(\pi - \alpha_k + k\Delta\alpha) - k^*(\pi - \alpha_j + j\Delta\alpha)$$
$$= (j^* - k^*)\pi$$

Hence

$$F_{kj}[\gamma] + F_{kj}[\gamma'] \equiv (j^* - k^*)\pi \pmod{2\pi} \text{ when } \gamma' = \text{shift (reflect } (\gamma)) \tag{14}$$

Since axial symmetry means invariance under a reflection and the appropriate shift we can let $\gamma' = \gamma$ in the above condition and obtain

$$F_{kj}[\gamma] \equiv (j^* - k^*)\,\frac{\pi}{2} \pmod{\pi} \text{ when } \gamma \text{ is axially symmetric} \tag{15}$$

(We get mod π because $2\alpha \equiv \beta \pmod{2\pi}$ means $2\alpha - \beta = 2\pi n$ so that $\alpha - \beta/2 = \pi n$ which is equivalent to $\alpha = \beta/2 \pmod{\pi}$).

The choice of j^* and k^* as multipliers was made because they are the smallest pair of positive integral values whose ratio is j/k. The ratio condition is all that is really required to prove the above relationships and nonintegral values are avoided because every time we divide our values $\{\alpha_n\}$ we divide the modulus a corresponding amount (e.g., if $\alpha_1 \equiv \alpha_3 \pmod{2\pi}$ then $\alpha_1/3 \equiv \alpha_3/3 \pmod{2\pi/3}$).

A simple example helps to explain why we choose (j^*, k^*) over (j, k) as multipliers. Suppose that $j = 2$, $k = 4$ so that $j^* = 1$, $k^* = 2$. If we calculate values of $\alpha_4 - 2\alpha_2$ over two classes representing distinct shapes it is possible that the values would fall in disjoint intervals $\left[0, \frac{2\pi}{3}\right]$ and $\left[\frac{4\pi}{3}, 2\pi\right]$ with a significant gap between. Unfortunately, the intervals for the values of $2\alpha_4 - 4\alpha_2$ would be $\left[0, \frac{4\pi}{3}\right]$ and $\left[\frac{2\pi}{3}, 2\pi\right]$ which overlap considerably. The change in scale involves loss of discrimination because the phase angles are only known modulo 2π.

We must add a warning about applying these properties to real data where the geometric properties are not exact. Whenever a harmonic amplitude A_k is small then the significance of α_k is greatly diminished and therefore one cannot expect the measure F_{kj} based on α_k to reveal anything significant about the geometry of the shape.

Analysis of Shape Symmetries and Similarities

If we are given the FDs $\{A_n, \alpha_n\}$ for a closed curve γ we might like to know how nearly symmetric γ is in the rotational or axial sense. For k-rotational symmetry we propose the following measure

$$R_k[\gamma] = \sum_{j \not\equiv 0 \,(\mathrm{mod}\ k)} A_j \tag{16}$$

where the sum is finite depending on what frequency has been chosen to truncate the sequence of FDs. A curve with perfect k-rotational symmetry has $R_k[\gamma] = 0$ and we expect that $R_k[\gamma]$ near zero will generally indicate a shape which would be judged nearly symmetric.

For axial symmetry we propose

$$X\gamma] = \sum \left| F_{kj} - \frac{\pi}{2}(j^* - k^*) \right| \tag{17}$$

where the sum is taken over some subset of pairs (k, j) preferably avoiding those pairs where either or both of A_k, A_j are small. Another possibility is to weight each term with a weight $w_{kj} = \frac{1}{2}(A_k + A_j)$ or $w_{kj} = \min\{A_k, A_j\}$. The latter seems to be what is required since <u>one</u> small harmonic amplitude suffices to destroy the measure F_{kj}.

Given curves γ and γ' with Fourier Descriptors $\{A_n, \alpha_n\}$ and $\{A'_n, \alpha'_n\}$ how similar are they? Or how close are they to being mirror images? For similarity the following measures are suggested by the previous sections

$$S_A\ [\gamma, \gamma'] = \sum \left| A_k - A'_k \right| \tag{18}$$

$$S_\alpha\ [\gamma, \gamma'] = \sum \left| F_{kj} - F'_{kj} \right| \tag{19}$$

where, as before, weights can be inserted in the formula for S_{α}.

To measure the extent to which γ' is a mirror image of γ we use S_A and the following measure based on the phase angles.

$$M[\gamma, \gamma'] = \sum \left| F_{kj} + F'_{kj} - (j^* - k^*)\pi \right| \tag{20}$$

In computing these absolute differences care must be taken to get the smallest magnitude change modulo the appropriate uncertainty. The modulus is 2π for all α-based measures except $X[\gamma]$ which uses π.

If we have found two curves to be almost identical in shape by virtue of $S_A[\gamma, \gamma']$ and $S_{\alpha}[\gamma, \gamma']$ being nearly zero then what is a good estimate for the shift $\Delta\alpha$ between their respective starting points? A least squares fit of $f(n) = n\Delta\alpha$ to $(\alpha'_n - \alpha_n)$ suggests itself but there is a serious difficulty. The quantities $(\alpha'_n - \alpha_n)$ are not single-valued because phase angles are modulo 2π. In practice we have found that for quite similar shapes one can fix the $(\alpha'_n - \alpha_n)$ at what seem clearly to be the best choices modulo 2π and then use least squares to get the finely-tuned estimate of $\Delta\alpha$. It is helpful that $|\Delta\alpha| < \pi$ can be assumed!

This pseudo-least-squares can be used (with analogous caution) to find the approximate axis of best axial symmetry and the shift between analogous starting points for mirror images.

To get a feel for how well the values $\{A_n, \alpha_n\}$ reflect shape properties we have calculated in Table 1 some FDs for the eight shapes in Fig. 4. In particular, the four shapes X, Y, Z, W are in form quite similar except that Y, W are reflected versions of X, Z. More precisely, the mirror image of X is more like Y than X itself is like Y. We expect to find that

$$\left| F_{21}[X] + F_{21}[Y] - \pi \right| \ll \left| F_{21}[X] - F_{21}[Y] \right|$$

In actuality we get .112 ≪ 1.169 so that X is very nearly the same as Y without reflection. Performing the corresponding test on F_{32} we get .339 ≪ 2.828 so both F_{21} and F_{32} strongly support the hypothesis that X and Y are nearly mirror images rather than similar without reflection.

Turning our attention to Y and W we find that

$$\left| F_{21}[Y] - F_{21}[W] \right| = .210 \ll 1.071 = \left| F_{21}[Y] + F_{21}[W] - \pi \right|$$

and correspondingly for F_{32} we get .628 ≪ 3.117 so Y and W are more nearly identical than they are mirror images.

According to formulas of the previous section axial symmetry is revealed by F_{21} and F_{32} being near $\pi/2$ or $3\pi/2$ (i.e., nearly $\pi/2 \bmod \pi$). The reader is invited to check for himself the extent to which Table 1 reflects such geometric properties of the shapes in Fig. 4.

Fourier Descriptors Based on the Curvature Function

In the original report on Fourier Descriptors Cosgriff [1] suggests two alternate ways of defining the descriptors. One way is to expand the angle-versus-length function $\theta(\ell)$ in a Fourier series as we have done in this paper. The second way is to expand the curvature function $k(\ell) = \frac{d\theta}{d\ell}$ instead. As is well-known, $k(\ell) = \frac{-1}{R_\ell}$ where R_ℓ is the radius of curvature of curve θ at the point ℓ. The minus sign is not normal but simply reflects our convention that clockwise is negative.

One might have expected $k(\ell)$ to be the prime choice for basing Fourier Descriptors but there is a problem. For a polygonal curve, $\theta(\ell)$ is a step function so $k(\ell) = \frac{d\theta}{d\ell}$ is zero almost everywhere and infinite at the discrete jumps of $\theta(\ell)$. This makes $k(\ell)$ a poor candidate for polygonal curves and therefore of little use for

TABLE 1

SHAPE	X	Y	Z	W	R	S	T	U
A_1	.366	.223	.440	.363	.024	.468	0	1.558
A_2	.866	.729	.872	.717	.116	.840	.004	1.081
A_3	.333	.289	.500	.513	.272	.384	0	.592
A_4	.250	.217	.108	.243	.025	.229	.023	.122
A_5	.273	.105	.135	0	.139	.244	0	.161
α_1	2.099	1.043	4.361	2.509	4.414	6.269	—	3.867
α_2	2.099	1.156	.274	3.878	1.252	4.702	5.473	.033
α_3	3.142	0	3.670	4.397	3.844	6.237	—	2.583
α_4	1.571	1.113	4.222	.827	1.969	4.723	6.237	6.057
α_5	1.043	2.099	6.108	—	3.467	6.181	—	4.562
F_{21}	4.184	5.353	4.118	5.143	4.990	4.730	—	4.865
F_{32}	-.013	-3.468	.235	-2.840	3.932	-1.638	—	5.067
F_{42}							-4.709	

$\pi/2 = 1.571$

$3\pi/2 = 4.713$

practical shape discrimination. We can however investigate what such Fourier Descriptors would be like if the curve were smooth.

If we take polygonal curve γ and compute the Fourier Descriptors $\{A_n, \alpha_n\}$ based on the angle-versus-length function $\theta(\ell)$ then (see integrand in Eq. (5))

$$\hat{\theta}(\ell) = \frac{-2\pi\ell}{L} + \mu_0 + \sum_{k=1}^{N} A_k \cos\left(\frac{2\pi k\ell}{L} - \alpha_k\right) + \delta_0$$

is a reasonable approximation to θ and what's more it's smooth so $\frac{d\hat{\theta}}{d\ell}$ is well defined and easily calculated!

If we define $k(\ell) = \frac{d\hat{\theta}}{d\ell}$ and $k^*(t) = k\left(\frac{Lt}{2\pi}\right)$ for $t \in [0, 2\pi]$ we find

$$k^*(t) = B_0 + \sum_{k=1}^{N} B_k \cos(kt - \beta_k)$$

where

$$B_0 = -\frac{2\pi}{L} \qquad B_k = \frac{2\pi k A_k}{L} \quad \text{and} \quad \beta_k = \left(\alpha_k - \frac{\pi}{2}\right)$$

In other words, $\{B_k, \beta_k\}$ are the curvature based Fourier Descriptors for a smooth curve which approximates the curve for which $\{A_k, \alpha_k\}$ are the angle-versus-length based Fourier Descriptors. The fact that B_k depends on L shows that $k(\ell)$ is not a size invariant function the way $\theta(\ell)$ is. To fix this we should probably define $k^*(t) = Lk\left(\frac{Lt}{2\pi}\right)$. Note that the constant term $B_0 = -\frac{2\pi}{L}$ is correct for the average curvature. In particular, for a circle of radius R we get constant curvature $k^*(t) = \frac{-1}{R}$ so the average curvature is also $\frac{-1}{R}$. But $L = 2\pi R$ so $B_0 = -\frac{2\pi}{L} = \frac{-1}{R}$ making B_0 the average curvature.

The most important thing we find from calculating the $\{B_k, \beta_k\}$ is their extremely simple relationship to the angle-versus-length based Fourier Descriptors $\{A_k, \alpha_k\}$. The new harmonic amplitudes $\{B_k\}$ differ from $\{A_k\}$ only in the sense that the $\{B_k\}$ give greater weight to higher frequencies. The descriptors based on curvature are therefore essentially equivalent to those based on the angle-versus length function and there is no need to further consider these alternative descriptors.

Synthetic Generation of Curves from Fourier Descriptors

According to the Reconstruction Theorem proved in an earlier section a curve can be reconstructed from a knowledge of its Fourier Descriptors $\{A_k, \alpha_k, \mu_0\}$ and the triple (L, δ_0, Z_0) giving length, initial tangential direction and position of starting point. We repeat the formula here for ease of reference.

$$Z(\ell) = Z_0 + \frac{L}{2\pi} \int_0^{2\pi\ell/L} e^{i\left\{-t+\delta_0+\mu_0+\sum_{j=1}^{N} A_j \cos(jt-\alpha_j)\right\}} dt \tag{21}$$

This formula suggests the possibility of making up a fictitious set of Fourier Descriptors $\{A_k, \alpha_k\}$ and seeing what sort of curve $Z(\ell)$ results. We should mention that μ_0 is not really a free variable but is given by $\{A_k, \alpha_k\}$ as

$$\mu_0 = \sum_{k=1}^{N} A_k \cos(-\alpha_k) = \sum_{k=1}^{N} A_k \cos \alpha_k$$

Now given any $\{A_k, \alpha_k\}_1^N$ and a triple (L, δ_0, Z_0), $Z(\ell)$ becomes a well-defined continuous function on $[0, L]$ and hence traces out a curve in the plane. Almost at once we would like to know which $\{A_k, \alpha_k\}$ generate closed curves?

We have obtained some simple conditions under which $Z(L) = Z_0$ and they are presented in the following theorem (see Appendix for proof).

Closure Theorem:

The function $Z(\ell)$ represents a closed curve if one of the following conditions holds:

1. A_1 is a zero of the first Bessel function $J_1(x)$ and $A_n = 0$ for $n \geq 2$.
2. $A_n = 0$ for all $n \not\equiv 0 \pmod{k}$ where $k \geq 2$.

As a special case of condition 2 in the Closure Theorem we see that if only one $A_k (k \geq 2)$ is nonzero then $Z(\ell)$ is a closed curve with k-rotational symmetry. After seeing a number of curves reconstructed from single terms we can hazard the following interpretation of the resulting curve. Starting with a circle you push in toward the center at k equally spaced points on the circle and how hard you push is determined by the magnitude of A_k.

Using relationships stated earlier (see Eq. (11)) one can generate curves with axial symmetry simply by putting $\alpha_n = \pi/2$ for all n. This will not however insure a closed curve but an axially symmetric closed curve could be made by adding a straight line between Z(0) and Z(L).

It should be pointed out that our closure theorem does not insure that you get a simple closed curve. The conditions for simplicity are probably very complicated.

Optimality of Fourier Basis for Generating Curve Descriptors

In earlier sections we introduced the angle-versus-length function $\theta(\ell)$ and its normalized cumulative variant $\phi^*(t)$. We then expanded $\phi^*(t)$ as a Fourier series and used its coefficients as descriptors of the shape of the curve. The familiar real-variable forms of the Fourier series are

$$\phi^*(t) = \mu_0 + \sum_{k=1}^{\infty} \left(a_k \cos kt + b_k \sin kt\right)$$

and

$$\phi^*(t) = \mu_0 + \sum_{k=1}^{\infty} A_k \cos(kt - \alpha_k)$$

We can, however, express the Fourier Series in a complex form [10], where the family of functions generating the expansion is $\{e^{int}\}_{-\infty}^{\infty}$ and the expansion is

$$\phi^*(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}$$

with coefficients given by

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} \phi^*(t)\, e^{-ikt}\, dt$$

These coefficients c_k are complex numbers related to the real a_k, b_k and A_k, α_k by the equations

$$c_0 = \mu_0, \; 2c_k = a_k - ib_k = A_k e^{-i\alpha_k} \text{ and } 2c_{-k} = a_k + ib_k = A_k e^{i\alpha_k}$$

Since there are other families $\{f_n(t)\}$ of functions upon which such an expansion could be based we are led to question whether or not the Fourier basis $\{e^{int}\}_{-\infty}^{\infty}$ has any advantage over other families. One feature of Fourier series which we have already found to be extremely valuable is the invariance of harmonic-amplitudes under shifts of starting point and the simple relationship $\alpha'_m = \alpha_m + m\Delta\alpha$ which holds under shifts of starting point. We shall soon show that among all possible expansions $\phi^*(t) = \Sigma c_n f_n(t)$ only the Fourier series allows a simple functional relationship between $c_n(\phi^*_{\Delta\alpha})$, $c_n(\phi^*)$ and $\Delta\alpha$ where $\phi^*_{\Delta\alpha}, \phi^*$ are the functions derived from a single curve with starting points differing by $\Delta\alpha$. For the Fourier series expressed in complex form we obtain the simple relation

$$c_n(\phi^*_{\Delta\alpha}) = c_n(\phi^*)\, e^{in\Delta\alpha} \qquad n \neq 0$$

embodying the relations $A_n(\phi^*_{\Delta\alpha}) = A_n(\phi^*)$ and $\alpha_n(\phi^*_{\Delta\alpha}) = \alpha_n(\phi^*) - n\Delta\alpha$ which we have already discovered (Eq. (9)). The earlier work of Raudseps [3] contains some empirical evidence favoring Fourier series but no theoretical results on its uniqueness with respect to a desirable property.

Let $\phi^*(t)$ and $\phi^*_{\Delta\alpha}(t)$ be the normalized cumulative variants of the angle-versus-length functions for two curves differing only in starting point $\Delta\alpha$. Using the definition of ϕ^* (Eq. (1)) and the relationships $\theta_{\Delta\alpha}(\ell) = \theta(\ell - \Delta\alpha)$ and $\theta(\ell) = \phi(\ell) + \delta_0$

we find that

$$\phi^*_{\Delta\alpha}(t) = \phi^*(t - \Delta\alpha) + \Delta\alpha + \delta_0 - \delta_0^{\Delta\alpha} \tag{22}$$

Now suppose $\{f_n(t)\}$ is a basis of linearly independent functions in the Hilbert space $\mathscr{L}^2[0, 2\pi]$ of complex-valued square-summable periodic functions on the interval $[0, 2\pi]$ employing Lebesgue measure on the real line. Let ϕ^* and $\phi^*_{\Delta\alpha}$ be expanded using $\{f_n(t)\}$ as

$$\phi^*(t) = \Sigma c_n f_n(t) \qquad \phi^*_{\Delta\alpha} = \Sigma c_n^{\Delta\alpha} f_n(t)$$

Although $\{f_n(t)\}$ is not necessarily an orthogonal basis we can still compute the coefficients c_n and $c_n^{\Delta\alpha}$ by selecting a family $\{g_m(t)\}$ in such a way that each g_m is orthogonal to the entire subspace spanned by all functions $\{f_n\}$ save the one f_m. The existence of each g_m is assured by well-known facts about Hilbert space [11]. We can in fact choose $g_m(t)$ so that

$$\int_0^{2\pi} f_n(t)\,\overline{g_m(t)}\,dt = \delta_{nm} \quad \text{where bar means complex conjugate}$$

Furthermore, $g_m(t)$ can be chosen periodic in $[0, 2\pi$.

With this family $\{g_m(t)\}$ we can compute coefficients

$$c_m = \int_0^{2\pi} \phi^*(t)\,\overline{g_m(t)}\,dt \qquad c_m^{\Delta\alpha} = \int_0^{2\pi} \phi^*_{\Delta\alpha}(t)\,\overline{g_m(t)}\,dt$$

Now we ask the question, "Is there any family $\{f_n(t)\}$ such that $c_m^{\Delta\alpha} = c_m$ for all save one value of m (e.g., m=0), <u>for all closed curves</u> ϕ^*?" "If not then is there some $\{f_n(t)\}$ such that $c_m^{\Delta\alpha}$ depends only on c_m and $\Delta\alpha$, (i.e. is independent of c_k, $k \neq m$) for all ϕ^*?"

The answer is provided by the following theorem whose proof is in the Appendix.

Optimality Theorem

1. There is no family $\{f_n(t)\}$ such that $c_m^{\Delta\alpha} = c_m$ for all save one value of m, for all closed curves ϕ^*.

2. The Fourier family $\{e^{int}\}$ is the unique family for which $c_m^{\Delta\alpha}$ depends only on c_m and $\Delta\alpha$ independent of curve ϕ^* and other coefficients c_k, $k \neq m$.

We must point out that the criterion of optimality used here concerns the problem of different starting points for identically shaped curves and that if a canonical preferred starting point could be determined in some other fashion then expansion of $\phi^*(t)$ with respect to other orthogonal functions may well be quite useful.

Usefulness of Fourier Descriptors for Shape Discrimination

We present in this section a sampling of the evidence indicating that Fourier Descriptors are very good features for use in shape discrimination. Our own experiments involve numerals from the hand-printed character sets collected by Munson [24]. Each character of this set is represented by a 24 × 24 binary matrix or black and white digital picture. Figure 5a is one such picture of a hand-printed numeral '4'. The curvaturepoint method [8, 9] was used to obtain a polygonal boundary description as shown in Fig. 5b. It was then an easy matter to calculated the Fourier Descriptors $\{A_k, \alpha_k\}$ using formulas derived earlier. Figure 5c shows the first ten pairs of harmonic amplitude and phase angle. Figure 5d depicts a reconstruction of the '4' based on the 10 terms of Fig. 5c. This reconstruction was generated by the reconstruction formula of an earlier section in conjunction with numerical integration. The shape of Fig. 5d is visual proof that the 'fourness' of the digital picture is contained in the twenty numbers $\{A_k, \alpha_k\}_1^{10}$. Brill [6] contains examples of five numerals reconstructed from 5, 10, and 15 FD pairs indicating that approximately 7 pairs should be sufficient to discriminate the five numerals.

Using 39 samples each of numerals 1, 2, 3 from the Munson data [24] representing the work of 13 separate authors we plotted these 117 numerals in the 2-dimensional space of features (A_1, A_2) to observe how well the numeral classes separate, if at all. Figure 6 shows the degree to which these three classes cluster in (A_1, A_2) space. Knowing that harmonic amplitudes alone cannot discriminate shapes which are nearly reflections of one another we tested the feature $F_{42} = \alpha_4 - 2\alpha_2$ on similar samples of the numerals 2 and 5. The results were that with few exceptions F_{42} was in the interval [4.7, 5.6] for numeral '2' and in [3.1, 4.4] for numeral '5'. Of the three exceptions out of 80, two fives were outside both intervals and one two was in the interval populated by fives. As can be seen from Fig. 7 these three samples are not very standard looking numerals.

It would be convenient if we could see visually what the numeral classes of Fig. 6 looked like in the 5-dimensional space $(A_1, A_2, A_3, A_4, A_5)$ but there is no way to directly display such higher dimensional spaces. An indirect method has been recently reported [23] for characterizing the degree to which classes of points in a general metric space cluster internally compared to the separation between classes. A tree is constructed with the points as nodes so that the total of all edge lengths in the tree is a minimum among all possible trees spanning the point set. In case the class membership of each point is known this minimal spanning tree or MST can be used to calculate the relative compactness of the classes with respect to each other -- that is, the tendency for points of one class to be near points of the same class rather than other classes. The number of MST edges with end nodes in different classes measures this tendency rather well. We have performed such an analysis on the 117 points used earlier to see how well the classes would separate in the space $(A_1, \ldots, A_5)$. The results are that the MST contains one edge linking a '1' and '2' and three edges linking '2's and '3's. With three classes two of these edges were absolutely necessary and the extra two (out of 114) are a reflection of the tendency toward intermixture of classes. The tendency is quite minimal. Looking closely at the nodes causing the crossover edges we were able to determine that they were due entirely to one author whose '2's curled downward making their shapes approximate more closely to '3's.

The recognition experiments carried out by Brill [6] give impressive indications of the power of Fourier Descriptors. We briefly summarize the results. In one experiment 600 machine printed numerals from 50 different fonts were recognized with an error rate of 1.5%. In another, 400 hand printed numerals from 40 different styles were recognized with an error rate of 9.5%. These are very good if not spectacular error rates. What is rather surprising is that the recognition was accomplished using lower order Fourier Descriptors and a reference set containing about <u>2 samples</u> per numeral class. In other words, numerals from 50 fonts were recognized by an algorithm whose training set consisted of at most two fonts -- similarly, for the hand printed numerals. Brill *et al.*, [7] report on further recognition experiments with machine-printed characters which are known to make trouble.

Experiments in Shape Generation Using Fourier Descriptors

Programs have been written in PL/I to reconstruct and plot (on line printer or digital Calcomp plotter) curves from arbitrary Fourier Descriptors. We have experimented very slightly with the generation of curves using these programs and Fig. 8 contains a small sample of some of the more interesting results. In line with the Closure Theorem of an earlier section we reconstructed the presumably closed curves from a single FD A_1 where A_1 is one of the zeros of the first

Bessel function. Figure 8(a) and 8(b) are from $A_1 = 3.83171$ and 7.01559 respectively. Most of the remaining curves generated have both k-rotational and axial symmetries which can be forced by putting $A_j = 0$ for $j \not\equiv 0 \pmod{k}$ and $\alpha_{nk} \equiv \pi/2 \pmod{\pi}$. Rather attractive curves are generated by a very small set of FDs; for example, Fig. 8(c) is generated by $A_5 = 4$, $\alpha_5 = 0$ and Fig. 8(d) is generated by $A_4 = 3$, $\alpha_4 = \pi/2$, $A_8 = 2$, $\alpha_8 = 3\pi/2$. The very interesting shape of Fig. 8(e) involves only two nonzero harmonic amplitudes A_6 and A_9. Table 2 lists the FD input to generate each of the curves in Fig. 8.

TABLE 2

Figure	Generating Fourier Descriptors
8(a)	$A_1 = 3.83171$
8(b)	$A_1 = 7.01559$
8(c)	$A_5 = 4.0$
8(d)	$A_4 = 3.0$, $\alpha_4 = \pi/2$, $A_8 = 2.0$, $\alpha_8 = 3\pi/2$
8(e)	$A_6 = 1.0$, $\alpha_6 = \pi/2$, $A_9 = 2.0$, $\alpha_9 = \pi/2$
8(f)	$A_{2k} = 1.2 - (.2)k$, $\alpha_{2k} = \pi/2$ for $k = 1, 2, 3, 4, 5$
8(g)	$A_8 = 1.0$, $\alpha_8 = 3\pi/2$, $A_{10} = 2.0$, $\alpha_{10} = \pi/2$
8(h)	$A_{2k} = (.5)k$, $\alpha_{2k} = 1.0(k-1)$ for $k = 1, 2, 3$
8(j)	$A_k = .6 - (.1)k$ for $k = 1$ to 5 and $\alpha_1 = \alpha_3 = \alpha_5 = \pi/2$, $\alpha_2 = \alpha_4 = 3\pi/2$
8(k)	$A_3 = 2.5$, $\alpha_3 = \pi/2$, $A_6 = 1.0$, $\alpha_6 = 3\pi/2$, $A_9 = .5$, $\alpha_9 = \pi/2$
8(l)	$A_6 = 3.0$, $\alpha_6 = \pi/2$, $A_{12} = 2.0$, $\alpha_{12} = 3\pi/2$
8(m)	$A_3 = 1.5$

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Appendix

Theorem 1:

If two closed curves γ and γ' differ only in starting point by $\Delta\ell$ in units of arclength (clockwise from Z_0 to Z_0') then

a) $A_k' = A_k$

b) $\alpha_k' = \alpha_k + k\Delta\alpha$ where $\Delta\alpha = \dfrac{-2\pi\Delta\ell}{L}$

c) $\mu_0' = \mu_0 + \delta_0 - \delta_0' + \Delta\alpha$

Conversely if these three conditions hold then γ and γ' differ only in a shift of starting point by $\Delta\ell$.

Proof:

If γ and γ' differ by $\Delta\ell$ in starting point then $\theta'(\ell) = \theta(\ell + \Delta\ell)$ and so

$$\phi'(\ell) + \delta_0' = \phi(\ell + \Delta\ell) + \delta_0$$

or employing Eq. (1)

$$\phi^{*\prime}\left(\frac{2\pi\ell}{L}\right) - \frac{2\pi\ell}{L} + \delta_0' = \phi^*\left(\frac{2\pi\ell}{L} + \frac{2\pi\Delta\ell}{L}\right) - \frac{2\pi\ell}{L} - \frac{2\pi\Delta\ell}{L} + \delta_0$$

Letting $t = \dfrac{2\pi\ell}{L}$ and $\Delta t = \dfrac{2\pi\Delta\ell}{L}$ we get

$$\phi^{*\prime}(t) - \phi^*(t + \Delta t) = \delta_0 - \delta_0' - \Delta t\ .$$

The two functions $F(t) = \phi^{*\prime}(t) = \mu_0' + \Sigma A_k' \cos(kt - \alpha_k')$ and $G(t) = \phi^*(t + \Delta t) = \mu_0 + \Sigma A_k \cos(k(t + \Delta t) - \alpha_k)$ differ by a constant and therefore their Fourier coefficients must match term by term except that $\mu_0' - \mu_0 = \delta_0 - \delta_0' - \Delta t$. The term by term equality of harmonic amplitudes and phase angles assures that

$$A_k' = A_k$$

and

$$\alpha_k' = \alpha_k - k\Delta t$$

Letting $\Delta\alpha = -\Delta t$ we obtain condition b. Reading the sequence of equations backwards we obtain the converse.

Theorem 2:

If γ and γ' differ only in the sense that γ' is a mirror image of γ with the same starting point then

a) $A_k' = A_k$

b) $\alpha_k' + \alpha_k = \pi$

c) $\mu_0' + \mu_0 = -\Delta\phi_0$ where $\Delta\phi_0$ is the bend at Z_0.

Conversely if these three conditions hold then γ and γ' are mirror images of each other with the same starting point.

Proof:

Referring to Fig. 9 we see that the curvature of γ' at $Z'(L - \ell)$ is identical to that of γ at $Z(\ell)$. This is because reflection of the curve changes the sign of curvature but the change of orientation also changes the sign resulting in no net change. As a result of this the cumulative change of direction ϕ' between $(L - \ell)$ and L is the same as the change in ϕ between 0 and ℓ.

Hence we can write

$$\phi'(L - \ell) + \phi(\ell) + \Delta\phi_0 = -2\pi$$

where $\Delta\phi_0$ is the change in direction at Z_0 to take care of situations where γ is polygonal and Z_0 is a vertex.

Transforming to the normalized functions ϕ^* and $\phi^{*'}$ (see Eq. (1)) and variable $t = \frac{2\pi\ell}{L}$ we get

$$\phi^{*'}\left(\frac{2\pi(L-\ell)}{L}\right) - \frac{2\pi(L-\ell)}{L} + \phi^*\left(\frac{2\pi\ell}{L}\right) - \frac{2\pi\ell}{L} + \Delta\phi_0 = -2\pi$$

or

$$\phi^{*'}(2\pi - t) - (2\pi - t) + \phi^*(t) - t + \Delta\phi_0 = -2\pi$$

or

$$\phi^{*\prime}(-t) + \phi^{*}(t) + \Delta\phi_0 = 0$$

since $\phi^{*\prime}$ is periodic $[2\pi]$.

If f(t) and g(t) satisfy

$$f(t) + g(-t) = c$$

and if

$$g(t) = \mu'_0 + \sum_{k=1}^{\infty} (a'_k \cos kt + b'_k \sin kt)$$

while a_k, b_k and μ_0 are the analogous coefficients for f(t), a straightforward calculation yields

$$a'_k = -a_k \qquad b'_k = b_k \qquad \mu'_0 = -\mu_0 + c.$$

For $g = \phi^{*\prime}$, $f = \phi^{*}$ and $c = -\Delta\phi_0$ we get

$$a'_k = -a_k \qquad b'_k = b_k \qquad \text{and} \qquad \mu'_0 = -\mu_0 - \Delta\phi_0$$

Since A_k, α_k are polar coordinates for a_k, b_k we obtain conditions a) and b) immediately.

The converse follows from observing that if the Fourier coefficients of $\phi^{*\prime}$ and ϕ^{*} are related as above

$$\phi^{*\prime}(-t) + \phi^{*}(t) + \Delta\phi_0 = 0$$

and this implies

$$\phi'(L-\ell) + \phi(\ell) + \Delta\phi_0 = -2\pi.$$

Theorem 3

A curve γ has n-fold rotational symmetry if and only if $A_k = 0$ for $k \not\equiv 0 \pmod{n}$.

Proof:

First assume n-fold rotational symmetry. This implies immediately that $\phi^{*}(t)$ has period $\frac{2\pi}{n}$ and we must show that Fourier coefficients a_k, b_k are zero except when $k \equiv 0 \pmod{n}$ for a function f which is periodic $\left[\frac{2\pi}{n}\right]$. It is easy to show this by direct calculations.

The converse is almost immediate because a function of the form

$$f(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} A_{nj} \cos(njt - \alpha_{nj})$$

is periodic with a period $\frac{2\pi}{n}$.

Closure Theorem

Let

$$Z(\ell) = Z_0 + \frac{L}{2\pi} \int_0^{2\pi\ell/L} e^{i\left\{-t + \delta_0 + \mu_0 + \sum_{j=1}^{N} A_j \cos(jt - \alpha_j)\right\}} dt$$

Then $Z(\ell)$ represents a closed curve if one of the following conditions holds:

1. A_1 is a zero of the first Bessel function $J_1(x)$ and $A_n = 0$ for $n > 1$.
2. $A_n = 0$ for all $n \not\equiv 0 \pmod{k}$ where $k \geq 2$.

Proof:

If condition 2 holds then $f(t) = \sum_{j=1}^{N} A_j \cos(jt - \alpha_j)$ is periodic with period $2\pi/k$ and we must show that

$$\int_0^{2\pi} e^{-it + if(t)} dt = 0 \ .$$

A simple calculation shows that when f(t) is periodic$\left(\frac{2\pi}{k}\right)$ this expression has as a factor the sum of all the kth roots of unity and so it is zero.

If condition 1 holds then A_1 is the only nonzero A_j and we need to show

$$\int_0^{2\pi} e^{-it + i A_1 \cos(t - \alpha_1)} dt = 0$$

A simple change of variable $t' = t - \alpha_1$ gets rid of α_1 and our requirement on A_1 becomes

$$\int_0^{2\pi} e^{-it'} e^{iA_1 \cos t'} dt' = 0$$

This equation holds for A_1 such that $J_1(A_1) = 0$ where J_1 is the first Bessel function [13] and the theorem is proved.

Optimality Theorem

1. There is no family $\left\{ f_n(t) \right\}$ such that $c_m^{\Delta\alpha} = c_m$ for all save one value of m, for all closed curves ϕ^*.

2. The Fourier family $\left\{ e^{int} \right\}$ is the unique family for which $c_m^{\Delta\alpha}$ depends only on c_m and $\Delta\alpha$ independent of curve ϕ^* and the other coefficients c_k, $k \neq m$.

Proof:

Suppose a family $\left\{ f_n(t) \right\}$ as described in 1 did exist. If such a situation did hold we would have

$$\int_0^{2\pi} \left[\phi^*(t) - \phi^*_{\Delta\alpha}(t)\right] \overline{g_m(t)}\, dt = 0 \qquad \text{for } m \neq 0$$

The function $F(t) = \phi^*(t) - \phi^*_{\Delta\alpha}(t)$ in square brackets can also be expanded as

$$F(t) = \Sigma b_m f_m(t)$$

where

$$b_m = \int_0^{2\pi} F(t)\, \overline{g_m(t)}\, dt$$

From the expression above $b_m = 0$ for $m \neq 0$ and so

$$F(t) = b_0 f_0(t)$$

or

$$\phi^*(t) - \phi^*_{\Delta\alpha}(t) = b_0(\Delta\alpha)\, f_0(t)$$

which expresses the possible dependence of b_0 on $\Delta\alpha$.

Substituting for $\phi^*_{\Delta\alpha}(t)$ from Eq. (22), dividing by $\Delta\alpha$ and taking the limit as $\Delta\alpha \to 0$ we obtain

$$\lim_{\Delta\alpha \to 0} \frac{\phi^*(t) - \phi^*(t - \Delta\alpha)}{\Delta\alpha} - 1 + \frac{\delta_0 - \delta_0^{\Delta\alpha}}{\Delta\alpha} - \frac{b_0(\Delta\alpha)}{\Delta\alpha} f_0(t) = 0$$

The first quotient approaches the derivative of ϕ^* at t, the second approaches the derivative of θ at its starting point and the third quotient approaches $b_0'(0)$ since $b_0(0) = 0$. We finally obtain

$$f_0(t) = D\, k(t) + E$$

where

$$k(t) = \frac{d\phi^*}{dt}$$

is the curvature of the curve being expanded and D, E are constants not depending on t. Hence $f_0(t)$ depends on the curvature of the curve whose function ϕ^* is being expanded and therefore the expansion couldn't work for arbitrary curves. We conclude that no family $\{f_n(t)\}$ can be chosen to obtain $c_n^{\Delta\alpha} = c_n$ for all but $n = 0$, for all closed curves ϕ^*.

Since we cannot arrange to have $c_n^{\Delta\alpha} = c_n$ the next best thing would be to have $c_n^{\Delta\alpha}$ depend in some simple way on c_n and $\Delta\alpha$. We shall now see what happens when we assume the basis $\{f_n(t)\}$ is such that $c_n^{\Delta\alpha} = h_n(c_n, \Delta\alpha)$ for all n and h_n independent of $\phi^*(t)$. In general, $c_n^{\Delta\alpha}$ would depend on all the c_i and not just on c_n. This condition becomes

$$c_n^{\Delta\alpha} = \int_0^{2\pi} \phi^*_{\Delta\alpha}(t)\, \overline{g_n(t)}\, dt = h_n\left(\int_0^{2\pi} \phi^*(t)\, \overline{g_n(t)}\, dt,\ \Delta\alpha\right)$$

Since the above equation must hold for arbitrary ϕ^* we can let ϕ^* approach a delta function $\phi^*(t) \approx \lambda\delta(t - \mu)$. Substituting for $\phi^*_{\Delta\alpha}$ and then $\lambda\delta(t - \mu)$ for ϕ^* we get

$$\lambda\, \overline{g_m(\Delta\alpha + \mu)} + \left(\Delta\alpha + \delta_0 - \delta_0^{\Delta\alpha}\right) \int_0^{2\pi} \overline{g_m(t)}\, dt = h_m\left(\lambda\, \overline{g_m(\mu)},\ \Delta\alpha\right)$$

Putting

$$y = \lambda\, \overline{g_m(\mu)}$$

and

$$D = \left(\Delta\alpha + \delta_0 - \delta_0^{\Delta\alpha}\right) \int_0^{2\pi} \overline{g_m(t)}\, dt$$

we get

$$y \frac{\overline{g_m(\Delta\alpha + \mu)}}{\overline{g_m(\mu)}} + D = h_m(y, \Delta\alpha)$$

Treating y and $\Delta\alpha$ as variables, we see that the right side does not depend on μ, and therefore neither does the left side, so that

$$G_m(\Delta\alpha) = \frac{\overline{g_m(\Delta\alpha + \mu)}}{\overline{g_m(\mu)}}$$

is independent of μ for all $\Delta\alpha$. The only reasonable (e.g., anywhere continuous) solution [12] of such a functional equation is

$$g_m(x) = A_m e^{\sigma_m x}$$

Since $g_m(x)$ is periodic on $[0, 2\pi]$,

$$\sigma_m = in(m)$$

where n(m) is an integer, and since $\{g_m\}$ is a complete set, the set $\{n(m)\}$ must include all integers. Therefore the family $\{g_m(t)\}$ is in fact $\{e^{int}\}_{-\infty}^{\infty}$ and aside from appropriate multiplicative constants the family $\{f_n(t)\}$ is precisely the Fourier basis $\{e^{-int}\}_{-\infty}^{\infty}$.

There is a slight flaw in the above argument where we approximated ϕ^* to a delta function but the problem is simple enough to repair. We were wrong in saying that ϕ^* was arbitrary because ϕ^* comes from a closed curve and must therefore satisfy the integral formula (see Reconstruction Theorem, Eq. (4))

$$\int_0^{2\pi} e^{i(\phi^*(t) - t)}\, dt = 0$$

If we define an infinite sequence of functions $\delta^*_{\lambda, m, \pm}$ by

$$\delta^*_{\lambda, m, \pm}(t) = \underline{\text{if}}\ |t - \mu| \le \frac{\lambda}{4\pi m}\ \underline{\text{then}}\ \pm 2\pi m\ \underline{\text{else}}\ 0$$

where $\lambda > 0$ and m is a positive integer then each $\delta^*_{\lambda, m, \pm}$ satisfies the integral formula for closure and for fixed λ, $\pm$ it is easily verified that

$$\lim_{m \to \infty} \delta^*_{\lambda, m, \pm}(t) = \pm \lambda \delta(t - \mu).$$

The proof of the optimality of Fourier series is now complete.

References

[1] R. L. Cosgriff, "Identification of shape," Report No. 820-11 of the Ohio State University Research Foundation, December 1960, ASTIA identification AD 254 792.

[2] D. L. Fritzsche, "A systematic method for character recognition," Report No. 1222-4 of the Ohio State University Research Foundation, November 1961, ASTIA identification AD 268 360.

[3] J. G. Raudseps, "Some aspects of the tangent-angle vs arc length representation of contours," Report No. 1801-6 of the Ohio State Universtiy Research Foundation, March 1965, ASTIA identification AD 462 877.

[4] R. J. Borel, "A mathematical pattern recognition technique based on contour shape properties," Report No. 1801-11 of the Ohio State University Research Foundation, December 1965, ASTIA identification AD 476 113.

[5] E. L. Brill, "The development and evaluation of three recognition criteria for pattern recognition," Master Thesis (Ohio State University), 1967.

[6] E. L. Brill, "Character recognition via Fourier Descriptors," WESCON Technical Papers, Session 25, Qualitative Pattern Recognition Through Image Shaping, Los Angeles, California, August 1968.

[7] E. L. Brill et al., "Some approaches to character recognition for postal address reader applications," Conference on Automatic Pattern Recognition, U. S. Post Office, May 1969.

[8] C. T. Zahn, "Two-dimensional pattern description and recognition via curvaturepoints," Technical Report No. SLAC-TN-66-70, Stanford Linear Accelerator Center (USAEC), Stanford University, December 1966.

[9] C. T. Zahn, "A formal description for two-dimensional patterns," Proceedings of the International Joint Conference on Artificial Intelligence, May 1969, pp. 621 - 628, available also as Report No. SLAC-PUB-538, Stanford Linear Accelerator Center, Stanford University.

[10] W. Kaplan, Advanced Calculus, (Addison-Wesley, 1968); p. 434.

[11] P. R. Halmos, Introduction to Hilbert Space, Chelsea, p. 23.

[12] J. Aczel, Lectures on Functional Equations and Their Applications, (Academic Press, New York, 1966); p. 143.

[13] Bateman Manuscript Project, Higher Transcendental Functions, Vol. II, p. 14.

[14] A. Rosenfeld, Picture Processing by Computer, (Academic Press).

[15] F. Attneave and M. D. Arnoult, "The quantitative study of shape and pattern perception" in Pattern Recognition edited by L. Uhr, John Wiley.

[16] F. Attneave, "Some informational aspects of visual perception," Psychol Rev. 183-193 (1954).

[17] L. Uhr, "Pattern recognition" in Pattern Recognition edited by L. Uhr, John Wiley.

[18] J. Y. Lettvin et al., "What the frog's eye tells the frog's brain," Proc. IRE 47, 1940-1951 (1959).

[19] D. H. Hubel and T. N. Wiesel, "Receptive fields, binocular interaction, and functional architecture in the cat's visual cortex," J. Physiol. 160, 106ff (1962).

[20] R. S. Ledley, Use of Computers in Biology and Medicine, (McGraw-Hill, New York); p. 340.

[21] R. C. Buck, Advanced Calculus, (McGraw-Hill, New York); p. 252.

[22] Ibid. p. 261.

[23] C. T. Zahn, "Graph-theoretical methods for detecting and describing Gestalt clusters," IEEE Transactions on Computers, January 1971.

[24] J. H. Munson, "Experiments in the recognition of handprinted text: Part I — character recognition," AFIPS Conference Proceedings, 1968 Fall Joint Computer Conference, Vol. 33, Part Two, pp. 1125-1138.

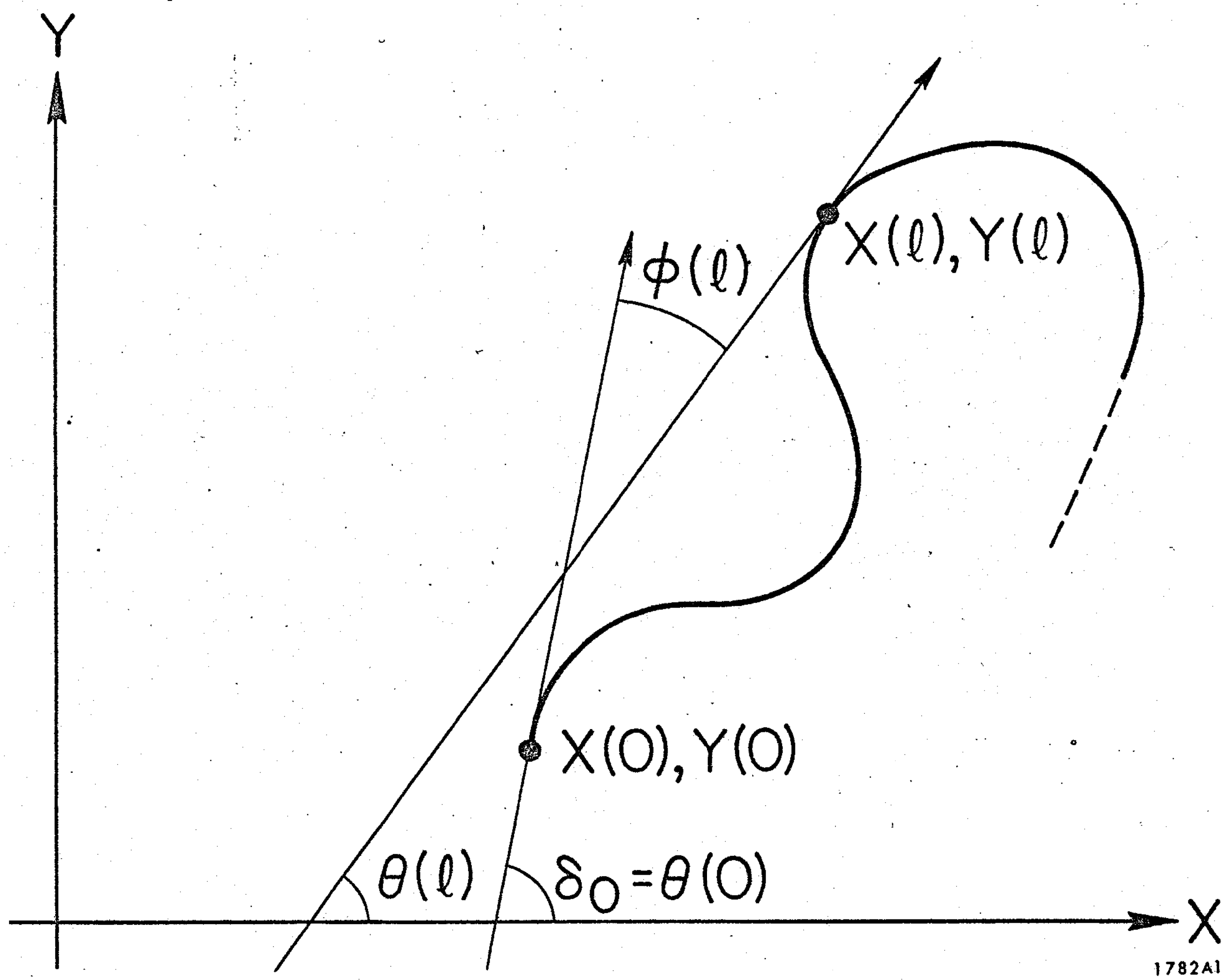


Fig. 1a

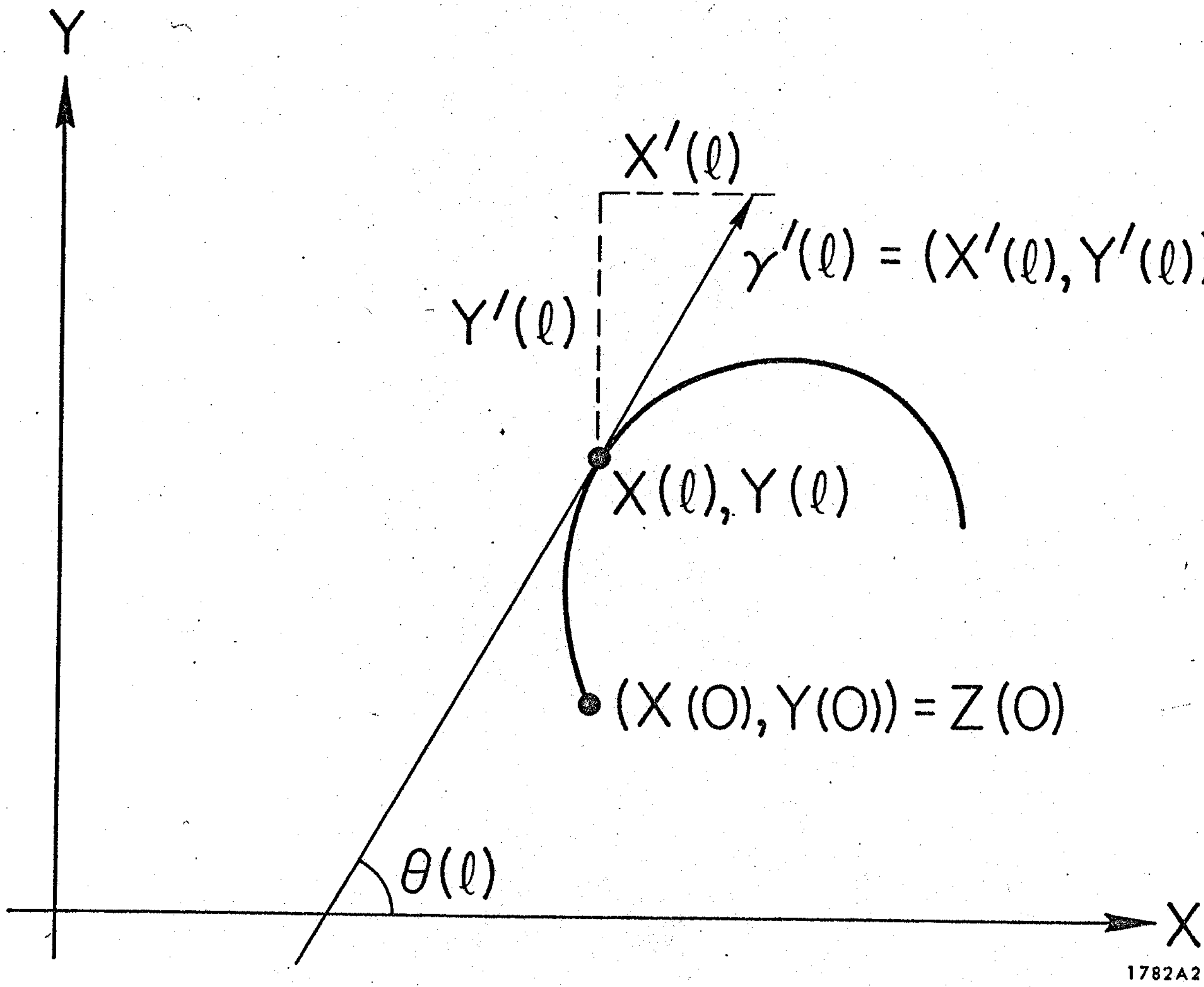

Y
X
$X'(\ell)$
$\gamma'(\ell) = (X'(\ell), Y'(\ell))$
$Y'(\ell)$
$X(\ell), Y(\ell)$
$(X(0), Y(0)) = Z(0)$
$\theta(\ell)$
1782A2

Fig. 1b

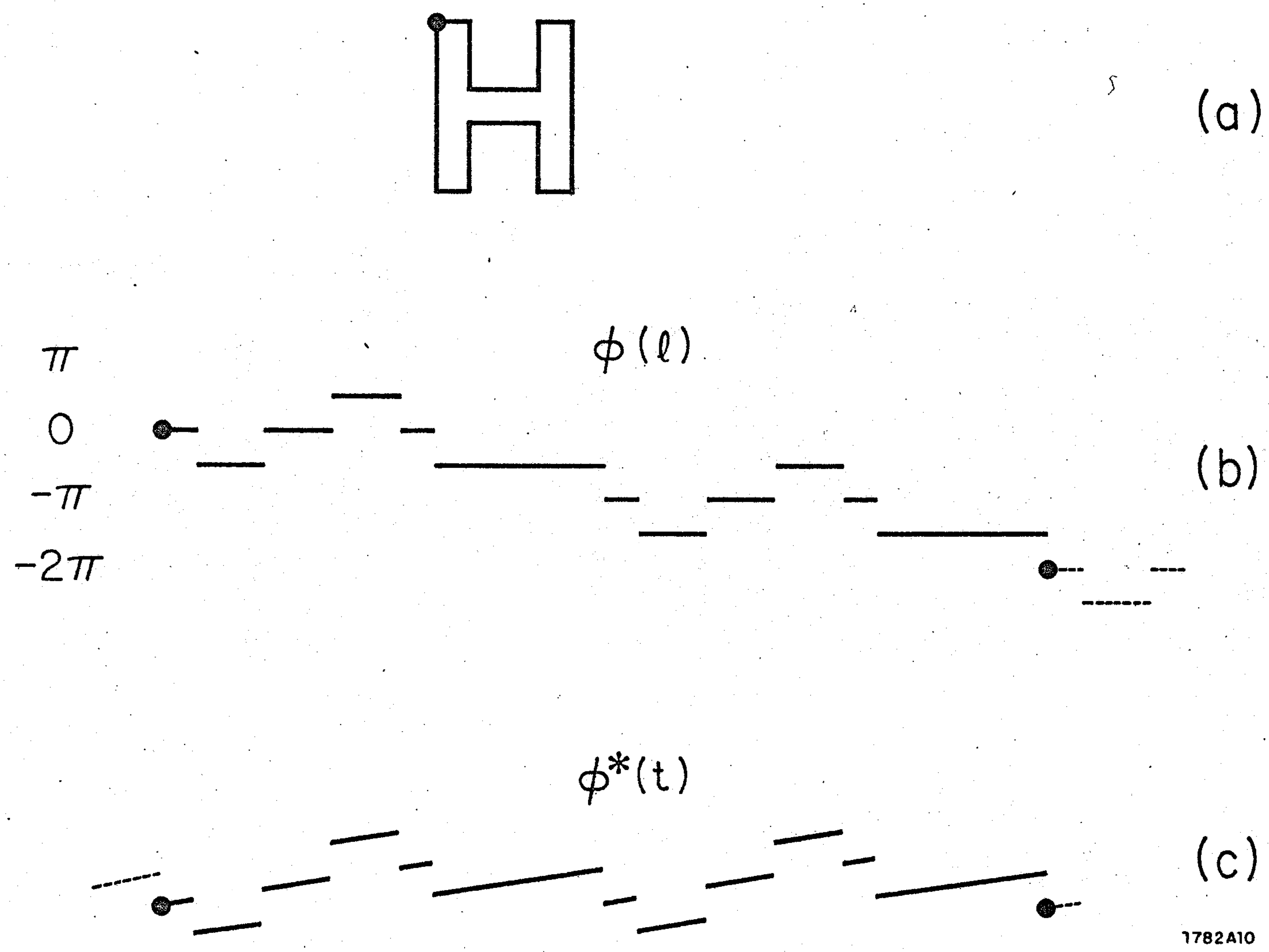
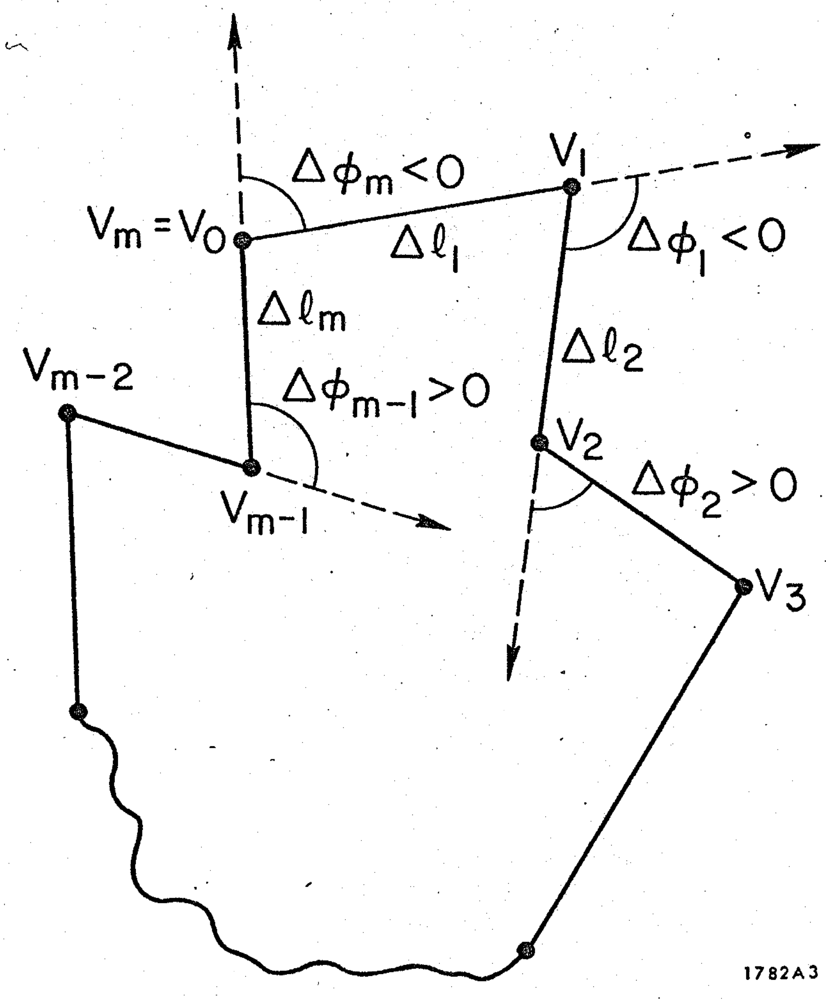


Fig. 2



$\Delta\phi_m < 0$
V_1
$V_m = V_0$
$\Delta\ell_1$
$\Delta\phi_1 < 0$
$\Delta\ell_m$
$\Delta\ell_2$
V_{m-2}
$\Delta\phi_{m-1} > 0$
V_2
$\Delta\phi_2 > 0$
V_{m-1}
V_3
1782A3

Fig. 3

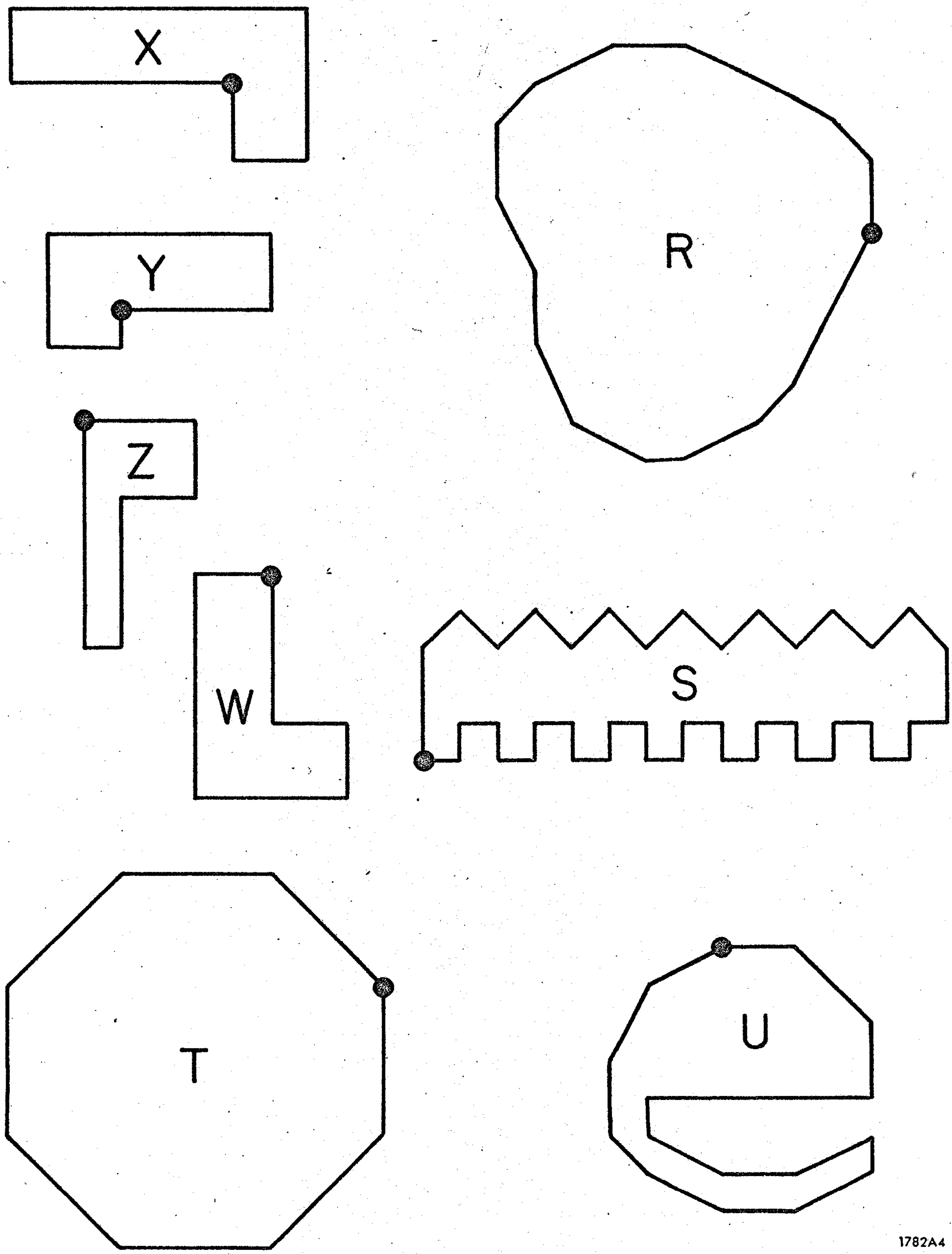
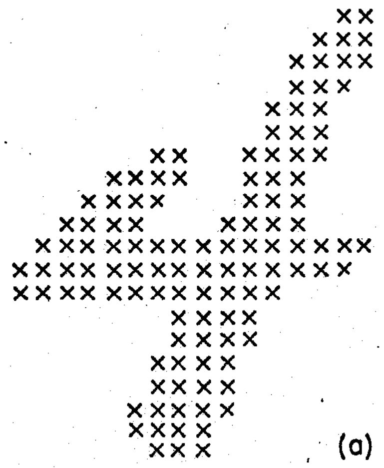


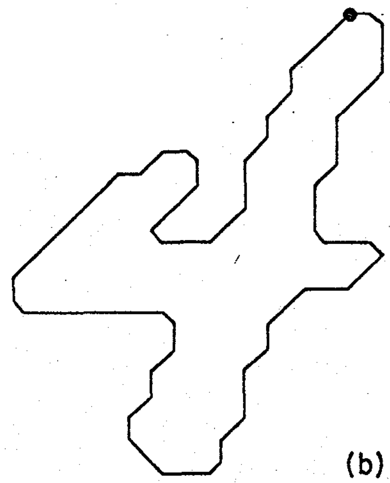
Fig. 4

BINARY IMAGE



(a)

BOUNDARY DESCRIPTION



(b)

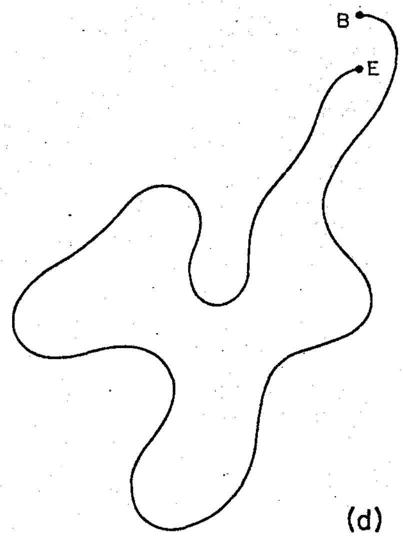
FOURIER DESCRIPTORS

i	A_i	α_i
1	.347	.698
2	.371	5.687
3	1.001	4.989
4	.384	3.279
5	.575	4.588
6	.423	1.084
7	.422	.004
8	.128	4.283
9	.240	4.084
10	.081	4.708

$\mu_0 = -0.75$

(c)

RECONSTRUCTED BOUNDARY



(d)

1782C5

Fig. 5

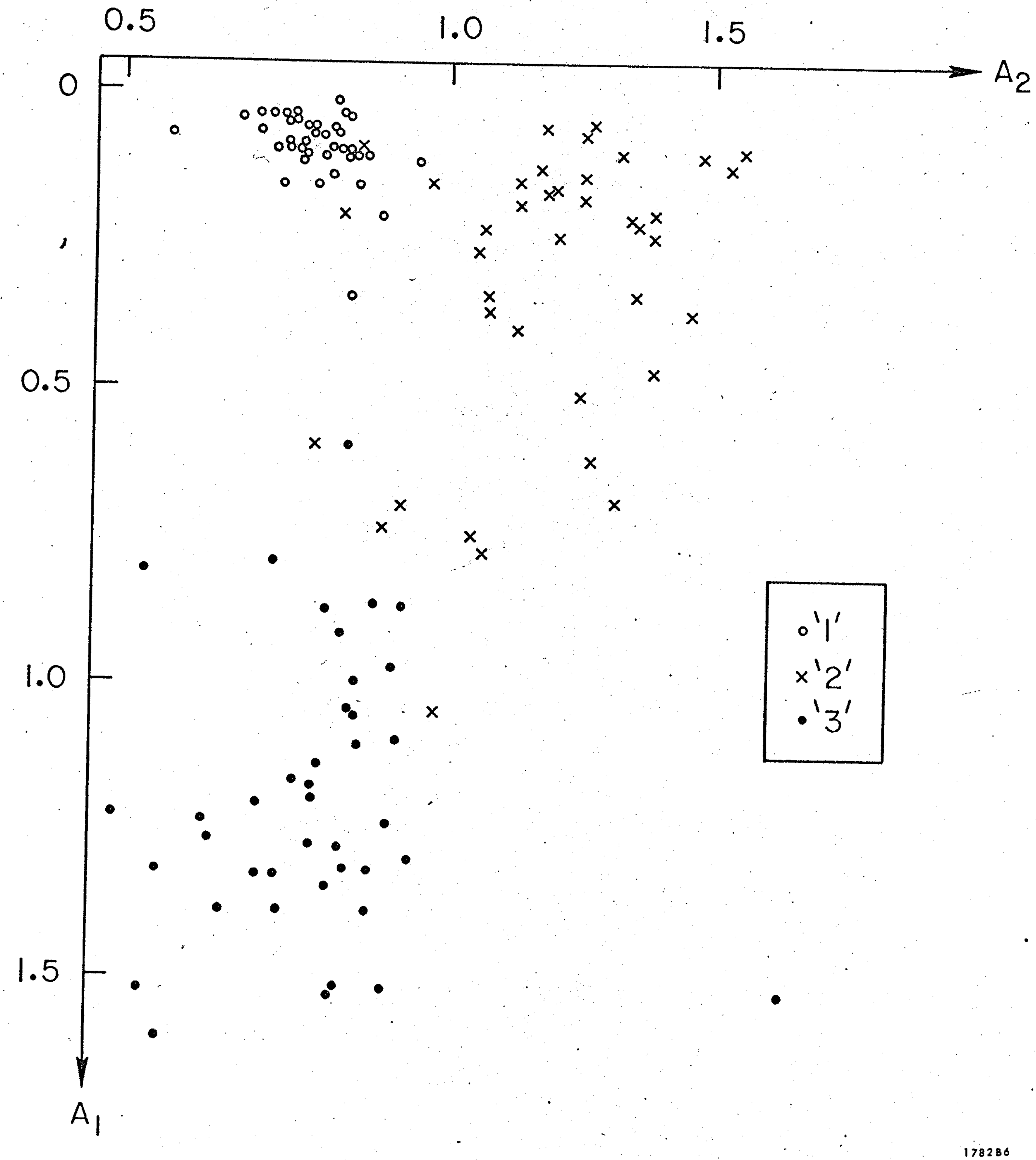


Fig. 6

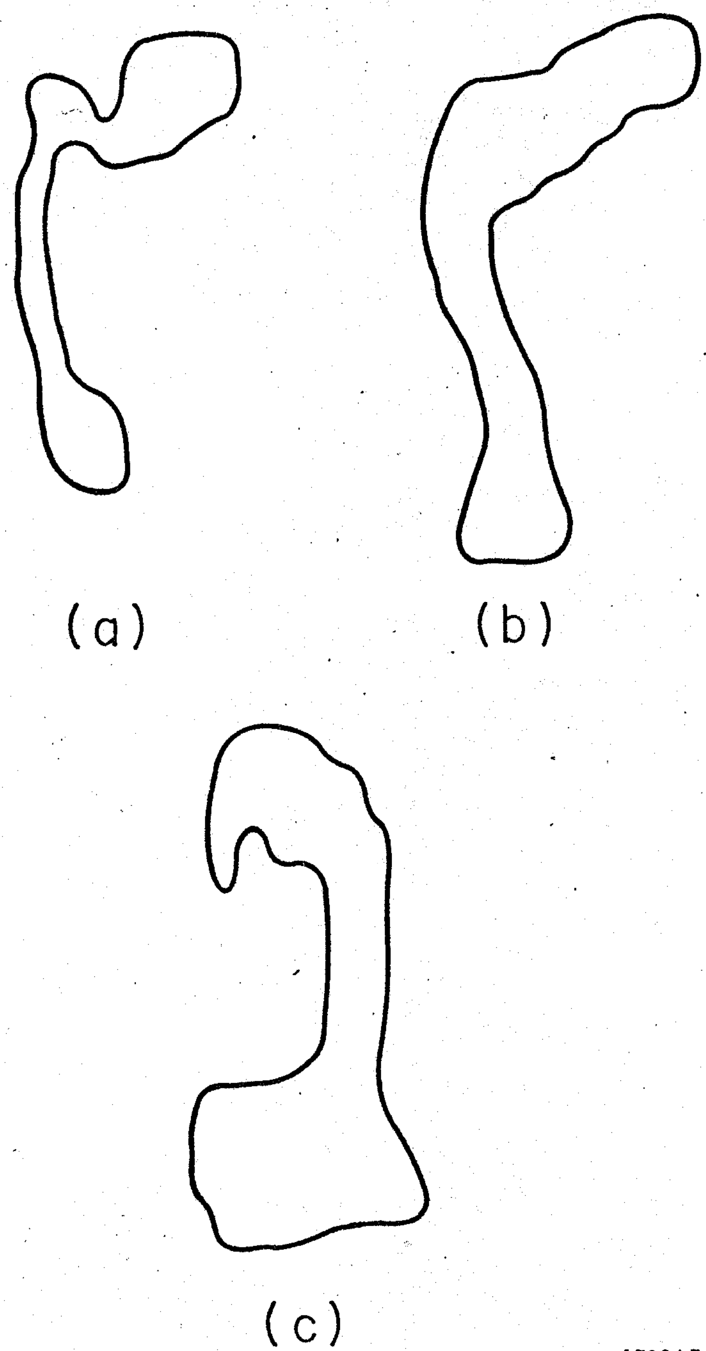


Fig. 7

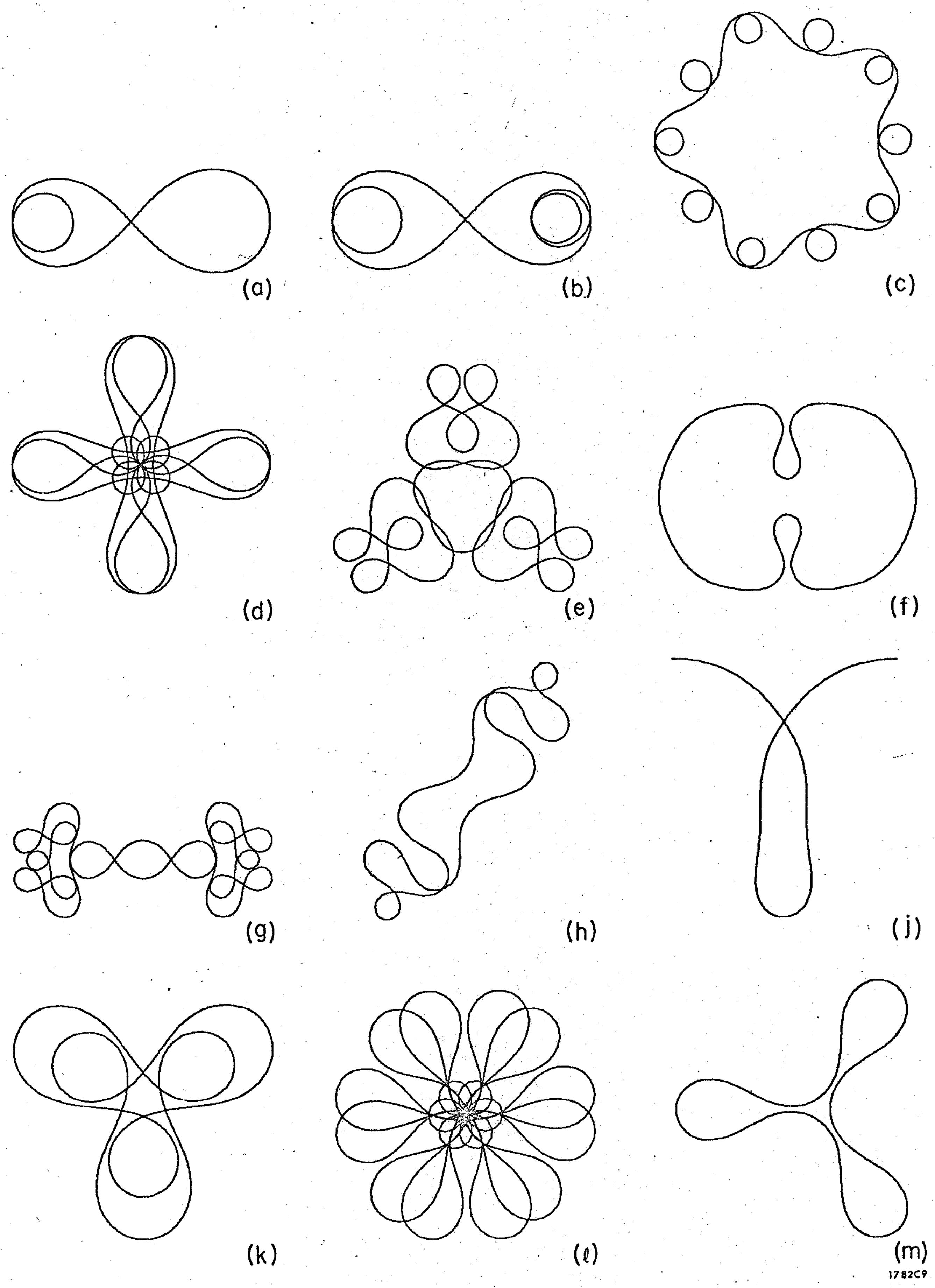


Fig. 8

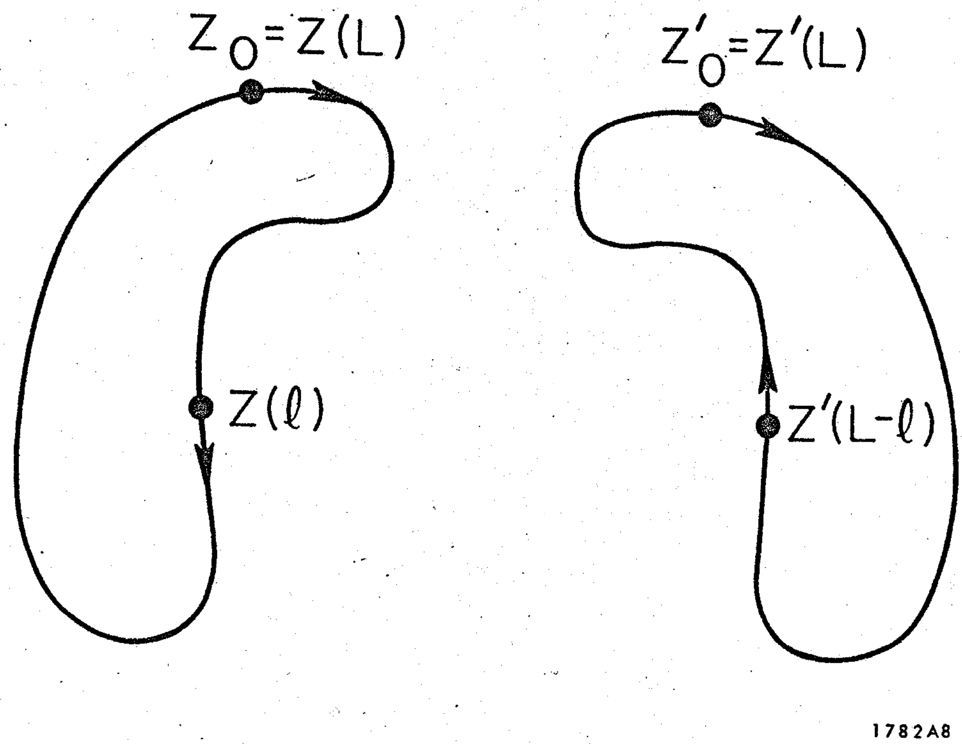


Fig. 9