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ANOTHER BOUND ON THE ABSORPTIVE PART OF ELASTIC SCATTERING AMPLITUDES*

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ABSTRACT

Using Lagrange multipliers for inequality constraints, an upper bound on the absorptive part of elastic scattering amplitudes is derived assuming unitarity, a fixed total and elastic cross section, and the condition that the partial waves decrease monotonically with increasing angular momentum. Numerical results are given.

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1. INTRODUCTION

Consider the elastic scattering of equal mass particles of spin zero. Given the total cross section and the elastic cross section, as well as the unitarity requirement on the partial wave amplitudes, how large can the absorptive part of the elastic scattering amplitude become at any given scattering angle? This problem has been solved by Singh and Roy[1] and the maximum value has been compared with experimental differential cross sections at high energies and for small scattering angles on the basis of several further assumptions: (1) At high energies, the equal mass assumption can be relaxed. (2) The unpolarized differential cross sections are independent of the spin of the external particles and, hence, the spin zero bound applies. (3) The amplitude, in the region of the diffraction peak, is purely imaginary. The comparison[1] with experimental data is rather good for small angles, but for larger angles the data falls far below the calculated bound.

The distribution of partial wave amplitudes which achieves this bound looks very much like a Fresnel zone plate, carefully constructed to maximize the scattering in the given direction. The distribution is illustrated as the shaded region of Fig. 1, the details of which will be explained later. The larger the angle, the more zones are required. More conventional models of matter would have a central core surrounded perhaps by successively less absorptive regions. A particularly simple way to implement this intuition is to require the imaginary parts of the partial wave amplitudes to decrease monotonically with increasing angular momentum.[2] This is not unreasonable for energies above resonances. Adding this assumption to those given above should yield a better bound at larger angles, precisely where the preceding one fails. It is to the solution of this problem that this paper is devoted. The approach used in the construction of the solution is the method of Lagrange multipliers generalized to include inequality constraints.[3]

In Section 2, the mathematical problem is formulated and solved exactly. In Section 3, the same problem is simplified by approximating the discrete partial wave series by a continuum and by assuming the scattering angle is small. Section 4 compares the improved bound with experimental data and interpretes the results. Finally, in Section 5, we summarize our conclusions. In an appendix, a number of sums are tabulated.

2. MATHEMATICAL DETAILS

The mathematical statement of the problem is as follows[4]: Maximize

$$A = \sum (2\ell+1)\; a_\ell P_\ell(z)$$

given the total cross section

$$A_0 = \frac{k^2}{4\pi}\,\sigma_T = \sum (2\ell+1)\; a_\ell$$

the elastic cross section

$$\sum_{el} = \frac{k^2}{4\pi}\,\sigma_{el} = \sum (2\ell+1)\left(a_\ell^2 + r_\ell^2\right)$$

and unitarity

$$u_\ell \equiv a_\ell - a_\ell^2 - r_\ell^2 \geq 0 \qquad\qquad \ell = 0, 1, 2, \ldots$$

In addition, we require that the partial waves decrease monotonically, i.e.,

$$a_\ell \geq a_{\ell+1} \tag{1}$$

However, thinking of the requirements of statistics as well as of most dynamical models, one would like to impose this requirement separately on the even and odd partial waves, i.e.

$$a_\ell \geq a_{\ell+2} \tag{2}$$

Fortunately, this is a minor complication. To avoid notational confusion, we solve the problem first assuming (1) and then state the result assuming (2). From a mathematical point of view, these requirements are interesting since they impose a relation between neighboring partial waves, unlike those examples discussed in EB.

To this end, we introduce the auxiliary function

$$\mathscr{L} = A + \alpha\left(A_0 - \sum (2\ell+1)\, a_\ell\right) + \frac{1}{2a}\left(\sum_{el} - \sum (2\ell+1)\left(a_\ell^2 + r_\ell^2\right)\right)$$

$$+ \sum (2\ell+1)\, \lambda_\ell u_\ell + \sum \omega_\ell \left(a_\ell - a_{\ell+1}\right) \tag{3}$$

where, on the basis of the theorem[3] that the multipliers are the rate of change of the maximum with respect to the constraint, we anticipate that

$$0 < \alpha < 1 \qquad \text{and} \qquad a > 0 \ .$$

Of course, $\lambda_\ell \geq 0$ and $\omega_\ell \geq 0$ for all ℓ. Varying with respect to r_ℓ, we find

$$\frac{\delta \mathscr{L}}{\delta r_\ell} = (2\ell+1) \left[-\frac{1}{a} - 2\lambda_\ell \right] r_\ell = 0 \ .$$

This implies $r_\ell = 0$. Also,

$$\frac{\delta \mathscr{L}}{\delta a_\ell} = (2\ell+1) \left[P_\ell - \alpha - \frac{a_\ell}{a} + \lambda_\ell (1-2a_\ell) + (\omega_\ell - \omega_{\ell-1})/(2\ell+1) \right] = 0 \ . \qquad (4)$$

For convenience, we have defined $\omega_{-1}=0$. First, we must find what are the <u>necessary</u> conditions on a local maximum. The most general form for a local maximum under these assumptions is

$$x = (1 = 1 = \ldots = a_{N_0} > a_{N_0+1} > \ldots > a_{L_1} = a_{L_1+1} = \ldots = a_{L_1+N_1} >$$

$$a_{L_1+N_1+1} > \ldots > a_{L_2} = a_{L_2+1} = \ldots = a_{L_2+N_2} > a_{L_2+N_2+1}$$

$$a_{L_N} = 0 = 0 = \ldots)$$

In words, the solution is a series of plateaus, on which at least two partial waves have the same value, and regions where successive partial waves are strictly decreasing. Our problem is to determine where the transition points and jumps occur as well as the values of a_ℓ. As in EB, we define three partitions of the partial waves

$$B_1 = \{\ell \,|\, a_\ell = 1\}$$

$$I = \{\ell \,|\, 0 < a_\ell < 1\}$$

$$B_0 = \{\ell \,|\, a_\ell = 0\} \ .$$

A priori, B_1, I, or B_0 could be empty. It follows from our variational equations (4) that, in B_1,

$$\lambda_\ell = P_\ell - \alpha - \frac{1}{a} + \frac{\omega_\ell - \omega_{\ell-1}}{2\ell+1} \geq 0 \quad , \tag{5a}$$

in I,

$$\lambda_\ell = 0 \; , \qquad \frac{\omega_{\ell-1} - \omega_\ell}{2\ell+1} = P_\ell - \alpha - \frac{a_\ell}{a} \quad , \tag{5b}$$

and in B_0,

$$\lambda_\ell = \alpha - P_\ell + \frac{\omega_{\ell-1} - \omega_\ell}{2\ell+1} \geq 0 \quad . \tag{5c}$$

Let us first determine the properties of each of the transition points where a_ℓ actually decreases.

1. $a_{N_0} > a_{N_0+1}$ implies $\omega_{N_0} = 0$, hence, from (5a)

$$\lambda_{N_0} = P_{N_0} - \alpha - \frac{1}{a} - \frac{\omega_{N_0-1}}{2N_0+1} \geq 0 \quad ,$$

consequently,

$$P_{N_0} - \alpha - \frac{1}{a} \geq \frac{\omega_{N_0-1}}{2N_0+1} \geq 0 \quad .$$

In particular, this means that $a\,P_{N_0} - \alpha \geq 1$ so, as in the case without monotonicity,[3] B_1 is empty if $a(1-\alpha) < 1$.

2. In I, if, for any ℓ, one has $a_{\ell-1} > a_\ell > a_{\ell+1}$, then $\omega_{\ell-1} = \omega_\ell = 0$. Hence, from (b)

$$a_\ell = a(P_\ell - \alpha) \quad .$$

Of course, by the definition of I, one must have $1 > a(P_\ell - \alpha) > 0$.

3. $a_{L_N-1} > a_{L_N} = 0$ implies $\omega_{L_N-1} = 0$. So, from (5c),

$$\lambda_{L_N} = \alpha - P_{L_N} - \frac{\omega_{L_N}}{2L_N+1} \geq 0 \quad ,$$

and consequently,

$$\alpha - P_{L_N} \geq \frac{\omega_{L_N}}{2L_N+1} \geq 0 \quad .$$

In particular, this means that $\alpha \geq P_{L_N}$.

Now we come to the heart of the problem, how to determine the plateaus of constancy. The three points immediately above suggest that the multipliers a and α determine where B_1 stops and the value of largest nonzero partial wave amplitude, just as in the case without the monotonicity requirement. In I, we shall find that the transition points are determined <u>independently</u> of a and α. In other words, σ_T and σ_{el} determine the size of the core (B_1) and the value of the largest contributing partial wave (B_0). However, in the intermediate region (I), the shape of the distribution is determined solely by the requirement that the strength of the partial wave amplitudes should be monotonically decreasing. To see this, consider any particular plateau

$$a_{L_j-1} > a_{L_j} = \ldots = a_{L_j+N_j} > a_{L_j+N_j+1} \quad .$$

We know that $\omega_{L_j-1} = 0$. Using the difference equation (5b), one can then solve for

$$-\omega_{L_j+n} = \sum_{L_j}^{L_j+n} (2\ell+1) \left(P_\ell - \alpha - \frac{a_{L_J}}{a} \right) \quad n=0, \ldots, N_j \; .$$

Using $\omega_{L_j+N_j} = 0$, we find

$$0 = \sum_{L_j}^{L_j+N_j} (2\ell+1) \left(P_\ell - \alpha - \frac{a_{L_j}}{a}\right) \quad .$$

It is convenient at this point to define a weighted average of the Legendre polynomials

$${}_{N}\langle P_\ell \rangle_{M} \equiv \sum_{\ell=M}^{N} (2\ell+1)\, P_\ell \Big/ \sum_{\ell=M}^{N} (2\ell+1) \quad .$$

Notice that ${}_{N}\langle P_\ell \rangle_{N} = P_N$ and that

$${}_{N}\langle P_\ell + c \rangle_{M} = {}_{N}\langle P_\ell \rangle_{M} + c \quad \text{for any constant } c.$$

We have found in Eq. (4) that

$$a_{L_j} = a \left({}_{L_j+N_j}\langle P_\ell \rangle_{L_j} - \alpha \right)$$

Thus the value of the partial wave amplitude is related to the weighted average of the Legendre polynomials across the plateau. The requirement $a_{L_j} > 0$ implies ${}_{L_j+N_J}\langle P_\ell \rangle_{L_j} > \alpha \geq 0$. Therefore, α determines the largest nonzero partial wave. Inserting this solution for a_{L_j} into the equation for ω_{L_j+n}, we have

$$-\omega_{L_j+n} = \sum_{L_j}^{L_j+n} (2\ell+1) \left(P_\ell - {}_{L_j+N_j}\langle P_\ell \rangle_{L_j}\right) \qquad n=0, \ldots, N_j \tag{6}$$

Note that this can also be written as

$$\omega_{L_j+n} = \sum_{L_j+n+1}^{L_j+N_j} (2\ell+1) \left(P_\ell - {}_{L_j+N_j} P_{\ell\; L_j}\right) \cdot \qquad n=0, \ldots, N_j-1 \tag{7}$$

The condition $\omega_{L_j+n} > 0$ therefore implies

$$ {}_{L_j+N_j}\langle P_\ell \rangle_{L_j} \geq {}_{L_j+n}\langle P_\ell \rangle_{L_j} \qquad \text{for } n=0, \ldots, N_j $$

or

$$ {}_{L_j+N_j}\langle P_\ell \rangle_{L_j+N_j-n} \geq {}_{L_j+N_j}\langle P_\ell \rangle_{L_j} \qquad \text{for } n=0, \ldots, N_j $$

It can be shown that the conditions that $\omega_{L_j-1}=0$ and $a_{L_j-1} > a_{L_j}$ imply $P_{L_j-1} > {}_{L_j+N_j}\langle P_\ell \rangle_{L_j}$, while the conditions $\omega_{L_j+N_j}=0$ and $a_{L_j} > a_{L_j+N_j+1}$ imply

$$ {}_{L_j+N_j}\langle P_\ell \rangle_{L_j} > P_{L_j+N_j+1} \quad . $$

We may summarize all of these inequalities as

$$ P_{L_j-1} > {}_{L_j+N_j}\langle P_\ell \rangle_{L_j} > P_{L_j+N_j+1} \tag{8a} $$

and

$$ {}_{L_j+N_j}\langle P_\ell \rangle_{L_j+m} \geq {}_{L_j+N_j}\langle P_\ell \rangle_{L_j} \geq {}_{L_j+n}\langle P_\ell \rangle_{L_j} \qquad n, m=0, 1, \ldots, N_j \tag{8b} $$

As the derivation shows, either of the two inequalities in (8b) implies the other. Notice that all reference to α and a have disappeared so that the plateau interval $[L_j, L_j+N_j]$ may be determined solely by properties of the Legendre polynomials. Since successive plateaus must be monotonically decreasing, the condition ${}_{L_j+N_j}\langle P_\ell \rangle_{L_j} \geq \alpha$ is significant only for the last interval, i.e., only for determining B_0. Thus we may determine all possible plateau intervals independently of the multipliers α and a. Since for a maximum α is greater than zero, we may restrict our determination to those for which ${}_{L_j+N_j}\langle P_\ell \rangle_{L_j} > 0$.

We have now completely characterized the necessary conditions on a local maximum. In summary, the last value N_0 in B_1 must satisfy $a(P_{N_0}-\alpha) \geq 1$. If $a_{\ell-1} > a_\ell > a_{\ell+1}$, then $1 > a_\ell = a(P_\ell - \alpha) > 0$. The plateau interval must satisfy the sets of inequalities expressed by Eqs. (8a) and (8b). On a plateau, $1 \quad a_{L_j} = a\left(L_j + N_j \langle P_\ell \rangle_{L_j} - \alpha\right) > 0$. Finally, the first partial wave L_N for which a_ℓ vanishes must satisfy $\alpha \geq P_{L_N}$. To determine the local maximum, one must determine the sufficiency of these many conditions. We have not been able to show that these conditions uniquely determine the local maximum; indeed, we suspect that one can probably find some angles for which the local maximum is not unique.[5] Given any given scattering angle, one can use the inequalities to determine the plateau intervals. Then, given σ_T and σ_{el}, one can try to determine α and a to satisfy the equality constraints. In practice, it is easier and faster to choose α and a and to then calculate the corresponding σ_T and σ_{el}.

It is a simple matter now to solve the problem where the monotonicity requirement is applied to the even and odd partial waves separately (Eq. (2)). The general form of the local maximum is

$$x^{\text{even}} = \Big(1 = \ldots = a_{2M_0} > a_{2M_0+2} > \ldots > a_{2K_1} = a_{2K_1+2} = \ldots = a_{2K_1+2M_1} >$$

$$a_{2K_1+2M_1+2} > \ldots > a_{2K_M} = 0 = 0 = \ldots$$

$$x^{\text{odd}} = \Big(1 = \ldots = a_{2N_0+1} > a_{2N_0+3} > \ldots > a_{2L_1+1} = a_{2L_1+3} = \ldots = a_{2L_1+2N_1+1} >$$

$$a_{2L_1+2N_1+3} > \ldots > a_{2L_N+1} = 0 = 0 = \ldots\Big)$$

Correspondingly, we are led to define weighted averages over the even and odd partial waves separately

$$_{2N}\langle P_\ell \rangle_{2M} \equiv \sum_{n=M}^{N} (4n+1)\, P_{2n} \Big/ \sum_{n=M}^{N} (4n+1)$$

$$_{2N+1}\langle P_\ell \rangle_{2M+1} \equiv \sum_{n=M}^{N} (4n+3)\, P_{2n+1} \Big/ \sum_{n=M}^{N} (4n+3)$$

The preceding solution then becomes the correct solution to this problem if one everywhere treats the even and odd partial waves separately. For example, for the even case, the last value in B_1 must satisfy $a(P_{2M_0} - \alpha) > 1$. The plateau intervals $[2K_j, 2K_j+2M_j]$ must satisfy

$$P_{2K_j-2} > {}_{2K_j+2M_j}\langle P_\ell \rangle_{2K_j} > P_{2K_j+2M_j+2}$$

$$_{2K_j+2M_j}\langle P_\ell \rangle_{2K_j+2M} > {}_{2K_j+2M_j}\langle P_\ell \rangle_{2K_j} > {}_{2K_j+2n}\langle P_\ell \rangle_{2K_j} \qquad m, n = 0, 1, \ldots, M_j$$

On the plateau, we find

$$a_{2K_j} = a\left({}_{2K_j+2M_j}\langle P_\ell \rangle_{2K_j} - \alpha \right)$$

Finally, the first partial wave which vanishes satisfies $\alpha \geq P_{2K_M}$. Corresponding statements hold for the odd integers. In an appendix, we record the values for a number of the sums and averages.

3. CONTINUUM APPROXIMATION

It is slightly more convenient in numerical applications to approximate ℓ by a continuous variable since many partial waves contribute in general, and to assume the scattering angle is small. This approximation also makes it possible to carry the solution further analytically and clarifies the result. Since one of our purposes here is to illustrate a somewhat unfamiliar mathematical method, we will solve the problem again in this approximation. Using the standard replacement of the Legendre function by a Bessel function, the auxiliary function, $\mathscr{L}$, of the last section becomes

$$2\sin^2\frac{\mu}{2}\,\mathscr{L} = \int_0^\infty x\,dx\, J_0(x)\; a(x) + \alpha\left[\frac{\Delta^2}{8\pi}\sigma_T - \int_0^\infty x\,dx\; a(x)\right]$$

$$+ \frac{1}{2a}\left[\frac{\Delta^2}{8\pi}\sigma_{el} - \int_0^\infty x\,dx\; a^2(x)\right] + \int_0^\infty x\,dx\; \lambda(x)\; u(x) - \int_0^\infty dx\; \omega(x)\,\frac{da(x)}{dx}\,,$$

where

$$\Delta^2 \equiv -t = 4K^2\sin^2\frac{\mu}{2}\,, \quad x \simeq (2\ell+1)\sin\frac{\mu}{2}\,, \quad \omega(x) \equiv 2\sin^2\frac{\mu}{2}\,\omega_\ell\,.$$

(For simplicity we have set $r_\ell=0$ at the outset.) One consequence of replacing the Legendre polynomials by Bessel functions has been to shift the dependence on the momentum transfer to the boundary conditions. (Compare Eq. (3).) It is important to remember that this approximation is best for small angles and when many partial waves contribute to the sums. Notice that the monotonicity Eq. (1) naturally translates into a negativity condition on the slope of the partial wave amplitude. One could, of course, also generalize the monotonicity condition separately applied to the even and odd partial waves by defining partial wave amplitudes of even and odd signature, but for simplicity, we will ignore this alternative. We will also assume that a(x) is a continuous function, so the most general behavior possible has regions where a(x) is strictly decreasing separated by regions of constant a(x).

Formally, $\mathscr{L}$ is a function of a(x) and its first derivative da/dx; thus the maximum satisfies the usual Euler-Lagrange equations. One finds

$$J_0(x) - \alpha - \frac{a(x)}{a} + \lambda(1 - 2a(x)) + \frac{1}{x}\frac{d\omega}{dx} = 0 \qquad (9)$$

As before, we label the three partitions of the solution B_1, I, and B_0. The obvious analogues of Eqs. (5a-c) are as follows:

$$B_1: \lambda(x) = J_0(x) - \alpha - \frac{1}{a} + \frac{1}{x}\frac{d\omega}{dx} \geq 0 \qquad (10a)$$

$$I: \quad \lambda(x) = 0 \qquad \frac{-1}{x}\frac{d\omega}{dx} = J_0(x) - \alpha - \frac{a(x)}{a} \qquad (10b)$$

$$B_0: \lambda(x) = \alpha - J_0(x) - \frac{1}{x}\frac{d\omega}{dx} - 0 \quad . \qquad (10c)$$

On any interval, $y_i \leq x \leq x_{i+1}$, on which a(x) is strictly decreasing, $\omega(x) = 0$ and hence, $d\omega/dx = 0$. Therefore,

$$a(x) = a(J_0(x) - \alpha) \qquad (11)$$

Continuity of a(x) then implies that

$$1 = a(J_0(y_0) - \alpha) \qquad (12)$$

where y_0 is the largest value of x in B_1 and that

$$0 = J_0(x_n) - \alpha \quad , \qquad (13)$$

where x_N is the smallest value in B_0. Thus, the equality multipliers a and α determine the sets B_1 and B_0. Of course, if $1 > a(1-\alpha)$, then B_1 is empty.

Let us further explore the intervals, $x_i \leq x \leq y_i$, in I, on which a(x) is constant. These are surrounded by intervals on which a(x) is strictly decreasing and, by Eq. (11) above, $a(x) = a(J_0(x) - \alpha)$ on the surrounding intervals. However, since $a(x_i) = a(y_i)$, continuity of $a(x)$ then implies $J_0(x_i) = J_0(y_i)$. It follows from (10b) that $\omega(x)$ is continuous in I and, since ω vanishes on the surrounding intervals, we have $\omega(x_i) = \omega(y_i) = 0$. Inside the plateau interval

$$\frac{d\omega}{dx} = x\left[\frac{a(x_i)}{a} + \alpha - J_0(x)\right] = x\left[J_0(x_i) - J_0(x)\right]$$

Using $\omega(x_i)=0$ this may be easily integrated to obtain $\omega(x)$. The condition $\omega(y_i)=0$ then leads to

$$J_0(x_i) = {}_{y_i}\langle J_0\rangle_{x_i} \quad ,$$

where, as in the previous section, we define the weighted average

$${}_y\langle J_0\rangle_x \equiv \int_x^y J_0(z)\, zdz \Big/ \int_x^y zdz \quad .$$

(See the Appendix for the explicit evaluation.) In summary, the end points of the plateau interval satisfy

$$J_0(x_i) = {}_{y_i}\langle J_0\rangle_{x_i} = J_0(y_i) \tag{14a}$$

which may be compared to Eq. (8a) for the discrete case. One may further exploit the condition $\omega(x) \geq 0$ inside the interval to obtain the analogue of Eq. (8b),

$${}_{y_i}\langle J_0\rangle_y \geq {}_{y_i}\langle J_9\rangle_{x_i} \geq {}_x\langle J_0\rangle_{x_i} \quad ; \qquad \text{for } x_i \leq x, \quad y \leq y_i \quad . \tag{14b}$$

One can show, however, that these inequalities in fact follow from (14a), provided that the plateau interval extends over only one cycle of $J_0(x)$. This makes finding the plateau intervals easier in the continuous case than in the discrete case. And, as before, the determination of the possible intervals do not involve a and α.

Given the multipliers a and α, the necessary conditions given above are also sufficient to determine to solution. These multipliers, in turn, are to be determined from the equality constraints

$$\frac{\Delta^2}{8\pi}\sigma_T = \frac{1}{2}y_0^2 + \int_{y_0}^{x_N} x dx\, a(x) \tag{15}$$

$$\frac{\Delta^2}{8\pi}\sigma_{el} = \frac{1}{2}y_0^2 + \int_{y_0}^{x_N} x dx\, a^2(x) \tag{16}$$

where

$$a(x) = \begin{cases} a(J_0(x) - \alpha) & y_i \le x \le x_{i+1} \\ a(J_0(x_i) - \alpha) & x_i \le x \le y_i \end{cases}$$

and, if B_1 is empty, $y_0=0$, whereas, if B_1 is not empty, $1=a(J_0(y_0)-\alpha)$. In any case, x_N satisfies $\alpha = J_0(x_N)$. A typical solutions is indicated in Fig. 1, where the dashed curve is $a(J_0(x)-\alpha)$ and the solid curve is the solution a(x). The shaded regions indicate the partial wave amplitudes in the problem without the monotonicity constraint, described in the Introduction.

It is interesting to compare the expressions for the end of B_1 and the start of B_0:

$$J_0(y_0) = \alpha + \frac{1}{a} \qquad \text{and} \qquad J_0(x_N) = \alpha \quad .$$

As a becomes large, $J_0(y_0) \simeq J_0(x_N)$. Since both these points must lie on falling part of the curve a(x), one has $y_0 \simeq x_N$ as a gets large. Thus the ratio σ_{el}/σ_T approaches unity as a gets large. On the other hand, if a is smaller than $(1-\alpha)^{-1}$, then B_1 is empty and one expects a small value of the ratio σ_{el}/σ_T.

Given values of σ_T and σ_{el} from experimental data, one can determine a and α and, subsequently, the maximum value of the absorptive part of the scattering amplitude may be computed. We now turn to the numerical computations.

4. NUMERICAL RESULTS AND CONCLUSIONS

Because it is somewhat easier to evaluate, we discuss in detail the results in the continuum approximation. We have compared this to the evaluation in the discrete case and found little difference for the momentum transfers with which we will be concerned below. We believe that for the entire range of data presented, the continuum approximation gives a bound within a few percent of the actual bound, and, as we shall see, it is of little interest to inquire into the precise discrepancy.

To find numerical values for the upper bound, the candidates for plateau intervals must first be determined. The end points (x_i, y_i) of each interval are all quite close to the odd zeroes of $J_0(x)$. For example, the first region of constant $a(x)$ extends from $x_1 \simeq 2.35$ to $y_1 \simeq 8.55$. For comparison, the first and third zeroes of J_0 occur at 2.40 and 8.65. Values of x_i and y_i for successive intervals lie even closer to the higher odd zeroes of $J_0(x)$.

Physical values of σ_T for processes of interest are around 40 mb, and realistic values of σ_{el}/σ_T are in the range 1/6 to 1/3. For values of the momentum transfer $\Delta^2 \lesssim 2\,(\mathrm{GeV/c})^2$, these constraints can be accommodated with $y_0 \lesssim 1.6$ and α in a range that allows between 2 and 8 plateaus. Since there are two equality constraints, the maximum value of $d\sigma/dt$ (neglecting the real part of the amplitude) will depend, in general, on two variables. It is convenient to choose these to be $\tau = \Delta^2\sigma_T$ and $R = \sigma_{el}/\sigma_T$. Figure 2 shows the maximum value of $16\pi \frac{d\sigma}{dt}/\sigma_T^2 = \frac{d\sigma}{dt}/\left(\frac{d\sigma}{dt}\right)_{t=0}$ versus τ for two values of the ratio R.

In Fig. 3, the solution to this problem is compared with data and with the solutions of related problems discussed by Singh and Roy[1] and by Ravenhall and Pardee.[2] To facilitate comparison with earlier results, we have used for the abscissa the variable $\rho = \left(\Delta^2\sigma_T^2/4\pi\sigma_{el}\right) = \frac{\tau R}{4\pi}$. Data for πp and pp scattering are indicated.[6] Curve A is the upper bound derived by Singh and Roy in the absence of monotonicity constraints. Curve B is the bound given by Ravenhall and Pardee for essentially the same problem as we have discussed.[7] The curves C and C' are the same bounds appearing in Fig. 2, only here plotted as a function of ρ rather than of τ. Notice that in Fig. 2 C' lies below C, while in Fig. 3 C' lies above C over most of its range.

Inasmuch as the variable ρ has been ascribed some significance,[1,2] let us comment on this variable. Suppose B_1 were empty, so that $y_0=0$. Then since $a(x)/a$ is independent of a for all x, the ratio

$$A/A_0 = \frac{\int_0^{x_N} xdx\, J_0(x)\, a(x)}{\int_0^{x_N} xdx\, a(x)}$$

is independent of a and depends only on α. However, setting $y_0=0$ in Eqs. (15) and (16), one sees that α is determined by the ratio

$$\left(\frac{\Delta^2\sigma_T}{8\pi}\right)^2 \Big/ \frac{\Delta^2\sigma_{el}}{8\pi} = \frac{1}{2}\rho \quad .$$

Therefore, if B_1 is empty, A/A_0 is a function of ρ only, a property which has been called "universality".[1,2] This result is independent of the monotonicity assumption. However, the actual values for the experimental data usually require that B_1 not be empty. Thus, in general, the upper bound derived here will depend on R as well as on ρ. In fact, one can show that

$$\frac{\partial}{\partial R}\left(\frac{A}{A_0}\right) = \frac{4\pi}{\Delta^2\sigma_{el}}\left[2y_0\, J_1(y_0) - y_0^2\, J_0(y_0)\right] ,$$

where the differentiation is performed for fixed ρ. While this is zero when B_1 is empty, it is not zero in general.

Comparing our upper bound with the data, we see that the addition of the monotonicity requirement has significantly improved the bound of Singh and Roy;[1] however, it still approximates the data only for a very small range of ρ. Already at $\rho = 10$, the bound exceeds the data by a factor of 2; for typical values of σ_T and σ_{el}, this corresponds roughly to $\Delta^2 \lesssim 0.3\,(\mathrm{GeV/c})^2$.

5. SUMMARY

One could have hoped that with such general considerations as crude unitarity ($u_\ell \geq 0$), and the values of the total and elastic cross sections, the shape of the diffraction peak might have been understood. Even assuming the real part of the scattering amplitude is negligible, we found that only for very small values of the momentum transfer does the bound approximate the data. An exponential fit to the data is a good approximation far beyond values of t for which our bound is relevant.

One should note that the values of the a_ℓ which realize the maximum at a particular angle depends on that angle. The upper bound plotted in our graphs is not a reflection of any one set of partial wave amplitudes, but rather, as the angle changes, the values of the a_ℓ also change. Thus, for example, the area under the upper bound could be much larger than σ_{el}, and this turns out to be the case.

We conclude that, if the shape of diffraction peaks is to be understood, it is <u>not</u> on the basis of the naive considerations discussed here. There is probably a deep dynamical reason both for the small real part (if, indeed, it is small for all t) and for the rapid decrease of the differential cross section with momentum transfer.

ACKNOWLEDGEMENTS

Most of this work was performed while one of us (MBE) was still at SLAC. We would like to thank Mr. Peete Baer of the SLAC Computer Center for illustrating the efficiency of the APL by finding the plateau intervals in the discrete case. We also wish to acknowledge interesting conversations with Dr. S. Nussinov concerning the statement of this problem.

APPENDIX

$$\sum_{\ell=0}^{L}(2\ell+1)\ P_\ell = P'_{L+1} + P'_L$$

$$\sum_{n=0}^{N}(4n+1)\ P_{2n} = P'_{2N+1} \qquad \sum_{n=0}^{N}(4n+1) = (N+1)\ (2N+1)$$

$$\sum_{n=0}^{N}(4n+3)\ P_{2n+1} = P'_{2N+2} \qquad \sum_{n=0}^{N}(4n+3) = (N+1)\ (2N+3)$$

Hence

$$P'_\ell = \frac{d}{dz}\ P_\ell(z)$$

Consequently,

$$_{2K+2M}\langle P_\ell\rangle_{2K} = \frac{P'_{2K+2M+1} - P'_{2K-1}}{(M+1)\ (4K+2M+1)}$$

$$_{2L+2N+1}\langle P_\ell\rangle_{2L+1} = \frac{P'_{2L+2N+2} - P'_{2L}}{(N+1)\ (4L+2N+3)}$$

$$_{L+N}\langle P_\ell\rangle_{L} = \frac{P'_{L+N+1} + P'_{L+N} - P'_L - P'_{L-1}}{(N+2)\ (2L+N)}$$

In the continuous case, we have, analogously,

$$_y\langle J_0\rangle_x = \frac{2\left[y\,J_1(y) - x\,J_1(x)\right]}{y^2 - x^2}$$

FOOTNOTES AND REFERENCES

1. V. Singh and S. M. Roy, Phys. Rev. Letters 24, 28 (1970); V. Singh and S. M. Roy, Phys. Rev. D1, 2638 (1970).
2. A similar generalization has been treated in Ravenhall and Pardee, Phys. Rev. D2, 589 (1970), although their solution is not the absolute bound.
3. M. B. Einhorn and R. Blankenbecler, Report No. SLAC-PUB-768 (TH), Stanford Linear Accelerator Center (July 1970), submitted to Ann. Phys. (N.Y.), hereinafter referred to as EB. This paper can be considered to be a sequel to Section III.B of EB.
4. The notation and conventions are as in EB.
5. However, such angles are probably rare. One could probably show that, among all scattering angles, the set on which the necessary conditions are not also sufficient is of measure zero; we have not tried.
6. Data was taken from the graphs appearing in Ravenhall and Pardee, Ref. 2.
7. There seems to be some confusion in Refs. 1 and 2 on how to treat B_1 and its potential effect on the result. One might wonder why the "upper bound" B determined by Ravenhall and Pardee lies below ours. The answer lies in their identifying the incorrect intervals as plateau intervals, and so their curve is not the upper bound.

FIGURE CAPTIONS

1. An example of a local maximum.
2. Upper bounds for R=0.20 and R=0.25.
3. Comparison of the present solution with previous bounds, and with data (see text for full explanation).

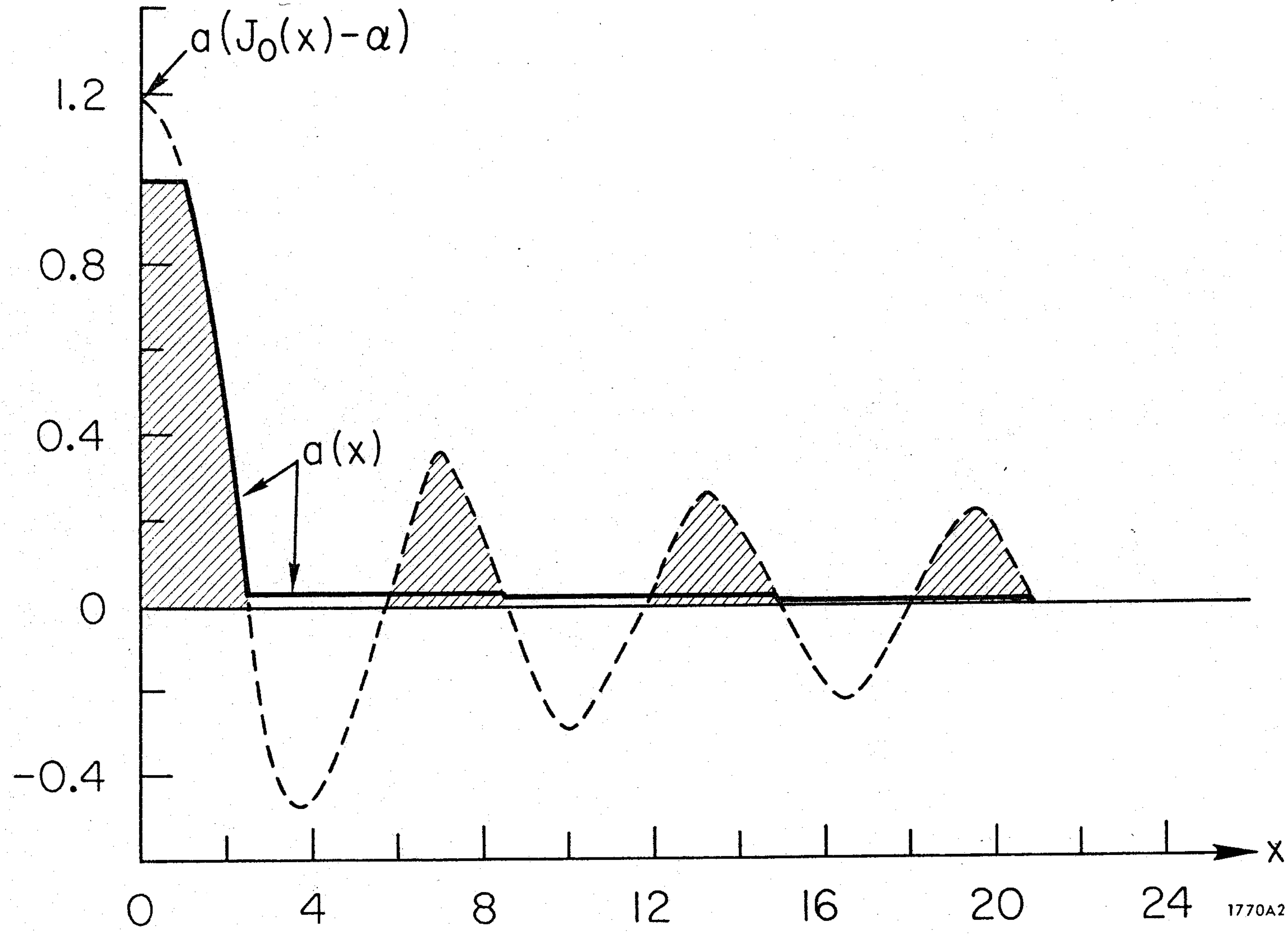

Fig. 1

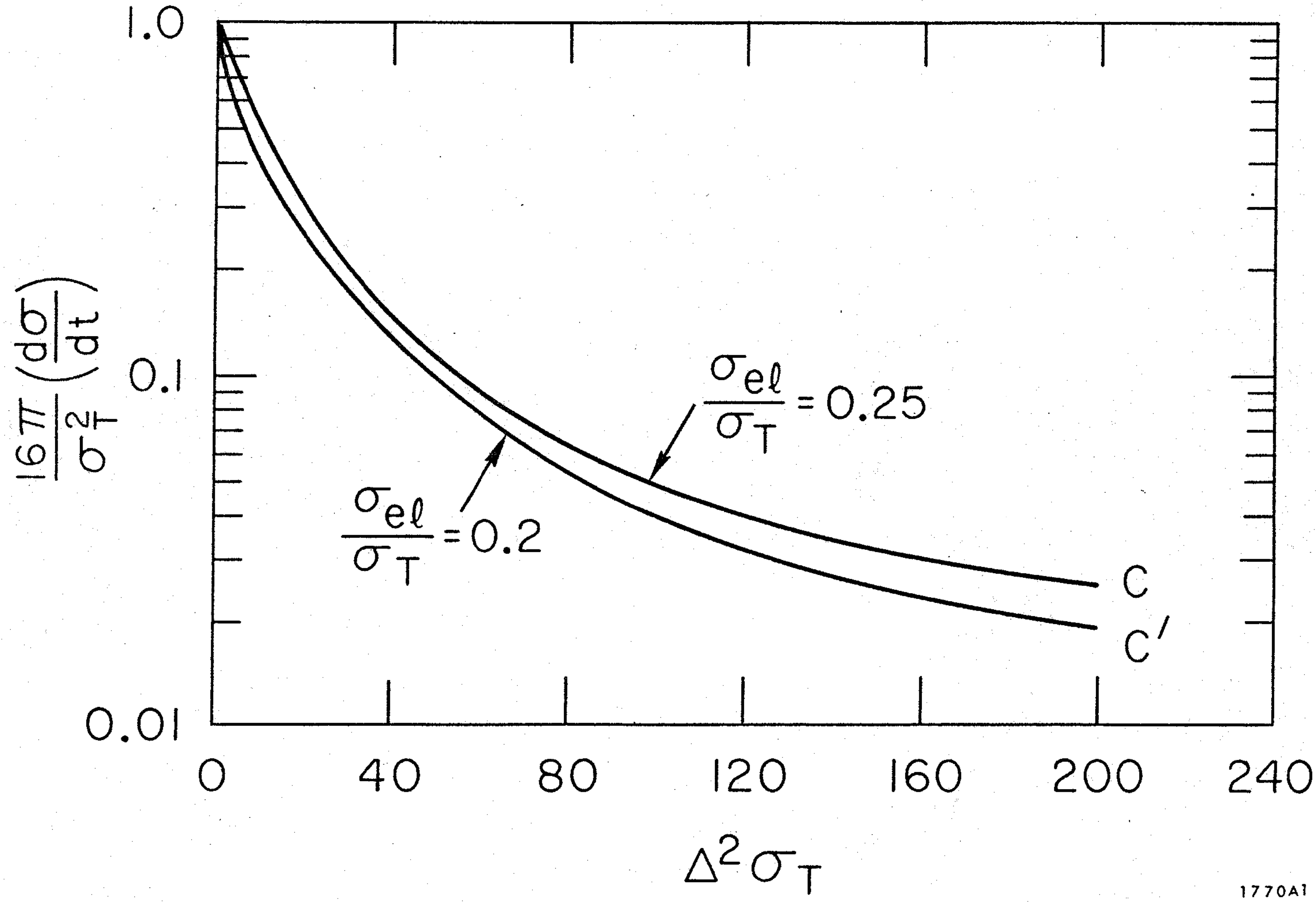
1.0
0.1
0.01
0
40
80
120
160
200
240
$\frac{16\pi}{\sigma_T^2}\left(\frac{d\sigma}{dt}\right)$
$\Delta^2 \sigma_T$
$\frac{\sigma_{el}}{\sigma_T} = 0.25$
$\frac{\sigma_{el}}{\sigma_T} = 0.2$
C
C′
1770A1

Fig. 2

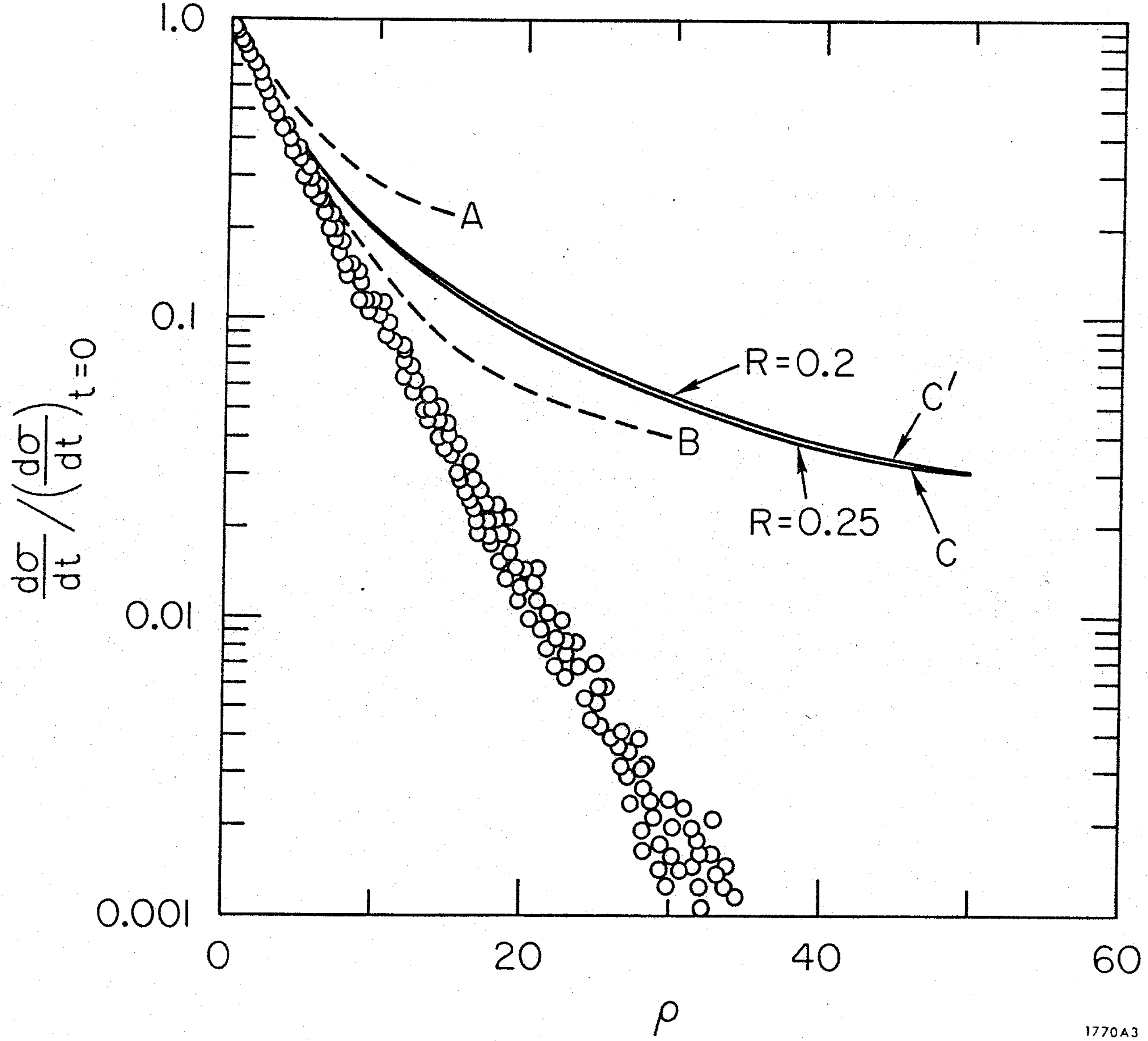

Fig. 3