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COMPUTATION OF THE PROPERTIES OF TRAVELING-WAVE LINAC STRUCTURES*

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ABSTRACT

A computer program which calculates the properties of traveling wave linac structures has been developed. Quantities which are computed include field distributions, quality factor (Q), shunt impedance (r), group velocity (v_g), and ratios of peak fields to average accelerating field ($\hat{E}/E_{eff}$ and $\hat{H}/E_{eff}$). The computation is limited to cylindrically symmetric structures, but the cross section of the cell boundary is otherwise quite arbitrary. The phase shift per cell ($k_0\ell$) may be chosen arbitrarily in the range $-\pi \leq k_0\ell \leq \pi$. The program employs a functional expansion of the fields, rather than the mesh method. Expansion coefficients (A_n) and frequency (ω) are adjusted by the program to fit the boundary conditions according to the principle that the Lagrangian of the field should vanish as a function of ω and A_n, and should be stationary with respect to the A_n. Comparisons with experimental results and with results of other computer programs are presented for several structures.

Introduction

The use of computer programs to aid in the design of standing-wave linac structures is now well established. This technique has proven to be valuable, and several programs have been written.[1-7] These programs have been used, for example, to optimize shunt impedance, minimize wall losses, reduce peak fields on the metal surfaces, and obtain the desired operating frequency by appropriate variations in cell geometry.

One advantage of the computer-aided design is the saving of a tremendous amount of painstaking, time-consuming model work so that the effects of variations in design parameters may be explored fully. Another advantage arises in the determination of peak fields along the metal boundaries. Here the experimental measurement is sufficiently difficult and uncertain that the computation may be more reliable.

Similar computations for traveling-wave structures are less common.[9-14] Nakamura[14] has developed a program for traveling-wave structures which permits circular rounding of disk edges but is restricted to a purely cylindrical pillbox for the outer cavity. Earlier works[9-13] were restricted to cases where the disk edge was also right angled. However we know, e.g., from the Los Alamos work[4] that optimization of the rf properties requires considerable freedom in choice of shapes of both the disk edges and the outer cavity walls.

Interest in a computer program for traveling-wave structures arose at SLAC from Neal's observation[15] that the traveling-wave regime should be significantly better than standing waves not only in shunt impedance but also in peak electric and magnetic fields, and therefore should be especially desirable for a superconducting linac. Subsequently a study of superconducting linacs was begun at SLAC,[16] and the program described here was developed as part of this study.

The capabilities of such a program should include the following: (1) arbitrary cell boundary; (2) arbitrary phase shift per cell; (3) ready modifications of the input parameters describing

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boundary shape and phase shift; (4) computation of frequency and field distribution; and (5) computation of parameters such as shunt impedance, quality factor, group velocity, and the peak electric and magnetic fields on metal surfaces.

The method which has been used is similar to that of Nakamura[14]: the fields are represented as expansions in series of appropriate functions, and a variational principle is used to adjust the expansion coefficients and frequency to satisfy the boundary conditions. The functional expansion method was chosen, instead of the more commonly used mesh method, mainly because it was believed that greater computational speed would be attained. The computation has been restricted to circularly symmetric TM modes in circularly symmetric structures.

Formulation

Field Models

It is assumed that the fields can be expanded in the form

$$\underset{\sim}{E} = \Sigma A_n \underset{\sim}{\epsilon}_n(\underset{\sim}{r},k); \qquad \underset{\sim}{H} = \Sigma A_n \underset{\sim}{\eta}_n(\underset{\sim}{r},k) \tag{1}$$

where $k = \omega/c$ and a factor of $e^{i\omega t}$ is understood. The $\underset{\sim}{\epsilon}_n$ and $\underset{\sim}{\eta}_n$ are functions which satisfy Maxwell's equations at the operating frequency;

$$\nabla\times\underset{\sim}{\epsilon}_n = -ik\underset{\sim}{\eta}_n; \qquad \nabla\cdot\underset{\sim}{\epsilon}_n = 0; \qquad \nabla\times\underset{\sim}{\eta}_n = ik\underset{\sim}{\epsilon}_n; \qquad \nabla\cdot\underset{\sim}{\eta}_n = 0 \tag{2}$$

Note that these are not normal mode expansions, since the retention of ω as a free parameter uses up one of the degrees of freedom necessary to specify orthogonality within a particular volume.

In a periodic structure the fields may be resolved into a forward wave and a backward wave. Either of these components satisfies a Floquet condition:

$$\underset{\sim}{E}(z+\ell) = e^{-i\Delta}\underset{\sim}{E}(z); \qquad \underset{\sim}{H}(z+\ell) = e^{-i\Delta}\underset{\sim}{H}(z) \tag{3}$$

where ℓ is the structure period and Δ (the phase shift per cell) is real if the structure is lossless. A positive Δ represents a wave traveling in the forward direction; the complex conjugate of (3) is a backward-wave solution. Thus it is only necessary to specify the fields within one cell in order to obtain a general solution.

Condition (3) may be considered a boundary condition to be applied at the periodic cell boundaries, or it may be incorporated explicitly in the field model.

Discussion of accelerator fields will be restricted to circularly symmetric TM modes in circularly symmetric structures, so that the only field components are E_z, E_r, and H_ϕ. A convenient representation is given by Fourier-analyzing Eq. (3). This results in the well-known space-harmonic expansion, which in the present case may be written

$$E_z = \Sigma E_n J_0(\kappa_n r)\, e^{-ik_n z}; \quad E_r = \Sigma E_n \frac{ik_n}{\kappa_n} J_1(\kappa_n r)\, e^{-ik_n z}; \quad H_\phi = \Sigma E_n \frac{ik}{\kappa_n} J_1(\kappa_n r)\, e^{-ik_n z} \tag{4}$$

where $k_n = \dfrac{\Delta + 2\pi n}{\ell}$ and $\kappa_n^2 = k^2 - k_n^2$, $n = 0, \pm 1, \pm 2 \ldots$ When $k_n^2 > k^2$ we have $J_0(\kappa_n r) = I_0(|\kappa_n| r)$ and $J_1(\kappa_n r)/\kappa_n = I_1(|\kappa_n| r)/|\kappa_n|$. For a "synchronous" space harmonic, $k_{ns} = k$ and $\kappa_{ns} = 0$, we have $J_0(\kappa_{ns} r) = 1$ and $J_1(\kappa_{ns} r)/\kappa_{ns} = r/2$.

Expansion (4), although very useful near the axis, may have poor convergence at large r, because of the $\exp(|\kappa_n| r)$ behavior of the I_0 and I_1 functions. Consequently, following Walkinshaw and Bell[9] and Nakamura[14], we may divide the cell into two annular regions and employ Eq. (4) in the inner regions and a different — hopefully better behaved — expansion in the outer region.

(See Fig. 1.) The expansion which has been used for the outer region is

$$E_z^{II}=\sum\left\{\left[B_m\cos\frac{m\pi z}{L}-iC_m\sin\frac{m\pi z}{L}\right]J_0(\mu_m r)+\left[F_m\cos\frac{m\pi z}{L}-iG_m\sin\frac{m\pi z}{L}\right]H_0^{(1)}(\mu_m r)\right\} \tag{5a}$$

$$E_r^{II}=\sum\frac{m\pi}{\mu_m L}\left\{\left[B_m\sin\frac{m\pi z}{L}+iC_m\cos\frac{m\pi z}{L}\right]J_1(\mu_m r)+\left[F_m\sin\frac{m\pi z}{L}+iG_m\cos\frac{m\pi z}{L}\right]H_1^{(1)}(\mu_m r)\right\} \tag{5b}$$

$$H_\phi^{II}=\sum\frac{k}{\mu_m}\left\{\left[iB_m\cos\frac{m\pi z}{L}+C_m\sin\frac{m\pi z}{L}\right]J_1(\mu_m r)+\left[iF_m\cos\frac{m\pi z}{L}+G_m\sin\frac{m\pi z}{L}\right]H_1^{(1)}(\mu_m r)\right\} \tag{5c}$$

where

$$\mu_m^2=k^2-\left(\frac{m\pi}{L}\right)^2, \qquad m=0,1,2\ldots$$

Here the expansion period 2L may be chosen appropriately for a particular cell geometry, but usually will be taken as $L\simeq\ell=$ the cell period. Note that for $m\pi/L>k$, we have $\mu_m=i|\mu_m|$, $H_0^{(1)}(\mu_m r)=-(2i/\pi)K_0(|\mu_m|r)$ and $H_1^{(1)}(\mu_m r)=-(2/\pi)K_1(|\mu_m|r)$.

The present representation [Eqs. (4) and (5)] is essentially the one used by Walkinshaw and Bell,[9] except that in the present case the geometry in the outer region is not simple so there are no explicit relationships between coefficients of a given order.

Boundary Conditions

The frequency (or propagation constant k) and the coefficients of the series expansions must be chosen to fit appropriate boundary conditions. The boundary types of interest here are (see Fig. 1) metal surfaces, periodic cell boundaries, and the interface between the inner and outer annular regions.

The metal boundary will be treated in the usual lossless approximation in which case it is sufficient to require

$$\underset{\sim}{E}\times\underset{\sim}{n}=\underset{\sim}{E}_t=0 \tag{6}$$

where $\underset{\sim}{n}$ is the outward normal at the surface. It is of course also necessary that the normal component of $\underset{\sim}{H}$ vanish. However by applying the "flux theorem," $\oint\underset{\sim}{E}\cdot d\underset{\sim}{\ell}=-ik\int\underset{\sim}{H}\cdot\underset{\sim}{n}\,dA$, to an infinitesimal surface element, we see that vanishing of $\underset{\sim}{E}\times\underset{\sim}{n}$ implies that $\underset{\sim}{H}\cdot\underset{\sim}{n}$ vanishes (but not the converse).

The condition at the interface between regions is simply continuity of the fields. However, by using the flux theorem (above) and its analogue, $\oint\underset{\sim}{H}\cdot d\underset{\sim}{\ell}=ik\int\underset{\sim}{E}\cdot\underset{\sim}{n}\,dA$, we readily see that continuity of the normal components is implied by continuity of the tangential components. Consequently we only need to require

$$(\underset{\sim}{E}^{II}-\underset{\sim}{E}^{I})\times\underset{\sim}{n}=0 \quad\text{and}\quad (\underset{\sim}{H}^{II}-\underset{\sim}{H}^{I})\times\underset{\sim}{n}=0 \tag{7}$$

at the interface.

At the periodic cell boundaries the fields must satisfy the Floquet condition, Eq. (3). Again as in the previous case, only the tangential components need by considered;

$$[\underset{\sim}{E}(+)-\underset{\sim}{E}(-)e^{-i\Delta}]\times\underset{\sim}{n}=0 \quad\text{and}\quad [\underset{\sim}{H}(+)-\underset{\sim}{H}(-)e^{-i\Delta}]\times\underset{\sim}{n}=0 \tag{8}$$

where $\underset{\sim}{E}(\pm)=\underset{\sim}{E}(z=\pm\ell/2)$, etc.

It will be necessary on account of practical computing limitations to approximate the field expansions with a finite number of terms; hence in general the boundary conditions can only be fit approximately. The variational approach which will be described next provides a rational means of choosing the expansion parameters to optimize the approximation.

Variational Principle

The steady-state fields within a lossless, bounded, charge-free region should satisfy the conditions that (1) the electric and magnetic energies are equal and (2) that the Lagrangian of the fields is stationary. This principle may be stated as

$$\text{a)} \quad L(k, A_n) = 0 \qquad \text{and} \qquad \text{b)} \quad \delta_n L(k, A_n) = \frac{\partial L}{\partial A_n^*} = 0 \tag{9}$$

(The derivative may of course be taken with respect to either A_n or A_n^* since the expression and its conjugate contain the same information.) The Lagrangian may be defined as either a volume integral or a surface integral:

$$\text{a)} \quad L = \frac{1}{4\pi}\int_V (\underset{\sim}{H}^* \cdot \underset{\sim}{H} - \underset{\sim}{E}^* \cdot \underset{\sim}{E}) dV \qquad \text{or} \qquad \text{b)} \quad L = \frac{1}{4\pi ik}\int_S (\underset{\sim}{H}^* \times \underset{\sim}{E}) \cdot \underset{\sim}{n}\, dS \tag{10}$$

where S is the surface which bounds the volume V and $\underset{\sim}{n}$ is the outward normal on S. Equation (10b) results from (10a) by application of Maxwell's equations and Gauss' theorem. The surface integral form (10a) clearly is related closely to the boundary conditions [Eqs. (6), (7), (8)].

In order to adapt the variational principle to a series expansion in a domain which may be divided into two regions, let us redefine the expansion (1) by finite series such that the terms which define $\underset{\sim}{E}^I$ and $\underset{\sim}{H}^I$ are enumerated by $1 \le n \le N_1$ and the terms which define $\underset{\sim}{E}^{II}$ and $\underset{\sim}{H}^{II}$ are enumerated by $N_1 < n \le N_1 + N_2$. The fields in either region will be designated simply

$$\underset{\sim}{E} = \sum_1^N A_n \underset{\sim}{\epsilon}_n \qquad \text{and} \qquad \underset{\sim}{H} = \sum_1^N A_n \underset{\sim}{\eta}_n \tag{11}$$

where $N = N_1 + N_2$, and with the understanding that

$$\underset{\sim}{\epsilon}_n = \underset{\sim}{\eta}_n = 0, \quad n > N_1 \quad \text{(in region I)};$$

$$\underset{\sim}{\epsilon}_n = \underset{\sim}{\eta}_n = 0, \quad 1 \le n \le N_1 \quad \text{(in region II)}.$$

Now since L is quadratic in the A_n and transcendental in k, it is appropriate to use the technique of linearized iterative fitting. Considering small variations in the parameters we write

$$k^{(i+1)} = k^{(i)} + \delta k^{(i)}; \qquad A_n^{(i+1)} = A_n^{(i)} + \delta A_n^{(i)} \tag{12}$$

$$\underset{\sim}{E}^{(i+1)} = \underset{\sim}{E}^{(i)} + \delta \underset{\sim}{E}^{(i)}; \qquad \underset{\sim}{H}^{(i+1)} = \underset{\sim}{H}^{(i)} + \delta \underset{\sim}{H}^{(i)} \tag{13}$$

$$\begin{aligned} \delta \underset{\sim}{E}^{(i)} &= \frac{\partial E^{(i)}}{\partial k}\, \delta k^{(i)} + \sum_1^N \underset{\sim}{\epsilon}_n^{(i)}\, \delta A_n^{(i)} = \sum_0^N \underset{\sim}{\epsilon}_n^{(i)}\, \delta A_n^{(i)} \\ \delta \underset{\sim}{H}^{(i)} &= \frac{\partial H^{(i)}}{\partial k}\, \delta k^{(i)} + \sum_1^N \underset{\sim}{\eta}_n^{(i)}\, \delta A_n^{(i)} = \sum_0^N \underset{\sim}{\eta}_n^{(i)}\, \delta A_n^{(i)} \end{aligned} \tag{14}$$

in which we treat k on the same basis as the A_n by making the identifications

$$A_0^{(i)} \equiv k^{(i)}; \qquad \underset{\sim}{\epsilon}_0^{(i)} \equiv \frac{\partial \underset{\sim}{E}^{(i)}}{\partial k}; \qquad \underset{\sim}{\eta}_0^{(i)} \equiv \frac{\partial \underset{\sim}{H}^{(i)}}{\partial k} \tag{15}$$

The superscript denotes the current value of the iteration index. The objective will be to start with an initial guess $k^{(0)}$, $A_n^{(0)}$, and to use the variational principle in a linearized form iteratively to obtain corrections for the parameters. If the initial guess is good enough, the procedure should converge to the correct values $\left(k^{(i)} \longrightarrow k \text{ and } A_n^{(i)} \longrightarrow A_n\right)$ after a few iterations.

In order to express the linearized form of the variational conditions, it will be convenient first to divide the Lagrangian into three parts:

$$L = L' + L'' + L''' \tag{16}$$

where L' represents the contribution of the metal surfaces; L'' represents the contribution of the interface between the inner and outer regions; and L''' represents the contributions of the periodic cell boundaries (see Fig. 1).

Treating the metal boundaries first, we expand the Lagrangian expressed by Eq. (10b) and obtain

$$L'^{(i+1)} = \frac{1}{4\pi i k^{(i)}} \int_{S'} (\underset{\sim}{H}^* \times \underset{\sim}{E} + \delta\underset{\sim}{H}^* \times \underset{\sim}{E} + \underset{\sim}{H}^* \times \delta\underset{\sim}{E} + \delta\underset{\sim}{H}^* \times \delta\underset{\sim}{E})^{(i)} \cdot \underset{\sim}{n}\, dS + \ldots$$

Now since the normal component of $\underset{\sim}{H}^* \times \underset{\sim}{E}$ can involve only tangential components of the fields, and since $\underset{\sim}{E}_t$ is supposed to vanish on the metal boundary, we consider $\underset{\sim}{E}$ to be a first-order small quantity in the above equation. Consequently the terms $(\delta\underset{\sim}{H}^* \times \underset{\sim}{E})$ and $(\delta\underset{\sim}{H}^* \times \delta\underset{\sim}{E})$ both are second order. The linearized form of L' then is

$$L'^{(i+1)} = L'^{(i)} + \frac{1}{4\pi i k^{(i)}} \int_{S'} \left(\sum_{n'=0}^{N} \underset{\sim}{H}^* \times \underset{\sim}{\epsilon}_{n'}\, \delta A_{n'} \right)^{(i)} \cdot \underset{\sim}{n}\, dS \tag{17a}$$

Similarly, the variational conditions [Eq. (9b)] may be expanded giving

$$\frac{\partial L'^{(i+1)}}{\partial A_n^*} = \frac{1}{4\pi i k^{(i)}} \int_{S'} \left(\underset{\sim}{\eta}_n^* \times \underset{\sim}{E} + \sum_{n'=0}^{N} \underset{\sim}{\eta}_n^* \times \underset{\sim}{\epsilon}_{n'}\, \delta A_{n'} \right)^{(i)} \cdot \underset{\sim}{n}\, dS \tag{17b}$$

Equations (17a, b) may be written as

$$L'^{(i+1)}_n = L'^{(i)}_n + \sum_{n'=0}^{N} L'^{(i)}_{nn'}\, \delta A^{(i)}_{n'}, \qquad 0 \le n \le N \tag{18}$$

with the definitions

$$L'_0 = L' = \frac{1}{4\pi i k} \int_{S'} (\underset{\sim}{H}^* \times \underset{\sim}{E}) \cdot \underset{\sim}{n}\, dS, \tag{19a}$$

$$L'_n = \frac{1}{4\pi i k} \int_{S'} (\underset{\sim}{\eta}_n^* \times \underset{\sim}{E}) \cdot \underset{\sim}{n}\, dS, \tag{19b}$$

$$L'_{0n'} = \frac{1}{4\pi i k} \int_{S'} (\underset{\sim}{H}^* \times \underset{\sim}{\epsilon}_{n'}) \cdot \underset{\sim}{n}\, dS, \tag{19c}$$

$$L'_{nn'} = \frac{1}{4\pi i k} \int_{S'} (\underset{\sim}{\eta}_n^* \times \underset{\sim}{\epsilon}_{n'}) \cdot \underset{\sim}{n}\, dS, \tag{19d}$$

in which the range of the indices is $1 \le n \le N$, $0 \le n' \le N$. A superscript (i) (iteration index) is understood.

To evaluate the integral along the interface boundary between the inner and outer regions, it is convenient to make the additional definition (at the interface)

$$\begin{aligned} \overline{\underset{\sim}{E}}^{(i)} &= \tfrac{1}{2}(\underset{\sim}{E}^{I} + \underset{\sim}{E}^{II})^{(i)}; \qquad \Delta\overline{\underset{\sim}{E}}^{(i)} = \tfrac{1}{2}(\underset{\sim}{E}^{II} - \underset{\sim}{E}^{I})^{(i)} \\ \overline{\underset{\sim}{H}}^{(i)} &= \tfrac{1}{2}(\underset{\sim}{H}^{I} + \underset{\sim}{H}^{II})^{(i)}; \qquad \Delta\overline{\underset{\sim}{H}}^{(i)} = \tfrac{1}{2}(\underset{\sim}{H}^{II} - \underset{\sim}{H}^{I})^{(i)} \end{aligned} \tag{20}$$

whence

$$\begin{aligned} \underset{\sim}{E}^{I,(i+1)} &= \left[\overline{\underset{\sim}{E}} - \Delta\overline{\underset{\sim}{E}} + \delta\underset{\sim}{E}^{I}\right]^{(i)}; \qquad \underset{\sim}{E}^{II,(i+1)} = \left[\overline{\underset{\sim}{E}} + \Delta\overline{\underset{\sim}{E}} + \delta\underset{\sim}{E}^{II}\right]^{(i)} \\ \underset{\sim}{H}^{I,(i+1)} &= \left[\overline{\underset{\sim}{H}} - \Delta\overline{\underset{\sim}{H}} + \delta\underset{\sim}{H}^{I}\right]^{(i)}; \qquad \underset{\sim}{H}^{II,(i+1)} = \left[\overline{\underset{\sim}{H}} + \Delta\overline{\underset{\sim}{H}} + \delta\underset{\sim}{H}^{II}\right]^{(i)} \end{aligned} \tag{21}$$

After some manipulation we obtain the equivalent of (18) with L' replaced by L'', where the definitions now are

$$L''_0 = \frac{2}{4\pi i k} \int_{S''} (-\overline{\underset{\sim}{H}}^* \times \Delta\overline{\underset{\sim}{E}} + \Delta\overline{\underset{\sim}{H}} \times \overline{\underset{\sim}{E}}^*) \cdot \underset{\sim}{n}^{I}\, dS \tag{22a}$$

and

$$L''_n = \frac{1}{4\pi ik}\int_{S''}\left[-\left(\underset{\sim}{\eta}^{I*}_n + \underset{\sim}{\eta}^{II*}_n\right)\times\Delta\overline{\underset{\sim}{E}} + \Delta\overline{\underset{\sim}{H}}\times\left(\underset{\sim}{\epsilon}^{I*}_n + \underset{\sim}{\epsilon}^{II*}_n\right)\right]\cdot\underset{\sim}{n}^{I}\,dS \tag{22b}$$

$$L''_{0n'} = \frac{1}{4\pi ik}\int_{S''}\left[\overline{\underset{\sim}{H}}^{*}\times\left(\underset{\sim}{\epsilon}^{I}_{n'} - \underset{\sim}{\epsilon}^{II}_{n'}\right) - \left(\underset{\sim}{\eta}^{I}_{n'} - \underset{\sim}{\eta}^{II}_{n'}\right)\times\overline{\underset{\sim}{E}}^{*}\right]\cdot\underset{\sim}{n}^{I}\,dS \tag{22c}$$

$$L_{nn'} = \frac{1}{4\pi ik}\int_{S''}\left[\left(\underset{\sim}{\eta}^{*}_n\times\underset{\sim}{\epsilon}_{n'}\right)^{I} - \left(\underset{\sim}{\eta}^{*}_n\times\underset{\sim}{\epsilon}_{n'}\right)^{II}\right]\cdot\underset{\sim}{n}^{I}\,dS \tag{22d}$$

where again the range of the indicies are $1 \le n \le N$, $0 \le n' \le N$ and the iteration index (i) is understood. Note that the outward normal vectors are related by $\underset{\sim}{n}^{II} = -\underset{\sim}{n}^{I}$ along the interface.

On the periodic boundary, we adopt the notation $\underset{\sim}{E}(z=\pm\ell/2) = \underset{\sim}{E}(\pm)$, etc., and in analogy with Eqs. (20) and (21) define

$$\begin{aligned}\widetilde{\underset{\sim}{E}}^{(i)} &= \frac{1}{2}\left[\underset{\sim}{E}(+) + \underset{\sim}{E}(-)\,e^{-i\Delta}\right]^{(i)}; \qquad \Delta\widetilde{\underset{\sim}{E}}^{(i)} = \frac{1}{2}\left[\underset{\sim}{E}(+) - \underset{\sim}{E}(-)\,e^{-i\Delta}\right]^{(i)}\\ \widetilde{\underset{\sim}{H}}^{(i)} &= \frac{1}{2}\left[\underset{\sim}{H}(+) + \underset{\sim}{H}(-)\,e^{-i\Delta}\right]^{(i)}; \qquad \Delta\widetilde{\underset{\sim}{H}}^{(i)} = \frac{1}{2}\left[\underset{\sim}{H}(+) - \underset{\sim}{H}(-)\,e^{-i\Delta}\right]^{(i)}\end{aligned} \tag{23}$$

whence

$$\underset{\sim}{E}^{(i+1)}(+) = \left[\widetilde{\underset{\sim}{E}} + \Delta\widetilde{\underset{\sim}{E}} + \delta\widetilde{\underset{\sim}{E}}(+)\right]^{(i)}; \quad \text{and} \quad \underset{\sim}{E}^{(i+1)}(-) = \left[\left(\widetilde{\underset{\sim}{E}} - \Delta\widetilde{\underset{\sim}{E}}\right)e^{i\Delta} + \delta\underset{\sim}{E}(-)\right]^{(i)} \tag{24}$$

and similar expressions for $H^{(i+1)}(+)$. The linear system equivalent to Eq. (18) again is found, with L' replaced by L''', and the following definitions:

$$L'''_0 = \frac{2}{4\pi ik}\int_{S'''}\left[\widetilde{\underset{\sim}{H}}^{*}\times\Delta\widetilde{\underset{\sim}{E}} - \Delta\widetilde{\underset{\sim}{H}}\times\widetilde{\underset{\sim}{E}}^{*}\right]_z\,dS \tag{25a}$$

$$L'''_n = \frac{1}{4\pi ik}\int_{S'''}\left[\left(\underset{\sim}{\eta}_n(+) + \underset{\sim}{\eta}_n(-)\,e^{-i\Delta}\right)^{*}\times\Delta\widetilde{\underset{\sim}{E}} - \Delta\widetilde{\underset{\sim}{H}}\times\left(\underset{\sim}{\epsilon}_n(+) + \underset{\sim}{\epsilon}_n(-)\,e^{-i\Delta}\right)^{*}\right]_z\,dS \tag{25b}$$

$$L'''_{0n'} = \frac{1}{4\pi ik}\int_{S'''}\left[\widetilde{\underset{\sim}{H}}^{*}\times\left(\underset{\sim}{\epsilon}_{n'}(+) - \underset{\sim}{\epsilon}_{n'}(-)\,e^{-i\Delta}\right) - \left(\underset{\sim}{\eta}_{n'}(+) - \underset{\sim}{\eta}_{n'}(-)\,e^{-i\Delta}\right)\times\widetilde{\underset{\sim}{E}}^{*}\right]_z\,dS \tag{25c}$$

$$L'''_{nn'} = \frac{1}{4\pi ik}\int_{S'''}\left[\underset{\sim}{\eta}^{*}_n(+)\times\underset{\sim}{\epsilon}_{n'}(+) - \underset{\sim}{\eta}^{*}_n(-)\times\underset{\sim}{\epsilon}_{n'}(-)\right]_z\,dS \tag{25d}$$

with $1 \le n \le N$, $0 \le n' \le N$; superscripts of (i) are understood. It is readily seen that Eqs. (25) all vanish identically for any model which explicitly satisfies the Floquet condition; e.g., the traveling-wave space harmonic expansion [Eq. (4)].

We may now put the parts together and state the linearized variational conditions as

$$L_n^{(i+1)} \cong 0 \cong L_n^{(i)} + \sum_{n'=0}^{N} L_{nn'}^{(i)}\,\delta A_{n'}^{(i)}, \qquad 0 \le n \le N \tag{26}$$

with $L_n^{(i)} = (L'_n + L''_n + L'''_n)^{(i)}$ and $L_{nn'}^{(i)} = (L'_{nn'} + L''_{nn'} + L'''_{nn'})^{(i)}$ given by Eqs. (19), (22) and (25), recalling the definition $A_0 = k$ [Eq. (15)]. Solutions of the linear system given by (26) with $L_n^{(i+1)}$ set equal to 0 gives the corrections $\delta A_n^{(i)} = (\delta k,\ \delta A_1 \ldots \delta A_N)^{(i)}$ in terms of the current approximation $k^{(i)}$, $A_1^{(i)}, \ldots, A_N^{(i)}$.

Other RF Properties

The quantities shunt impedance, quality factor, and group velocity are given by well-known formulae. For the present purposes we use the following definitions:

$$r_{tot}/Q = \frac{4Z_0\ell}{k}\,\frac{|E_{tot}|^2}{u}; \qquad r_0/Q = \frac{4Z_0\ell}{k}\,\frac{|E_{eff}|^2}{u} \tag{26); (27}$$

$$Q = \sqrt{\frac{k}{2}}\,\frac{u}{p_w}; \qquad v_g/c = \frac{2\ell p_z}{u} \tag{28); (29}$$

where r_{tot} = total shunt impedance; r_0 = synchronous shunt impedance; Q = quality factor; v_g = group velocity (actually "energy velocity"); Z_0 = the impedance of free space (377 Ω in practical units or $4\pi/c$ in gaussian units); and the remaining quantities are defined as follows:

$$|E_{tot}|^2 = \frac{1}{\ell}\int_{-\ell/2}^{\ell/2} |E_z(r=0)|^2 \, dz; \qquad E_{eff} \equiv \frac{1}{\ell}\int_{-\ell/2}^{\ell/2} E_z(r=0)\, e^{i\omega z/v_e} \, dz \qquad (30);(31)$$

$$u = \int_V (\underset{\sim}{E}^* \underset{\sim}{E} + \underset{\sim}{H}^* \underset{\sim}{H}) \, dV \propto \text{stored energy}; \qquad (32)$$

$$p_w = \int_{S'} \sqrt{\frac{\rho}{Z_0}} \, \underset{\sim}{H}^* \underset{\sim}{H} \, dS \propto \text{wall losses}; \qquad (33)$$

$$p_z = \text{Re} \int_A (\underset{\sim}{E} \times \underset{\sim}{H}^*)_Z \, dS \propto \text{power flow}; \qquad (34)$$

where v_e = particle velocity; ρ = resistivity of metal walls; $\int_{S'}$ denotes integration over the metal walls; and $\int_A$ denotes integration over any transverse cross section of the waveguide. Equations (30) through (34) presume gaussian units. The Q calculation is for normal (not superconducting) materials.

Computer Program

General Description

A program called TWAP (for Traveling Wave Accelerator Program) has been written in Fortran IV for the IBM 360/91. After reading in data which describe the problem, TWAP may perform the following functions:

1. Calculate the parameters (frequency and the expansion coefficients) using the variational fitting principle described previously;
2. Punch out these parameters on a card deck suitable for input into a subsequent run;
3. Perform the integrals necessary to evaluate r/Q, Q, and v_g, [see Eqs. (29)...(37)] above and search the metal boundaries for maximum electric and magnetic fields.

Each of these three functions may be performed in any order (or not at all) under input data control.

Input Data

Data are read in via punched cards. The input includes the "initial guess" for frequency and expansion coefficients; the number of expansion coefficients in each of the two annular regions; designation of which of the parameters will be allowed to vary in the fitting routine; auxilliary quantities such as phase shift per cell, particle velocity, the desired fitting accuracy, and the maximum number of iterations; and physical description of the cell. The cell is described by a sequence of data elements each of which defines a boundary segment of the longitudinal cross section. Optional elements are provided for segments of axis, periodic boundary, a plane of structure symmetry, interface between regions I and II, and several shapes of metal wall sections including straight lines, circular arcs, and elliptical arcs. An arbitrary number of mesh divisions is designated for each segment, and an arbitrary resistivity is specified for each metal segment.

Parameter Fitting

The vector L_n and matrix $L_{nn'}$ [Eq. (26)] are generated by numerical integration of Eqs. (19), (22), and (25) using Simpson's rule weighting of the values of the integrand evaluated at the boundary mesh points. A linear system routine then solves for the corrections δA_n; this routine

automatically reduces the dimension of the matrix if the matrix is singular or nearly singular, in which case some of the A_n are not varied. After correcting the A_n, the program decides whether to make another iteration; the iteration is terminated if the fit is sufficiently good, if the system seems to have become stationary, or if the maximum specified number of iterations has been performed.

Since the solution $A_n = 0, (1 \leq n \leq N)$ satisfies the fitting criteria, it is advisable to hold one of the coefficients, e.g., the dominant space harmonic, constant. Depending on symmetry or other structure properties it may be possible to hold some of the parameters at zero. Fitting of the structure geometry parameters must be done by changing the input data on successive runs; the program does not do this automatically.

Other RF Properties

The integrals given by Eqs. (30) through (34), and required to calculate r/Q, Q, and v_g/c [Eqs. (26) - (29)], are computed by Simpson's-rule numerical integration using the preset boundary mesh points. The volume integral, Eq. (32), is performed by first integrating over an annular shell associated with each boundary mesh point, and then doing the radial integral over the shells. The electric and magnetic energy integrals are performed separately, and the quantity $(u_H - u_E)/u$ (proportional to the Lagrangian) is calculated as an independent check on convergence of the fit.

In addition to the quantities r_{tot}/Q, r_{eff}/Q, Q, r_{eff}, and v_g/c, the integration subroutine also finds the magnitudes of the peak electric and magnetic fields, $\hat{E}$ and $\hat{H}$.

Typical Results

Test cases have been run for several structures for which the properties are available both from experiment and from other computer programs.

SLAC Constant Gradient Structure

Figure 2 shows the geometry of the cells in the existing SLAC constant gradient structure. Table I shows properties found by experiment and computed by Nakamura's program[14] and by TWAP. Here SI, SM, and S0 (Nakamura's designation)refer to cells from the input, middle, and output end of the tapered structure. The TWAP calculations used an expansion to order 20 $(0, \mp 1, \ldots, \mp 20)$ in the inner region and order 14 in the outer region; the corresponding numbers in Nakamura's calculation were 12 and 12. The agreement is seen to be fairly good, with the two computer programs generally agreeing slightly better with one another than with experiment.

In the TWAP calculation the maximum boundary errors (tangential E field) occurred at the sharp edge on the disk tip, and were about 10% of the peak electric field.

Los Alamos Structure

Figure 3 gives the dimensions of a particular cell ($\beta = 0.65$) of a design used in the Los Alamos proton linac.[4] Table II summarizes properties obtained by experiment, by the LALA program[4] and by TWAP. It proved impossible to obtain really satisfactory fits with the functional expansions in TWAP in this case; the maximum boundary errors (which occurred near the tip of the nose cone) were about 20% of the peak E field. Nevertheless, the agreement is fairly good (.05%) for frequency, although rather poor for r/Q (7.5%) and Q (4%).

Other Examples

Several examples related to the SLAC superconducting accelerator study are given in another paper [Herrmannsfeldt et al.[17]] in these Proceedings. As may be seen from Table I of the above reference, the TWAP results agree rather well with both the LALA computations and the experimental results. The boundary conditions are fit rather well by TWAP in smooth-walled structures such as those of Herrmannsfeldt et al.;[17] maximum boundary errors are 3% to 5% of the peak electric field.

Summary and Comments

The variational principle for electromagnetic fields has been used in a form which appears somewhat different from that used elsewhere,[9,10,7,14] although it presumably is equivalent. The advantage of the present formulation is that the frequency is found along with the other undetermined field parameters by a linearized fitting method which tends to converge in a very few iterations.

The use of functional expansions to represent the traveling wave fields has been extended to generalized (rotationally symmetric) boundaries, and is fairly successful if the boundary is not too complicated. The Los Alamos structure[4] with its prominant nose cones is a case in which the present model does not work very well. Methods for extending the applicability of this kind of model have been suggested e.g., by Walkinshaw and Bell[9] and by Nakamura[14] (following Gluckstern[8]). This matter is under further study.

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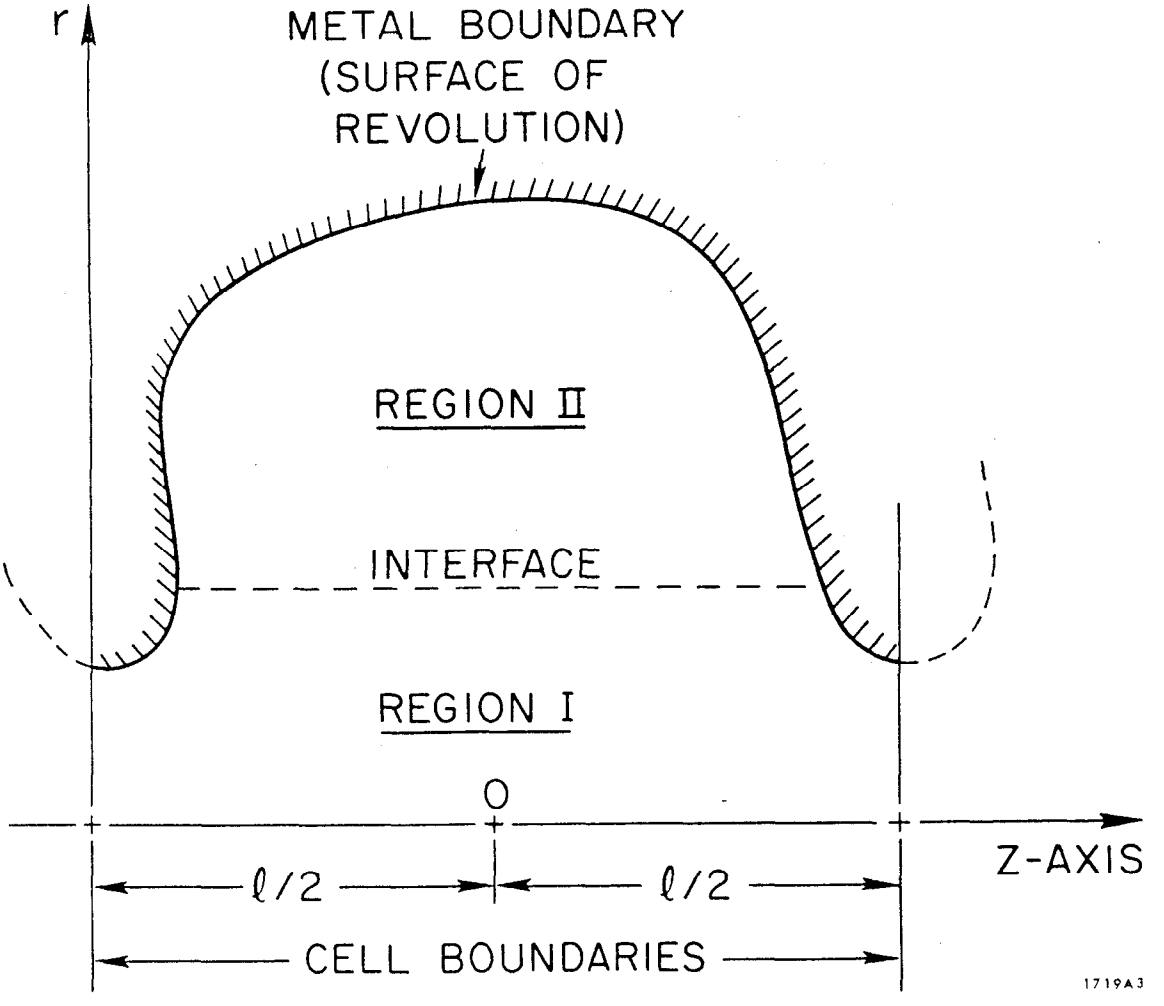
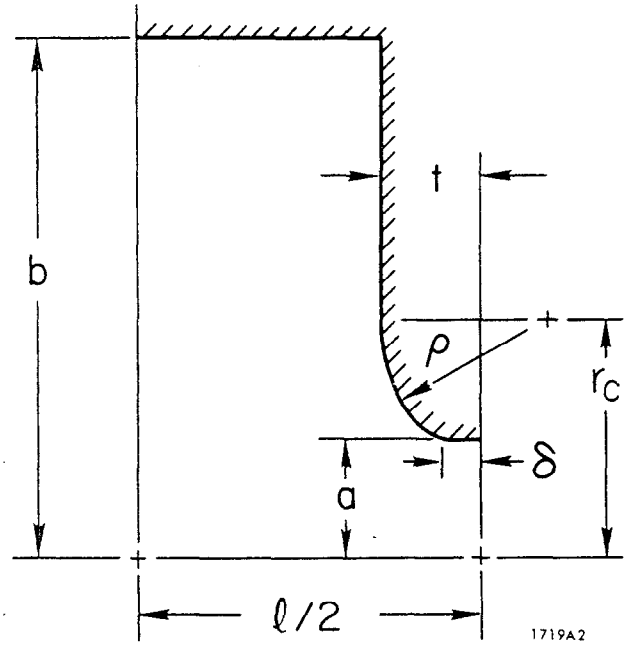


FIG. 1--Schematic illustration of cell geometry.



ℓ = 3.50012 cm
δ = 0.03936 cm
ρ = 0.30861 cm
r_c = a + 0.30351 cm
t = 0.29210 cm

FIG. 2--SLAC 2π/3 constant gradient structure. (See Table I, below, for the dimensions a and b.)

TABLE I

Rf Properties of SLAC 2π/3 Constant Gradient Structure

Cell Designation	Parameter	Experiment	Nakamura	TWAP
SI a = 1.3129 cm	b (cm)	4.1721	4.1733	4.1736
	v_g/c	.0204	.0206	.0206
	Q	14,160	13,863	13,917
	r_0/Q (Ω/cm)	37.5	38.5	38.9
SM a = 1.16605 cm	b (cm)	4.1328	4.1335	4.1338
	v_g/c	.0135	.0136	.0137
	Q	13,700	13,828	13,882
	r_0/Q (Ω/cm)	41.1	41.0	41.9
SO a = .96096 cm	b (cm)	4.0886	4.0892	4.0893
	v_g/c	.0065	.00652	.00672
	Q	13,230	13,791	13,845
	r_0/Q (Ω/cm)	45.1	43.1	46.0

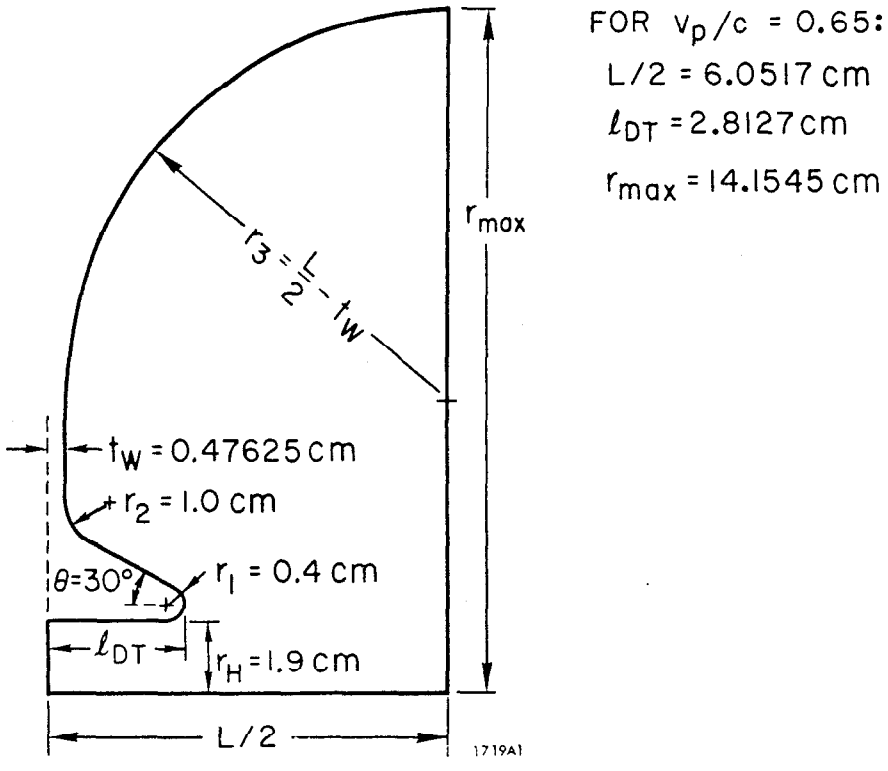


FIG. 3--Geometry of Los Alamos Cells. The structure is nominally designed for π-mode, but the results in Table II, below, are for 0-mode.

TABLE II

Rf Properties of LASL $\beta = 0.65$ Structure

Parameter	Experiment	LALA	TWAP
f (MHz)	804.24	804.33	803.88
r_0/Q (Ω/cm)	15.02	15.18	16.31
Q	25,000	28,960	27,858