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OPTIMAL EQUATIONS FOR THREE PARTICLE SCATTERING*

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ABSTRACT

In this paper we investigate two differing approaches to the three-body scattering problem: that of Faddeev and that of Lovelace. We find a simple operator connection between the two methods and use this connection to give a physical justification of Faddeev's residue prescription for determining three-body scattering amplitudes. Based on these results we present derivations for the integral equations, which directly give breakup and rearrangement amplitudes as solutions.

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I. INTRODUCTION

There exist in the literature of the three-body scattering problem essentially two different methods of defining the physical scattering amplitude. The first, widely used in the literature is that typified by the work of Lovelace [1,2]. The second, somewhat neglected because of its apparent complexity, is that developed by Faddeev [3]. Although it is known [1] that the two methods eventually lead to same physical S matrices the connection between the two approaches is somewhat obscure. In particular, although Faddeev's method is easy enough to describe mathematically, it does not have (at least in Faddeev's work) a direct physical motivation. This fact and the neglect of Faddeev's approach is unfortunate, since the method leads to the simplest available integral equations for describing three-body scattering when two-particle bound states are present in the initial and/or final configurations. Among the desirable properties of these integral equations is that solutions give the observed physical amplitudes without any integrations over the asymptotic channel wave functions such as are needed in Lovelace's approach in order to obtain the physical amplitudes of interest.

The principal aim of this paper is to clarify the interconnection between Lovelace's and Faddeev's approaches to defining three-body scattering amplitudes. To this end we find a simple operator connection between Faddeev's amplitudes and the Lovelace-type amplitudes in the form introduced by Alt, Grassberger and Sandhas [4]. This interconnection leads to a simple physical explanation of Faddeev's approach. We also provide a derivation of the integral equations which the physical breakup and rearrangement (including elastic scattering) amplitudes satisfy. Finally, we give a new set of integral equations for these amplitudes in which only the on-shell rearrangement amplitudes enter.

II. FADDEEV EQUATIONS

In this section we give a summary of Faddeev's results. In particular we recount Faddeev's method for determining the physical amplitudes for breakup, rearrangement and elastic scattering. Throughout we will use most of the same notation for operators and kinematic variables as one finds in Faddeev's book [3].

Let us denote by H_0 the three-particle kinetic energy operator. In momentum space [3, p. 6]

$$H_0 = \frac{1}{2\mu_\alpha} k_\alpha^2 + \frac{1}{2n_\alpha} p_\alpha^2 = \frac{p_\alpha^2}{2\mu_\beta} + \frac{p_\alpha \cdot p_\beta}{m_\gamma} + \frac{p_\beta^2}{2\mu_\alpha}, \tag{II.1}$$

where $\alpha\beta\gamma$ are the cyclic labels of the three particles, and m_α is the α particle mass; μ_α is a two-particle reduced mass $\mu_\alpha = (m_\rho m_\gamma)/(m_\rho + m_\gamma)$; n_α is a three-particle reduced mass, $n_\alpha = m_\alpha(m_\beta + m_\gamma)/(m_\alpha + m_\beta + m_\gamma)$. The p_α represent the individual particle momentum in the three-body center-of-mass; i.e., $k_\alpha = (m_\alpha p_\beta - m_\beta p_\alpha)/(m_\beta + m_\gamma)$. If we describe the interaction between any pair $(\beta\gamma)$ of particles by the potential V_α, then the total three-body potential is

$$V = \sum_{\alpha=1}^{n} V_\alpha \tag{II.2}$$

and the resulting complete Hamiltonian is

$$H = H_0 + V. \tag{II.3}$$

Faddeev analyzes the scattering solutions of the three-body problem by studying the behavior of the complete Green function $G(z) = (H - z)^{-1}$ in the complex z plane. $G(z)$ satisfies the well known Hilbert identities

$$G(z) = G_0(z) - G_0(z)\, V\, G(z), \tag{II.4}$$

$$= G_0(z) - G(z)\, V\, G_0(z), \tag{II.5}$$

where $G_0(z) = (H_0 - z)^{-1}$ is the unperturbed Green function. The study of this singular (as $z \rightarrow$ real axis) operator is facilitated by writing G(z) with the aid of Eqs. (4) and (5) in the form

$$G(z) = G_0(z) - G_0(z)\, T(z)\, G_0(z), \tag{II.6}$$

where T(z) is defined to be

$$T(z) = V - V\, G(z)\, V. \tag{II.7}$$

Clearly knowing T(z) determines G(z) via Eq. (II.6) and so the study of the singular G(z) can be replaced by the study of the less singular operator T(z).

In order to find a nonsingular linear integral equation from which T(z) may be determined, T(z) is broken up into components $M_{\alpha\beta}(z)$, which are suggested by the form of Eq. (II.7),

$$M_{\alpha\beta}(z) = \delta_{\alpha\beta}\, V_\alpha - V_\alpha G(z) V_\beta, \tag{II.8}$$

$$T(z) = \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} M_{\alpha\beta}(z). \tag{II.9}$$

The derivation of integral equations for $M_{\alpha\beta}$ is a straightforward algebraic exercise and one finds [3, p. 12]

$$M_{\alpha\beta}(z) = \delta_{\alpha\beta}\, T_\alpha(z) - \sum_{\gamma\neq\alpha} T_\alpha(z)\, G_0(z)\, M_{\gamma\beta}(z) \tag{II.10}$$

and

$$M_{\alpha\beta}(z) = \delta_{\alpha\beta} T_\beta(z) - \sum_{\gamma\neq\beta} M_{\alpha\gamma}(z)\, G_0(z)\, T_\beta(z). \tag{II.11}$$

In these equations the operator $T_\alpha(z)$ which determines both the driving term and the kernels is the two-body t-matrix defined in the three particle Hilbert space, namely, or in terms of matrix elements[1]

$$T_\alpha = V_\alpha + V_\alpha(z - H_0 - V_\alpha)^{-1} V_\alpha, \tag{II.12a}$$

$$T_\alpha(p_\alpha k_\alpha; p'_\alpha k'_\alpha; z) = \delta^3(p_\alpha - p'_\alpha)\, t_\alpha\left(k_\alpha, k'_\alpha; z - \frac{p_\alpha^2}{2n_\alpha}\right). \tag{II.12b}$$

The t_α appearing in the right-hand side of this relation is just the $(\beta\gamma)$ off-shell two-body t-matrix appropriate for the $(\beta\gamma)$ scattering in the $(\beta\gamma)$ center-of-mass system.

The Eqs. (II.10) and (II.11) are singular in the following sense. The driving term $T_\alpha(z)$ has a delta function in the p_α variable which will also appear in the solution $M_{\alpha\beta}$. Thus, it is advisable to consider the subtracted quantity

$$W_{\alpha\beta}(z) = M_{\alpha\beta}(z) - \delta_{\alpha\beta} T_\alpha(z) \tag{II.13}$$

which satisfies

$$W_{\alpha\beta}(z) = W^{(0)}_{\alpha\beta}(z) - \sum_{\gamma\neq\alpha} T_\alpha(z)\, G_0(z)\, W_{\gamma\beta}(z), \tag{II.14}$$

and

$$W_{\alpha\beta}(z) = W^{(0)}_{\alpha\beta}(z) - \sum_{\gamma\neq\beta} W_\alpha(z)\, G_0(z)\, T_\beta(z), \tag{II.15}$$

with

$$W^{(0)}_{\alpha\beta}(z) = - \bar{\delta}_{\alpha\beta}\, T_\alpha(z)\, G_0(z)\, T_\beta(z),\ \bar{\delta}_{\alpha\beta} = (1 - \delta_{\alpha\beta}). \tag{II.16}$$

The matrix elements of $W^{(0)}_{\alpha\beta}(z)$ are easily seen to be free of any delta-function singularities, though there may be infinities in $W^{(0)}_{\alpha\beta}(p_\alpha, k_\alpha; p'_\beta, k'_\beta; z)$ for some values of the arguments. The $W_{\alpha\beta}$ equations are the basic equations that Faddeev analyzes in detail. The mathematical nature of the solutions, $W_{\alpha\beta}$, is such that they are functions of z with just pole and branch cut singularities. Specifically, they are not distributions.

We now turn to the description of the physical amplitudes in terms of the $W_{\alpha\beta}$'s. The $W_{\alpha\beta}$ has two distinct types of singularities. When the three-body kinetic energy is such that the α-channel energy $z-p_\alpha^2/2n$ is at a two-body bound state energy then T_α has a pole in the variable $p_\alpha^2/2n$. This type of singularity arising from either T_α or T_β is called a primary singularity [3]. This type of singularity is present in the driving term, any finite iteration of the driving term, and the exact solution.

The existence of these singularities is associated with the different physically realizable asymptotic states of the system. The other type of singularity, called a secondary singularity [3], arises from the possibility in $W^{o}_{\alpha\beta}$ that the denominator of the $G_0(z)$ portion of the matrix element may vanish. This type of singularity does not persist for third or higher order iterates of Eqs. (II.15) and (II.16).

We now decompose $W_{\alpha\beta}$ by explicitly factoring out the primary singularities. To do this we need to expand t_α around its bound-state poles. This pole decomposition is in the case of only one bound state

$$t_\alpha(k, k'; z) = \frac{\phi_\alpha(k)\,\overline{\phi_\alpha(k')}}{z + \chi^2_\alpha} + \hat{t}_\alpha(k, k'; z), \qquad \text{(II. 17)}$$

where ϕ_α is a "vertex function" that is related to the two-body bound-state wave function, ψ_α, corresponding to the energy $-\chi^2_\alpha$, by

$$\phi_\alpha(k) = \left(k^2/2\mu_\alpha + \chi^2_\alpha\right) \psi_\alpha(k). \qquad \text{(II. 18)}$$

The only singularity the nonpole term $\hat{t}_\alpha$ will have will be the discontinuity across the scattering cut for positive energy values. The addition of more than one bound state in each channel complicates our formulae but the procedure remains unchanged.

Now $W_{\alpha\beta}$ may be written in the form

$$W_{\alpha\beta}(kp; k'p'; z) = \mathscr{F}_{\alpha\beta}(kp; k'p'; z) + \mathscr{G}_{\alpha\beta}(kp; p'_\beta; z)\, \frac{\overline{\phi_\beta(k'_\beta)}}{z + \chi^2_\beta - \frac{p'^2_\beta}{2n}}$$

$$+ \frac{\phi_\alpha(k_\alpha)}{z + \chi^2_\alpha - \frac{p^2_\alpha}{2n}}\, \tilde{\mathscr{G}}_{\alpha\beta}(p_\alpha; k'p'; z) + \frac{\phi_\alpha(k_\alpha)\,\mathscr{H}_{\alpha\beta}(p_\alpha; p'_\beta; z)\,\overline{\phi_\beta(k'_\beta)}}{\left(z + \chi^2_\alpha - \frac{p^2_\alpha}{2n}\right)\left(z + \chi^2_\beta - \frac{p'^2_\beta}{2n}\right)}. \qquad \text{(II. 19)}$$

The residue functions, $\mathscr{F}_{\alpha\beta}$, $\mathscr{G}_{\alpha\beta}$, $\tilde{\mathscr{G}}_{\alpha\beta}$, $\mathscr{H}_{\alpha\beta}$, in the expression above will not have any primary singularities, but may have secondary singularities.[2] It is easy to see that Eq. (II. 19) is mathematically correct. Clearly, $W^{o}_{\alpha\beta}(z)$ will, by its definition Eq. (II. 16), have an expansion of the form of Eq. (II. 19). Also any finite iterate, whose structure is sum of terms like

$$T_\alpha G_0 T_{\gamma_1} G_0 T_{\gamma_2} G_0 \cdots G_0 T_{\gamma_n} G_0 T_\beta, \quad (\alpha \neq \gamma_1 \neq \gamma_2 \neq \ldots \neq \gamma_n \neq \beta),$$

will have the representation, Eq. (II. 19). Thus it is plausible that the exact solution $W_{\alpha\beta}$ will have the form Eq. (II. 19). Faddeev in fact gives a proof of this [3, Ch. 6].

We now have all the ingredients necessary to write down Faddeev's identification of the various scattering amplitudes in terms of the quantities defined in (II. 19). Let the incoming asymptotic state of our system be in channel β - i. e., particle β is incident with momentum p'_β onto the bound state pair $(\alpha\gamma)$, bound with energy $-\chi^2_\beta$. Then the S-matrix for scattering into channel α (where α may be equal to β) with a final state described by momentum p_α and bound state energy $-\chi^2_\alpha$ is [3, p. 81]

$$S_{\alpha\beta}(p_\alpha; p'_\beta) = \delta_{\alpha\beta}\delta^3(p_\alpha - p'_\alpha) - -2\pi i\delta\left(-\chi^2_\alpha + \frac{p^2_\alpha}{2n_\alpha} + \chi^2_\beta - \frac{p'^2_\beta}{2n_\beta}\right)\mathscr{H}_{\alpha\beta}\left(p_\alpha; p'_\beta; -\chi^2_\beta + \frac{p'^2_\beta}{2n_\beta} + i0\right). \qquad \text{(II. 20)}$$

The amplitude for breakup is related to the following linear combination of $\mathscr{H}$ and $\mathscr{G}$

$$\mathscr{K}_{\alpha\beta}(kp; p'_\beta; z) = \mathscr{G}_{\alpha\beta}(kp; p'_\beta; z) + \frac{\phi_\alpha(k_\alpha)}{z + \chi^2_\alpha - \frac{p^2_\alpha}{2n_\alpha}}\,\mathscr{H}_{\alpha\beta}(p_\alpha; p'_\beta; z); \qquad \text{(II. 21)}$$

specifically, the $S_{0\beta}$ matrix for the breakup of channel β is

$$S_{0\beta}(kp; p'_\beta) = -2\pi i\delta\left(\frac{k^2}{2\mu} + \frac{p^2}{2n} - \frac{p'^2}{2n} + \chi^2_\beta\right)\sum_{\alpha=1}^{3}\mathscr{K}_{\alpha\beta}\left(k, p; p'_\beta; \frac{p'^2_\beta}{2n_\beta} - \chi^2_\beta + i0\right). \qquad \text{(II. 22)}$$

The final amplitude we shall write down is the scattering of three free particles to three free particles[3].

$$S_{00}(kp;k'p') = \delta^3(k-k')\delta^3(p-p')$$
$$- 2\pi i\delta\left(\frac{k^2}{2\mu} + \frac{p^2}{2n} - \frac{k'^2}{2\mu} - \frac{p'^2}{2n}\right) \sum_{\alpha,\beta} M_{\alpha\beta}\left(kp;k'p';\frac{k'^2}{2\mu} + \frac{p'^2}{2n} + i0\right). \tag{II.23}$$

Faddeev's method for establishing the validity of Eqs. (II.20), (II.22) and (II.23) is long and arduous.[4] One particularly disappointing aspect of Faddeev's approach is that there is no a priori physical reason why the functions introduced in Eq. (II.19) are (with appropriate linear combinations like Eq. (II.21)) the physical scattering amplitudes. In the next section we give a simple argument which allows one to understand the decomposition, Eq. (II.19), somewhat more physically as well as to prove the validity of Eqs. (II.20) - (II.23).

III. LOVELACE-ALT APPROACH

A second approach to setting up and defining three-body scattering amplitudes with a related set of integral equations is given in the work of Lovelace [1, 2] and later modifications of it by Alt, Grassberger, and Sandhas [4]. This method proceeds by determining, from the outset, a three-body operator whose matrix elements between the eigenstates of the asymptotic channel Hamiltonians, H_α, H_β, or H_0 is known to be the physical scattering amplitude. This section gives an outline of this method. The results are then used to obtain simple operator interconnection between the Lovelace and Faddeev viewpoints. Finally we use this interconnection to provide a simple independent justification of Faddeev's S-matrix equations.

We begin our discussion by reviewing a wave function description [5] of scattering general enough to account for the multichannel character of our three-body scattering problem. Let $\psi^{(\pm)}$ be an exact scattering eigenstate of the total

Hamiltonian, H,

$$H\psi^{(\pm)} = E\psi^{(\pm)} \tag{III.1}$$

which corresponds asymptotically to a state Φ. The + subscript on ψ indicates the scattering wave function that corresponds to Φ in the remote past, while the - superscript describes the wave function which corresponds to Φ in the infinite future. Total conservation of energy in the scattering process is satisfied by demanding that Φ be an eigenstate of the asymptotic Hamiltonian, H_0, with energy E,

$$H_0\Phi = E\Phi \ . \tag{III.2}$$

If one defines the scattered part of the wave function by

$$\psi^{(\pm)} = \Phi + \chi^{(\pm)}, \tag{III.3}$$

it follows from Eq. (III.1 - III.3) that

$$(H - E)\chi^{(\pm)} = -(H - E)\Phi. \tag{III.4}$$

Adding an $\pm\, i\epsilon$, $\epsilon > 0$, to H - E so that we may invert the operator H - E, we have

$$\chi^{(\pm)} = -G(E \pm i\epsilon)(H - E)\Phi \tag{III.5}$$

or, equivalently,

$$\psi^{(\pm)} = \Phi - G(E \pm i\epsilon)(H - E)\Phi \ . \tag{III.6}$$

In a multichannel scattering process the initial asymptotic Hamiltonian, H_{0i}, may be different from the final asymptotic Hamiltonian, H_{0f}. So the interaction, V_i, for the incoming wave is generally not the same as the interaction, V_f, for the outgoing wave. These channel potentials are defined by

$$H = H_{0i} + V_i = H_{0f} + V_f. \tag{III.7}$$

The channel wave functions, which are eigenfunctions of H, are given by

$$\psi^{(\pm)}_{i(f)} = \Phi_{i(f)} - G(E \pm i\epsilon)(H - E)\Phi_{i(f)}. \tag{III.8}$$

Potentials are introduced into this equation by using

$$(H - E)\Phi_{i(f)} = (H_{0i(f)} + V_{i(f)} - E)\Phi_{i(f)} = V_{i(f)}\Phi_{i(f)}. \qquad (III.9)$$

The S-matrix, S_{fi}, is defined as the inner product of $\psi_f^{(-)}$ with $\psi_i^{(+)}$. One now obtains the scattering amplitude by expanding the S-matrix about the diagonal element: Employing the identity

$$\psi_{f(i)}^{(\mp)} = \psi_{f(i)}^{(\pm)} + 2\pi i\delta(H - E_{f(i)})\, V_{f(i)}\Phi_{f(i)} \qquad (III.10)$$

which is a direct consequence of Eq. (III.8), one obtains

$$S_{fi} = \left(\psi_f^{(-)}, \psi_i^{(+)}\right) = \left(\psi_f^{(+)} + 2\pi i\,\delta(H - E_f)\, V_f\Phi_f, \psi_i^{(+)}\right),$$

$$= \delta_{fi} - 2\pi i\delta(E_i - E_f)\left(\Phi_f, V_f\psi_i^{(+)}\right) . \qquad (III.11)$$

The fi channel scattering amplitude is then $\left(\Phi_f, V_f\psi_i^{(+)}\right)$ and the transition operator, U_{fi}^{+}, corresponding to this amplitude is defined by

$$\left(\Phi_f, U_{fi}^{+}\Phi_i\right) = \left(\Phi_f, V_f\psi_i^{(+)}\right)$$

$$= \left(\Phi_f, \left[V_f - V_fG(E + i\epsilon)V_i\right]\Phi_i\right), \qquad (III.12)$$

or, equivalently,

$$U_{fi}^{+}(z) = V_f - V_fG(z)\, V_i. \qquad (III.13)$$

Instead of expanding ψ_f^- in Eq. (III.11), we could expand ψ_i^+ with Eq. (III.10). This leads to a second expression for the fi channel amplitude, namely

$$S_{fi} = \delta_{fi} - 2\pi i\delta(E_i - E_f)\, (\psi_f^-, V_i\Phi_i). \qquad (III.14)$$

Consequently, a second operator for the fi channel amplitude is

$$U_{fi}^{-}(z) = V_i - V_fG(z)\, V_i. \qquad (III.15)$$

The ambiguity in the choice of transition operators, $U_{fi}^{\pm}$ does not have any physical consequences since the difference of the two operators may be written as $H_{0i} - H_{0f}$ which vanishes when evaluated between the channel states ϕ_i and ϕ_f.

The $U^{\pm}_{fi}(z)$ are operators that Lovelace [1] uses to investigate the three-body problem. Specifically, in the three-body problem, with only pair-wise potentials present, we have $H_i \longrightarrow H_\alpha + V_\alpha$, so

$$V_i = (H - H_0 + V_\alpha) = V - V_\alpha = \overline{V}_\alpha, \quad \alpha = 0,1,2,3, \tag{III.16}$$

where $V_0 \equiv 0$. The Lovelace operators can then be written as

$$U^{+}_{\alpha\beta}(z) = \overline{V}_\alpha - \overline{V}_\alpha \, G(z) \, \overline{V}_\beta, \tag{III.17}$$

$$U^{-}_{\alpha\beta}(z) = \overline{V}_\beta - \overline{V}_\alpha \, G(z) \, \overline{V}_\beta. \tag{III.18}$$

The problem of determining $U^{\pm}_{\alpha\beta}(z)$ is made tractable by finding an integral equation whose solution is $U^{\pm}_{\alpha\beta}(z)$. By substituting the identities[5]

$$\overline{V}_\alpha \, G(z) \, V_\delta = U^{+}_{\alpha\delta}(z) \, G_0(z) \, T_\delta(z) \tag{III.19}$$

and

$$V_\delta \, G(z) \, \overline{V}_\beta = T_\delta(z) \, G_0(z) \, U^{-}_{\delta\beta}(z) \tag{III.20}$$

into Eqs. (III.17) and (III.18) one obtains integral equations for $U^{\pm}_{\alpha\beta}(z)$, viz.,

$$U^{+}_{\alpha\beta}(z) = \overline{V}_\alpha - \underset{\gamma\neq\beta}{} U^{+}_{\alpha\gamma}(z) \, G_0(z) \, T_\gamma(z), \tag{III.21}$$

$$U^{-}_{\alpha\beta}(z) = \overline{V}_\beta - \underset{\gamma\neq\alpha}{} T_\gamma(z) \, G_0(z) \, U^{-}_{\gamma\beta}(z). \tag{III.22}$$

These equations share with Faddeev's Eq. (II.11) a well-defined nonsingular mathematical behavior — that is acting a suitably restricted Banach space [3, Ch. 5] the kernels of these equations generate compact operators. Physically the Banach space is not very restrictive since it only requires that the momentum space functions fall off for large momentum (fast enough to be square integrable) and that they satisfy a Hölder smoothness property (so that the i prescriptions are well defined).

The disadvantages of Lovelace's operator equations Eqs. (III.21 - III.22) relative to Faddeev's are that Lovelace's equation involve two distinct off-shell extensions of the transition matrix and that they are not equations which give three-body amplitudes entirely in terms of the off-shell two-body t-matrix operators. Both of these disadvantages are removed by modifications introduced by Alt et al.,[4].

A more symmetric approach for defining a channel dependent transition operator is to define $U_{\alpha\beta}$ as

$$G(z) = G_\alpha(z) \ \delta_{\alpha\beta} - G_\alpha(z) \ U_{\alpha\beta}(z) \ G_\beta(z) \tag{III.23}$$

If one substitutes the identities

$$G(z) = G_\alpha(z) - G_\alpha(z) \ \overline{V}_\alpha \ G(z) \tag{III.24}$$

and

$$G(z) = G_\alpha(z) - G(z) \ \overline{V}_\alpha \ G_\alpha(z) \tag{III.25}$$

into themselves, it follows that,

$$U_{\alpha\beta}(z) = -\overline{\delta}_{\alpha\beta}(H_\beta - z) + U^+_{\alpha\beta}(z) = -\overline{\delta}_{\alpha\beta}(H_\alpha - z) + U^-_{\alpha\beta}(z). \tag{III.26}$$

For $z = E \pm i0$, the difference between $U^\pm$ and U vanishes on shell when acting on appropriate channel eignestates. Thus $U_{\alpha\beta}$ retains the interpretation of the physical transition operator in the α to β channel scattering process. Integral equations for $U_{\alpha\beta}$ follow from Eqs. (III.21) and (III.22) and the relation, Eq. (III.26), between $U_{\alpha\beta}$ and $U^\pm_{\alpha\beta}$:

$$U_{\alpha\beta}(z) = -\overline{\delta}_{\alpha\beta}(H_0 - z) - \sum_{\gamma\neq\beta} U_{\alpha\gamma}(z) \ G_0(z) \ T_\gamma(z) \tag{III.27}$$

and

$$U_{\alpha\beta}(z) = -\overline{\delta}_{\alpha\beta}(H_0 - z) - \sum_{\gamma\neq\alpha} T_\gamma(z) \ G_0(z) \ U_{\gamma\beta}(z). \tag{III.28}$$

The Lovelace-Alt Eqs. (III.27 - III.28) involve only one off-shell extension and do not include any direct reference to potentials.

We shall now study the interconnection between the Lovelace-Alt formulation and Faddeev's.[6]

Consider the operator $W'_{\alpha\beta}(z)$ defined by

$$W'_{\alpha\beta}(z) = T_\alpha(z)\, G_0(z)\, U_{\alpha\beta}(z)\, G_0(z)\, T_\beta(z)\ . \tag{III.29}$$

We will show that $W_{\alpha\beta} = W'_{\alpha\beta}$. Let us premultiply Eq. (III.28) by $T_\alpha(z)\, G_0(z)$ and postmultiply by $G_0(z)\, T_\beta(z)$ then Eq. (III.28) becomes

$$W'_{\alpha\beta}(z) = -\bar{\delta}_{\alpha\beta} T_\alpha(z)\, G_0(z)\, T_\beta(z) - \sum_{\gamma\neq\alpha} T_\alpha(z)\, G_0(z)\, W'_{\gamma\beta}(z), \tag{III.30}$$

so $W'_{\alpha\beta}$ satisfies the same integral equation as $W_{\alpha\beta}$. Faddeev has proved that this equation (even in the limit $\pm\, i\epsilon \longrightarrow 0$) has unique solutions for any z not equal to a three-body bound state energy. Consequently, $W_{\alpha\beta}$ and $W'_{\alpha\beta}$ are equal. In passing we point out an important difference between the Alt et al., U's and the $U^\pm$ of Lovelace. If we had used Lovelace's $U^\pm$ in an equation of the kind (III.29), we would not be led to any simple connection to Faddeev's W's.

We may now use Eq. (III.29) to give a direct physical interpretation of Faddeev's primary singularity decomposition (II.19) of $W_{\alpha\beta}$. Equation (III.29) tells that residues of the primary singularities of $W_{\alpha\beta}$ are just the matrix elements of $U_{\alpha\beta}$. We already know from the Lovelace approach that the on-shell channel matrix elements of $U_{\alpha\beta}$ are the physical amplitudes. In detail, the formulae for elastic scattering and rearrangement, obtained from Faddeev's prescription Eq. (II.19) and Eq. (III.29) are

$$\mathscr{H}_{\alpha\beta}(p_\alpha, p'_\beta; z) = \int \frac{\overline{\phi_\alpha(k_\alpha)}\, U_{\alpha\beta}(p_\alpha, k_\alpha; p_\beta k'_\beta; z)\, \phi_\beta(k'_\beta)\, dk\, dk'}{\left(\frac{p_\alpha^2}{2n} + \frac{k_\alpha^2}{2\mu} - z\right)\left(\frac{p'^2_\beta}{2n} + \frac{k_\beta^2}{2\mu} - z\right)}\ . \tag{III.31}$$

Taken on shell, $z = \frac{p'^2_\beta}{2n} - \chi_\beta^2 \pm i0 = \frac{p_\alpha^2}{2n} - \chi_\beta^2 \pm i0$ this equation becomes just

$$\mathscr{H}_{\alpha\beta}\left(p_\alpha, p'_\beta; \frac{p'^2_\beta}{2n} - \chi_\beta^2 \pm i0\right) = \int \psi_\alpha(k_\alpha) U_{\alpha\beta}\left(p_\alpha k_\alpha; p'_\beta, k'_\beta; \frac{p'^2_\beta}{2n} - \chi_\beta^2 \pm i0\right) \psi_\beta(k'_\beta) dk_\alpha dk'_\beta\ . \tag{III.32}$$

The right-hand side of this relation is just $(\boldsymbol{\Phi}_\alpha, U_{\alpha\beta}(E \pm i0)\boldsymbol{\Phi}_\beta)$ since the asymptotic channel wave function is

$$\boldsymbol{\Phi}_\alpha(p''_\alpha, k_\alpha; p_\alpha) = \delta^3(p''_\alpha - p_\alpha)\ \psi_\alpha(k_\alpha) \qquad \text{(III.33)}$$

Here we have proved explicitly that the physical interpretation of $\mathscr{K}_{\alpha\beta}$ as the rearrangement or elastic amplitude is correct.

Next let us examine the breakup process. Faddeev's prescription tells us to construct $\mathscr{K}_{\alpha\beta}$. Doing so, we have

$$\mathscr{K}_{\alpha\beta}(kp; p'_\beta; z) = \int \frac{t_\alpha\left(k_\alpha, k''_\alpha; z - \frac{p_\alpha^2}{2n}\right) U_{\alpha\beta}(k''_\alpha p_\alpha; k'_\beta, p'_\beta;\ z)\, \phi_\beta(k'_\beta)\, dk''_\alpha\, dk'_\beta}{\left(\frac{p_\alpha^2}{2n} + \frac{k_\alpha^2}{2\mu} - z\right)\left(\frac{p'^2_\beta}{2n} + \frac{k_\beta^2}{2\mu} - z\right)}, \qquad \text{(III.34)}$$

where the sum over $\alpha = 1,2,3$ gives the entire breakup amplitude. On-shell, with $z = \frac{p'^2_\beta}{2n} - \chi^2_\beta \pm i0$, it is clear that the right-hand side of Eq. (III.34) when summed over α becomes equal to $(\boldsymbol{\Phi}_0, U_{0\beta}(z)\boldsymbol{\Phi}_\beta)$ where according to Eq. (III.28) the Lovelace-Alt breakup operator is

$$U_{0\beta}(z) = -(H_0 - z) - \sum_{\alpha=1}^{3} T_\alpha(z)\, G_0(z)\, U_{\alpha\beta}(z), \qquad \beta \neq 0 \qquad \text{(III.35)}$$

and

$$\boldsymbol{\Phi}_0(p'', k''; p, k) = \delta^3(p'' - p)\, \delta^3(k'' - k),$$

$$\frac{p^2}{2n} + \frac{k^2}{2\mu} = \frac{p'^2_\beta}{2n} - \chi^2_\beta = \text{Re } z \qquad \text{(III.36)}$$

The four equations (III.31 - III.34) provide an explicit justification of Faddeev's residue prescription for defining the physical amplitude.

IV. INTEGRAL EQUATIONS FOR THE PHYSICAL AMPLITUDES

Sections II and III have provided us with detailed interconnections between the Faddeev and Lovelace-Alt formalisms with particular attention to the definitions of the physical amplitudes. Here, by starting from the Lovelace-Alt equations we derive a set of integral equations for simple and natural extensions of the physical amplitudes $\mathscr{K}$ and $\mathscr{H}$ half-off-shell.

We shall confine our attention to the scattering problem in which the initial state is in the β-channel, i.e., the pair $(\alpha\gamma)$ is bound and particle β is free. The Lovelace-Alt equation appropriate to this problem is Eq. (III.28). A useful intermediate integral equation is obtained by multiplying Eq. (III.28) by $\phi_\beta(k'_\beta)\left(\frac{p'^2}{2n}+\frac{k'^2}{2\mu}-z\right)^{-1}$ and integrating over dk'_β. Defining $A_{\alpha\beta}(kp;p'_\beta z)$ by

$$A_{\alpha\beta}(kp;p'_\beta;z)=\int\frac{U_{\alpha\beta}(kp;k'p')\,\phi_\beta(k'_\beta)\,dk'_\beta}{\frac{p'^2}{2n}+\frac{k'^2}{2\mu}-z} \tag{IV.1}$$

we have,

$$A_{\alpha\beta}(kp'\,p'_\beta;z)=$$

$$-\bar{\delta}_{\alpha\beta}\delta^3(p_\beta-p'_\beta)\,\phi_\beta(k_\beta)-\sum_{\delta\neq\alpha}\int\frac{t_\delta\left(k_\delta,k''_\delta;z-\frac{p_\delta^2}{2n}\right)\delta^3(p_\delta-p''_\delta)\,A_{\delta\beta}(k''p'';k'_\beta;z)\,dk''dp''}{\left(\frac{p''^2}{2n}+\frac{k''^2}{2\mu}-z\right)}. \tag{IV.2}$$

The equation for A will become an equation for the physical amplitudes if we multiply from the left by operators like $T_\alpha(z)\,G_0(z)$. Let us introduce a notation sufficiently general to handle a variety of cases. We define a generalized bound state pole expansion of t_α by

$$t_\alpha\left(k_\alpha,k'_\alpha;z-\frac{p_\alpha^2}{2n}\right)=f_\alpha(p_\alpha,z)\,\phi_\alpha(k_\alpha)\,\overline{\phi_\alpha(k_\alpha)}+\bar{t}_\alpha\left(k_\alpha,k'_\alpha;\;z-\frac{p_\alpha^2}{2n}\right), \tag{IV.3}$$

where arbitrariness of the expansion depends on the choice of $f_\alpha(p_\gamma,z)$. The definition of $\bar{t}_\alpha$ is specified once f_α is chosen. For $f_\alpha(p_\alpha,z)=\left(z-\frac{p_\alpha^2}{2n}+\chi_\alpha^2\right)^{-1}$ Eq. (IV.3) just reverts to Eq. (II.17).

From the integral equation for $A_{\alpha\beta}$, we can obtain a relation involving the amplitudes $\mathscr{H}_{\alpha\beta}$ if we multiply by $\overline{\phi_\alpha(k_\alpha)}\left(\frac{p_\alpha^2}{2n}+\frac{k_\alpha^2}{2\mu}-z\right)^{-1}$ and use Eq. (III.31).

We obtain

$$\mathscr{H}_{\alpha\beta}(p_\alpha, p'_\beta; z) = \int \frac{\overline{\phi_\alpha(k_\alpha)}\, A_{\alpha\beta}(k_\alpha p_\alpha; p'_\beta; z)\, dk_\alpha}{\left(\frac{p_\alpha^2}{2n}+\frac{k_\alpha^2}{2n}-z\right)}\,, \tag{IV.4}$$

and Eq. (IV.2) becomes

$$\mathscr{H}_{\alpha\beta}(p_\alpha, p'_\beta; z) = \mathscr{H}^0_{\alpha\beta}(p_\alpha, p'_\beta; z) - \sum_{\delta\neq\alpha}\int \frac{\overline{\phi_\alpha(k''_\alpha)}\,\delta(p_\alpha - p''_\alpha)}{\left(\frac{p''^2}{2n}+\frac{k''^2}{2\mu}-z\right)}$$

$$\times\left\{\overline{A}_{\delta\beta}(k''_\alpha p''_\alpha; p'_\beta; z) + f_\delta(p''_\delta, z)\ \phi_\delta(k''_\delta)\mathscr{H}_{\delta\beta}(p''_\delta, p_\beta; z)\right\} dp''dk'', \tag{IV.5}$$

where

$$\mathscr{H}^0_{\alpha\beta}(p_\alpha, p'_\beta; z) = \frac{-\bar{\delta}_{\alpha\beta}\,\overline{\phi_\alpha(p_\alpha, p'_\beta)}\ \ \phi_\beta(p_\alpha, p'_\beta)}{\frac{p_\alpha^2}{2\mu_\beta}+\frac{p_\alpha\cdot p'_\beta}{m\gamma}+\frac{p_\beta^2}{2\mu_\alpha}-z} \tag{IV.6}$$

and

$$\overline{A}_{\alpha\beta}(kp; p'_\beta; z) = \int \frac{\bar{t}_\alpha\left(k_\alpha, k''_\alpha; z-\frac{p_\alpha^2}{2n}\right) A_{\alpha\beta}(k''_\alpha p; p'_\beta;\ z)\,dk''_\alpha}{\left(\frac{p_\alpha^2}{2n}+\frac{k''^2_\alpha}{2\mu}-z\right)}\,. \tag{IV.7}$$

In our notation for ϕ_α in Eq. (IV.6) we give p_α, p'_β as the argument. This is meant to indicate the k_α which is fixed by knowing p_α, p'_β when $\alpha\neq\beta$. The $\mathscr{H}^{(0)}_{\alpha\beta}$ is a driving term. We obtain a closed set of equations once we add to Eq. (IV.5) a linear equation giving $\overline{A}_{\alpha\beta}$ in terms of $\overline{A}_{\alpha\beta}$ and $\mathscr{H}_{\alpha\beta}$. The necessary equation is obtained by multiplying Eq. (IV.2) by $\bar{t}_\alpha\left(k_\alpha, k''_\alpha; z-\frac{p_\alpha^2}{2n}\right)\left(\frac{p_\alpha^2}{2\mu}+\frac{k''^2_\alpha}{2\mu}-z\right)^{-1}$ and integrating with respect

$$\overline{A}_{\alpha\beta}(kp' p'_\beta; z) = \overline{A}^0_{\alpha\beta}(kp; p'_\beta; z)$$

$$- \sum_{\delta \neq \alpha} \frac{\overline{t}_\alpha \left(k_\alpha, k''_\alpha; z - \frac{p_\alpha^2}{2n} \right) \delta^3 (p_\alpha - p''_\alpha)}{\left(\frac{p''^2_\alpha}{2n} + \frac{k''^2}{2\mu} - z \right)} \left\{ \overline{A}_{\delta\beta}(k''_\alpha p''_\beta; p'_\beta; z) \right.$$

$$\left. + f_\delta(p'', z)\, \phi_\delta(k'')\, \mathscr{H}_{\delta\beta}(p''_\alpha, p'_\beta; z) \right\} dp''\, dk'', \qquad \text{(IV.8)}$$

where

$$\overline{A}^0_{\alpha\beta}(kp; p'_\beta; z) = \frac{-\overline{\delta}_{\alpha\beta} \overline{t}_\alpha \left(k_\alpha, k_\alpha(p_\alpha, p'_\alpha); z - \frac{p_\alpha^2}{2n} \right) \phi_\beta(p_\alpha, p'_\beta)}{\frac{p_\alpha^2}{2\mu_\beta} + \frac{p_\alpha \cdot p'_\beta}{m\gamma} + \frac{p_\beta^2}{2\mu_\alpha}}. \qquad \text{(IV.9)}$$

Now Eq. (IV.5) and (IV.8) are sets of solvable coupled integral equations. Actually we have a family of such equations since we are free to choose f_α.

We consider three different choices for f_α:

$$f_\alpha(p_\alpha; z) = 0, \qquad \text{(IV.10a)}$$

$$f_\alpha(p_\alpha; z) = \left(\dot{z} - \frac{p_\alpha^2}{2n} + \chi_\alpha^2 \right)^{-1}, \qquad \text{(IV.10b)}$$

$$f_\alpha(p_\alpha; z) = -i\pi\delta \left(z - \frac{p_\alpha^2}{2n} + \chi_\alpha^2 \right). \qquad \text{(IV.10c)}$$

Let us discuss choice (a). In this case $\overline{t}_\alpha = t_\alpha$, and $\overline{A}_{\alpha\beta}$ becomes according to Eq. (III.34) just $\mathscr{K}_{\alpha\beta}$ — the off-shell breakup amplitude. With $\overline{A}_{\alpha\beta} = \mathscr{K}_{\alpha\beta}$ Eq. (IV.8) is a self-contained three component integral equation which predicts the breakup amplitude. The Eq. (IV.5) is just an auxiliary equation predicting the elastic or rearrangement amplitude, $\mathscr{H}_{\alpha\beta}(p_\alpha, p_\beta; z)$ from $\mathscr{K}_{\alpha\beta}$.

Now let us examine choice b.[7] Our two sets of equations, Eq. (IV.4) and Eq. (IV.8) together represent a six component set of integral equations for $\mathscr{H}_{\alpha\beta}$ and $\overline{A}_{\alpha\beta} = \mathscr{G}_{\alpha\beta}$. This six component set is somewhat asymmetric in that the $\mathscr{H}$

components have one active variable, p, while $\mathscr{G}$ has two, p and q. One obtains breakup amplitudes $\mathscr{K}$ from $\mathscr{G}$ and $\mathscr{H}$ by Eq. (II.21). Our final choice c, is one which only on-shell values of the elastic-rearrangement amplitude, $\mathscr{H}_{\alpha\beta}$, appear in the coupled equation.

The merits of using (numerically or analytically) the above coupled integral equations to study breakup or elastic-rearrangement scattering seem clear. Our integral equations, for all three cases, give as their solution the physical amplitudes. No subsequent integrations over asymptotic channel wave functions are required. The alternative Lovelace approach, having the same physical content, would be to solve Eq. (III.28) for $U_{\alpha\beta}$ then employ Eq. (III.31) and (III.34) to obtain $\mathscr{H}$ and $\mathscr{G}$. This second approach involves more work as well as being less direct. On first inspection it may seem that the six component equations are twice as difficult to solve as the three component equations. This is not really the case. The intractable nature of Eqs. (IV.8) and (IV.5) resides in that fact that they are integral equations in two vector variables (pk). The finite component structure is not a serious difficulty. For example in case (b), the three components to Eq. (IV.5) are functions of only one variable. If we imagine that Eq. (IV.8) where turned into a matrix equation by discretizing the variables k and p by N points each, then the increased complexity represented by Eq. (IV.5) is no more than if we had used N + 1 points for k in Eq. (IV.8) instead of N. Thus, our six component equation is negligibly more complicated than the three component one in case (a).

We complete this section by discussing the singularity structure of our integral equations. As remarked in Section II there is the possibility that the driving terms, $\mathscr{H}^0_{\alpha\beta}$ and $\overline{A}^0_{\alpha\beta}$, have secondary singularities. These singularities arise from the vanishing of the Green function denominator

$$\left(\frac{p_\alpha^2}{2\mu_\beta} + \frac{p_\alpha \cdot p'_\beta}{m\gamma} + \frac{p'^2_\beta}{2\mu_\alpha}\right) - z\,.$$

The scattering energy z for the incoming channel β is $\frac{p'^2_\beta}{2n_\beta} - \chi^2_\beta$. So the Green function denominator becomes

$$\left(\frac{k_\beta(p_\alpha \cdot p_\beta)^2}{2\mu_\beta} + \frac{p'^2_\beta}{2n_\beta}\right) - \left(\frac{p'^2_\beta}{2n_\beta} - \chi^2_\beta\right) \geq \chi^2_\beta > 0.$$

Thus the Green function denominators are bounded away from zero for all values of p_α. Consequently, for the scattering problem with an incoming asymptotic scattering Hamiltonian different from H_0 the secondary singularities never arise. The driving terms $\mathscr{H}_{\alpha\beta}$ and $\overline{A}_{\alpha\beta}$ are Hölder continuous as they stand and no iteration is necessary in order to obtain smooth driving terms. The only case where the secondary singularities are present is in the scattering of three free particles to three free particles. For this problem Faddeev showed that the singularities vanish after the third iteration. It is just these singularities that Amado and Rubin [6] recently studied at threshold.

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FOOTNOTES

1. The notation used here for matrix elements is Faddeev's [3]. The relation to the usual bra and ket notation is $\langle p_\alpha k_\alpha | T | p'_\alpha k'_\alpha \rangle = T(p_\alpha k_\alpha ; p'_\alpha k'_\alpha)$.

2. To be more explicit in the definition of the expansion, Eq. (II.19), the residue functions $\mathscr{F}_{\alpha\beta}$, $\mathscr{G}_{\alpha\beta}$, $\tilde{\mathscr{G}}_{\alpha\beta}$, $\mathscr{H}_{\alpha\beta}$ are assumed to be Hölder continuous in the kp variables. Aside from the known secondary singularities in the first three iterates of Eqs. (II.14 and II.15) Faddeev [3] gives estimates to prove this.

3. The one remaining physical amplitude of interest is the amplitude for $(3 \to 2)$, which can be written in a form similar to Eq. (II.22) and involves $\mathscr{H}_{\alpha\beta}$ and $\tilde{\mathscr{G}}_{\alpha\beta}$.

4. See for example, Chapters 5, 6, 7, 9, and 11 in Faddeev's book [3]. Actually no explicit proof is given in [3] for Eqs. (II.20) and (II.22); however the proofs are not much more difficult than the one given in [3, Ch. 9] for S_{00}.

5. These identities are proved by using the Green function identity $G(z) = G_\delta(z) - G(z)\,\overline{V}_\delta\, G_\delta(z)$ where $G_\delta(z) = (H_0 + V_\delta - z)^{-1}$, together with the two-body identity $V_\delta\, G_\delta(z) = T_\delta(z)\, G_0(z)$.

6. Lovelace proved [1, Appendix] that his and Faddeev's approaches both give the same on-shell physical amplitude. The connection we give here between the two formalisms is both simple and more general in that the relationship is an operator one and consequently has a full off-shell content.

7. The integral equations arising for choice (b) correspond to [3, Eq. (5.19)].