

THE RENORMALIZATION GROUP AND STRONG INTERACTIONS*

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ABSTRACT

The renormalization group method of Gell-Mann and Low is applied to field theories of strong interactions. It is assumed that renormalization group equations exist for strong interactions which involve one or several momentum-dependent coupling constants. The further assumption that these coupling constants approach fixed values as the momentum goes to infinity is discussed in detail. However, an alternative is suggested, namely that these coupling constants approach a limit cycle in the limit of large momenta. Some results of this paper are: 1) The $e^+ - e^-$ annihilation experiments above 1 GeV energy may distinguish a fixed point from a limit cycle or other asymptotic behavior. 2) If electrodynamics or weak interactions become strong above some large momentum Λ then the renormalization group can be used (in principle) to determine the renormalized coupling constants of strong interactions, except for $U(3) \times U(3)$ symmetry breaking parameters. 3) Mass terms in the Lagrangian of strong, weak and electromagnetic interactions must break a symmetry of the combined interactions with zero mass. 4) The $\Delta I = 1/2$ rule in nonleptonic weak interactions can be understood assuming only that a renormalization group exists for strong interactions.

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I. INTRODUCTION

At large momenta radiative corrections in quantum electrodynamics grow logarithmically. At an energy of about 10^{40} eV the radiative corrections are of order 1 instead of order α (α is the fine structure constant) and at infinite energy the radiative corrections are infinite. As a result the Born approximation to quantum electrodynamics is unreliable at energies of 10^{40} eV or higher. This raises a challenge: can one find an approximation to electrodynamics which is valid for these energies?

The academic nature of this challenge is evident. Quantum electrodynamics neglects the interactions of photons with hadrons and the weak interactions of electrons, not to mention interactions not yet discovered. Any of these interactions could appreciably alter the electron-photon interaction at high energies. In any case, 10^{40} eV is an energy hopelessly beyond the range of any conceivable accelerator. Nonetheless, some notable authors have tried to meet this challenge.^{1,2} This paper is concerned with the work of Gell-Mann and Low,¹ who studied in particular the behavior of the photon propagator in the limit of large k^2 , k being the photon momentum. To study the photon propagator, Gell-Mann and Low used a method which has since become known as the renormalization group approach. The renormalization group was invented by Stueckelberg and Petermann,³ its role in the Gell-Mann-Low analysis is discussed in the book of Bogoliubov and Shirkov.⁴

Gell-Mann and Low suggest that their analysis may apply to theories of strong interactions as well as electrodynamics, and that in strong interactions their results might apply at energies more accessible than 10^{40} eV. In practice the storage ring experiments to measure the total $e^+ - e^-$ annihilation cross section into hadrons above 1 GeV momentum transfer (and perhaps the SLAC deep inelastic scattering experiments) explore a range of momenta relevant to the Gell-Mann-Low theory.

Clearly the time has come to explore in detail the consequences of the Gell-Mann-Low theory for strong interactions.

The basic formula in the Gell-Mann-Low theory for electrodynamics is a differential equation for a quantity e_λ . Let the renormalized photon propagator be written $k^{-2} d_c(k^2/m^2, e^2)$ where k is the photon four-momentum, e is the renormalized electron charge, and m is the renormalized electron mass. Then e_λ is defined by the equation

$$e_\lambda^2 = e^2 d_c(-\lambda^2/m^2, e^2) \quad (\text{I. 1})$$

Gell-Mann and Low set up a generalization of the usual renormalization procedure in which e_λ is defined to be the renormalized coupling constant, for some arbitrary chosen value of λ , in place of e . They also argue that e_λ , considered as a function of λ , interpolates between the physical charge e and the bare charge, namely e_λ for $\lambda \rightarrow 0$ is e and e_λ for $\lambda \rightarrow \infty$ is the bare charge. The bare charge will be denoted e_0 in this paper. The Gell-Mann-Low formula is of the form

$$\frac{de_\lambda^2}{d \ln \lambda^2} = \psi\left(m^2/\lambda^2, e_\lambda^2\right) \quad (\text{I. 2})$$

Gell-Mann and Low suggest that $\psi\left(m^2/\lambda^2, e_\lambda^2\right)$ has a nonzero limit as $m \rightarrow 0$, i.e., $\psi\left(0, e_\lambda^2\right)$ exists and is not identically zero. If this is true, then for $\lambda \gg m$, e_λ satisfies approximately

$$de_\lambda^2/d(\ln \lambda^2) = \psi\left(0, e_\lambda^2\right) \quad (\text{I. 3})$$

Gell-Mann and Low assume that this equation holds for any value of e_λ^2 , although it can be justified, at best, only for small e_λ^2 ; they then discuss qualitative features of the large momentum behavior of e_λ based on qualitative features of ψ . In particular, they show that the limit e_∞ of e_λ must either be infinite or if finite must be a root of the equation $\psi\left(0, e_\infty^2\right) = 0$. In either case e_∞ is independent of the value of the physical charge e .

The purpose of this paper is to propose that the Gell-Mann-Low theory when applied to theories of strong interactions can produce some startling consequences. For example, it will be shown that if strong interactions are described by one of the popular renormalizable models (e. g., the gluon model⁵ or the σ -model⁶) then one can use the renormalization group equations (analogous to Eq. (I.2)) of the model to determine one or more of the renormalized coupling constants of the model. In order to derive this result a physical assumption is made, namely that strong interactions will cease to be isolated from weak or electromagnetic interactions at some cutoff momentum Λ much larger than 1 GeV, and that any model theory of strong interactions is valid only below the cutoff. There will be small cutoff-dependent errors in the prediction of the renormalized coupling constants.

This is a consequence of the Gell-Mann-Low theory of theoretical interest. An experimental consequence which is a possible but far from certain prediction of the Gell-Mann-Low theory is that the cross section $\sigma_{TOT}(q^2)$ for $e^+ - e^-$ annihilation into hadrons at large momentum transfers q will oscillate as a function of q^2 . To be precise the oscillations would have the form

$$\sigma_{TOT}(q^2) = (q^2)^{-1} f(\ln q^2) \quad (I.4)$$

where $f(t)$ is a periodic function of t with period t_0 . The period t_0 and the amplitude of the oscillations cannot be predicted. This behavior is only one of the alternatives made possible by the Gell-Mann-Low theory, and it is possible only for theories with at least two renormalized coupling constants. For the gluon model which has only one renormalized coupling constant the prediction for $\sigma_{TOT}(q^2)$ at large q^2 is that $q^2 \sigma_{TOT}(q^2)$ be a constant. If $q^2 \sigma_{TOT}(q^2)$ is constant at large q^2 it is likely (according to the Gell-Mann-Low theory) that strong interactions are scale invariant at short distances. The hypothesis of broken scale invariance has been extensively discussed elsewhere.⁷

The renormalization group differential equations such as Eq. (I.3) are best understood by setting up an analogy with equations of motion in classical mechanics or electric circuit theory. Let $t = \ln \lambda^2$ and $x = e_{\lambda}^2$; then Eq. (I.3) becomes

$$\frac{dx}{dt} = \psi(0, x) \quad (\text{I.5})$$

This is a simple equation of motion with ψ being the analogue of a time — independent force (except the equation involves dx/dt not d^2x/dt^2). If the field theory has two renormalized coupling constants, say x and y , the corresponding renormalization group equations have the form (neglecting masses, as in Eq. (I.3))

$$\frac{dx}{dt} = \psi_1(x, y) \quad (\text{I.6})$$

$$\frac{dy}{dt} = \psi_2(x, y) \quad (\text{I.7})$$

i. e., they are two coupled equations with time-independent forces.

The crucial feature of the renormalization group equations is that they must be solved over a large or infinite range of t , namely from λ of order m to $\lambda = 1$ or ∞ , leading to the range $\ln m^2 < t < (\ln \Lambda^2 \text{ or } \infty)$ for t . Furthermore the equations are nonlinear (in perturbation theory $\psi(0, x)$, or $\psi_1(x, y)$ and $\psi_2(x, y)$, have power series expansions in x and y and are not linear in x and y). The essential question in solving the renormalization group equations is to determine the behavior of the solution in the limit of large t . This analogous to the central problem of nonlinear mechanics.⁸ Nonlinear mechanics is concerned with finding equilibrium points or other asymptotic solutions of equations like Eqs. (I.5) or (I.6) and (I.7) and studying the stability of these solutions. The predictions for strong interactions cited earlier arise from applying the theory of asymptotic solutions and their stability to the renormalization group equations.

The most serious drawback of the Gell-Mann-Low theory is that it is highly speculative; it requires extrapolation of functions like $\psi(0, x)$ from perturbation

theory (small x) to strongly interactions ($x \sim 1$) and there is no way to check the validity, even qualitatively, of this extrapolation. Furthermore, as one studies the consequences of the theory for strong interactions it becomes clear that there is a glaring omission in the Gell-Mann-Low equations. The omission is the omission of coupling constants associated with nonrenormalizable interactions. In perturbation theory there are well known reasons for distinguishing renormalizable interactions from nonrenormalizable interactions, but in strong coupling this distinction becomes blurred, as will be shown.

It is beyond the scope of this paper to discuss how to incorporate nonrenormalizable interactions into the renormalization group, but one can see that however this is done the resulting differential equations will be very complicated. This is because there are an infinite number of nonrenormalizable interactions. Hence it is important to understand the practical importance of the renormalization group for strong interactions, and to understand the reasons why one must add nonrenormalizable interactions to the renormalization group. For these reasons it is worth discussing the renormalization group without nonrenormalizable interactions even if nonrenormalizable interactions are necessary for a correct treatment. The emphasis in this paper will be on qualitative features of solutions of the renormalization group equations resulting from the existence of equilibrium or other asymptotic solutions; this analysis is generalizable to any number of coupling constants.

Anyone who has studied the Gell-Mann-Low theory, either in the original paper of Gell-Mann and Low¹ or in the review of Bogoliubov and Shirkov,⁴ has found it extraordinarily difficult to understand. Accordingly, Section II and the Appendix of this paper give a thorough review of the Gell-Mann-Low theory for electrodynamics. However there will be no attempt here to show that the limit of $\psi(m^2/\lambda^2, e_\lambda^2)$ for $m \rightarrow 0$ exists beyond fourth order in e_λ ; there will only be a brief review of work done on this question.

In Section III solutions of the renormalization group equations will be discussed in detail, including applications both to strong and electromagnetic interactions. It will be shown that if the asymptotic solution is a fixed point then the corresponding field theory exhibits broken scale invariance. In Section IV the possible mechanisms for breaking scale invariance will be discussed, and the renormalization group will be extended to include a λ -dependent mass m_λ . In Section V the renormalization group for strong interactions will be discussed based on the work of Sections III and IV. It will be argued that mass terms in strong, electromagnetic, and weak interactions must break a symmetry common to all three interactions. In Section VI it will be argued that the $\Delta I = 1/2$ rule in weak interactions can be understood if strong interactions have a renormalization group, independently of the type of asymptotic solution of the renormalization group equations. Section VII contains final remarks.

II. DERIVATION OF THE RENORMALIZATION GROUP EQUATIONS

However one defines a renormalization program for quantum electrodynamics to all orders in perturbation theory, there is one stage in the program where one makes infinite or cutoff-dependent subtractions in the n^{th} order vacuum polarization, electron self energy, and vertex function. In addition to the cutoff-dependent subtractions, one makes finite subtractions which can be chosen arbitrarily. This arbitrariness is customarily removed by specifying ad hoc normalizations for the renormalized fields and by specifying that the renormalized mass and charge parameters are the physical mass and charge of the electron. Gell-Mann and Low define alternative conditions for removing the arbitrariness: Gell-Mann and Low specify unconventional normalization conditions for the fields and define a charge parameter e_λ which is not the physical electron charge. However they use the conventional definition for the renormalized mass. The Gell-Mann-Low conditions involve a "renormalization parameter" λ which one can choose arbitrarily. The renormalized fields of Gell-Mann and Low for any given value of λ are related to the conventional fields through a finite renormalization. The reason for considering the unconventional renormalization conditions of Gell-Mann and Low is that they apparently define fields which are finite off the mass shell when the electron mass is zero.

The renormalization group equation results from comparing renormalized theories for two different values of the renormalization momentum λ ; if these theories both exist for zero electron mass then the Eq. (I.2) derived from these theories also exists for zero electron mass.

In this section a modified form of the Gell-Mann-Low renormalization conditions will be defined; then the Gell-Mann-Low renormalized theory will be expressed in terms of the conventionally renormalized theory, and finally the renormalization

group differential equation, Eq. (I.2) will be derived. In the Appendix the motivation for setting up the Gell-Mann-Low renormalization conditions will be reviewed in detail with illustrations from low orders of perturbation theory.

The renormalization conditions of Gell-Mann and Low can be applied to any method of renormalization. This means one does not have to use or understand Ward's renormalization program⁹ used in Gell-Mann and Low's paper. One can have the Bogoliubov-Parasiuk-Hepp¹⁰ or any other method in mind in reading this section.

The Gell-Mann-Low conditions consist of one restriction on vacuum polarization, two on the electron self-energy and one on the vertex function. There is considerable arbitrariness in how these conditions are formulated but it seems to be inevitable that they look awkward due to spin complications. The conditions proposed here will not be the ones given by Gell-Mann and Low; Bogoliubov and Shirkov evade the problem of stating precise conditions for the electron self-energy. For comparison purposes the restrictions used in conventional renormalization theory will be stated also.

It is convenient to use the following notation. Let $\Pi_{c\mu\nu}(k)$, $\Sigma_c(p)$, and $\Gamma_{c\mu}(p, q, k)$ be the vacuum polarization, electron self-energy, and vertex function for conventionally renormalized electrodynamics. In the vertex function p and q are electron momenta and $k = q - p$ is the photon momentum. Let $\Pi_{\lambda\mu\nu}(k)$, $\Sigma_\lambda(p)$, and $\Gamma_{\lambda\mu}(p, q, k)$ be the corresponding renormalized functions satisfying the Gell-Mann-Low restrictions. It is convenient to define invariant functions, as follows.

$$\Pi_{c\mu\nu}(k) = (g_{\mu\nu} k^2 - k_\mu k_\nu) \Pi_c(k^2) \quad (\text{II. 1})$$

$$\Pi_{\lambda\mu\nu}(k) = (g_{\mu\nu} k^2 - k_\mu k_\nu) \Pi_\lambda(k^2) \quad (\text{II. 2})$$

$$\Sigma_c(p) = \not{p} A_c(p^2) + B_c(p^2) \quad (\text{II. 3})$$

$$\Sigma_\lambda(p) = \not{p} A_\lambda(p^2) + B_\lambda(p^2) \quad (\text{II. 4})$$

$$\Gamma_{c\mu}(p, p, 0) = \gamma_\mu \Gamma_{c1}(p^2) + p_\mu \Gamma_{c2}(p^2) + \gamma_\mu \not{p} \Gamma_{c3}(p^2) + p_\mu \not{p} \Gamma_{c4}(p^2) \quad (\text{II. 5})$$

$$\Gamma_{\lambda\mu}(p, p, 0) = \gamma_\mu \Gamma_{\lambda1}(p^2) + p_\mu \Gamma_{\lambda2}(p^2) + \gamma_\mu \not{p} \Gamma_{\lambda3}(p^2) + p_\mu \not{p} \Gamma_{\lambda4}(p^2) \quad (\text{II. 6})$$

(where \not{p} is $\gamma^\mu p_\mu$). One looks at the vertex function for $k = 0$ because it is connected to the electron self-energy by Ward's identity:

$$\Gamma_{c\mu}(p, p, 0) = \gamma_\mu - \partial \Sigma_c(p) / \partial p^\mu = \gamma_\mu - \gamma_\mu A_c(p^2) - 2 \not{p} p_\mu \partial A_c(p^2) / \partial p^2 - 2 p_\mu \partial B_c(p^2) / \partial p^2 \quad (\text{II. 7})$$

$$\Gamma_{\lambda\mu}(p, p, 0) = \gamma_\mu - \gamma_\mu A_\lambda(p^2) - 2 \not{p} p_\mu \partial A_\lambda(p^2) / \partial p^2 - p_\mu \partial B_\lambda(p^2) / \partial p^2 \quad (\text{II. 8})$$

These identities give

$$\Gamma_{c1}(p^2) = 1 - A_c(p^2) \quad (\text{II. 9})$$

$$\Gamma_{c2}(p^2) = -2 \partial B_c(p^2) / \partial p^2 \quad (\text{II. 10})$$

$$\Gamma_{c3}(p^2) = 0 \quad (\text{II. 11})$$

$$\Gamma_{c4}(p^2) = -2 A_c(p^2) / \partial p^2 \quad (\text{II. 12})$$

and analogously

$$\Gamma_{\lambda1}(p^2) = 1 - A_\lambda(p^2), \text{ etc.} \quad (\text{II. 13})$$

There are four subtraction constants to be fixed, namely 1 constant independent of momentum in each of the following functions: $\Pi_c(k^2)$, $A_c(p^2)$, $B_c(p^2)$, and $\Gamma_{c1}(p^2)$. These subtraction constants are conventionally determined by the following conditions:

$$\Pi_c(0) = 0 \quad (\text{II. 14})$$

$$\Sigma_c(p) \Big|_{\not{p}=m} = m A_c(m^2) + B_c(m^2) = 0 \quad (\text{II. 15})$$

$$\left[\partial \Sigma_c(p) / \partial \not{p} \right] \Big|_{\not{p}=m} = A_c(m^2) + \left[2m^2 \partial A_c(p^2) / \partial p^2 + 2m \partial B_c(p^2) / \partial p^2 \right] \Big|_{p^2=m^2} = 0 \quad (\text{II. 16})$$

and

$$\Gamma_{c1}(m^2) + m\Gamma_{c2}(m^2) + m^2\Gamma_{c4}(m^2) = 1 \quad (\text{II. 17})$$

The conditions on the electron propagator are chosen so that neither the mass nor the residue of the pole of the electron propagator at $\not{p} = m$ are changed by interaction. The subtraction constant in $A_c(p^2)$ is determined by Eq. (II. 16) (note that the derivatives $\partial A_c(p^2)/\partial p^2$ and $\partial B_c(p^2)/\partial p^2$ do not involve the subtraction constant); the subtraction constant in $B_c(m^2)$ is then fixed by Eq. (II. 15). The peculiar equation for $\Gamma_{c1}(m^2)$ is dictated by the requirement that $\Gamma_{c1}(m^2)$ satisfy the Ward identity. Due to Eqs. (II. 9) to (II. 12), the condition for $\Gamma_{c1}(m^2)$ is a consequence of Eq. (II. 16). (As part of any renormalization program one must prove that all Ward identities are satisfied if the subtraction constant in the vertex function is chosen so that the vertex function satisfies one Ward identity at one value of the momentum.)

The alternative conditions of Gell-Mann and Low (somewhat modified by the author) are as follows:

$$\Pi_\lambda(-\lambda^2) = 0 \quad (\text{II. 18})$$

$$\Sigma_\lambda(p)|_{\not{p}=m} = mA_\lambda(m^2) + B_\lambda(m^2) = 0 \quad (\text{II. 19})$$

$$A_\lambda(-\lambda^2) = 0 \quad (\text{II. 20})$$

$$\Gamma_{\lambda 1}(-\lambda^2) = 1 \quad (\text{II. 21})$$

The first restriction on $\Sigma_\lambda(p)$ ensures that the mass m in the free electron propagator is the physical electron mass. This restriction determines the subtraction constant in $B_\lambda(p^2)$. The third equation (II. 20) (which was one of many possible choices) fixes the subtraction constant in $A_\lambda(p^2)$. The last equation fixes the subtraction constant in $\Gamma_{\lambda 1}(p^2)$ and is consistent with the Ward identity (Eq. (II. 13)) for $\Gamma_{\lambda 1}(p^2)$.

These conditions are chosen so as not to introduce singularities in the renormalized functions when the renormalized electron mass is zero. This is explained in detail in the Appendix for low orders of perturbation theory. The Gell-Mann-Low renormalized amplitudes are also free of infrared divergences when the photon mass is zero and the electron mass is nonzero. This is also explained in the Appendix.

Will the Gell-Mann-Low subtraction conditions combined with a renormalization program to all orders give renormalized amplitudes which are finite for $m = 0$ in all orders? This is a hard question and cannot be pursued here. Baker and Johnson¹¹ have an indirect proof that the photon propagator as renormalized by Gell-Mann and Low is finite for $m = 0$; the author has not checked their proof. Kinoshita¹² did an extensive study of mass singularities in field theory but did not study the effects of renormalizing according to the Gell-Mann-Low specifications. A full proof of the existence of the zero mass limit does not exist to the author's knowledge. If the zero mass limit does not exist in higher orders the conclusions of this paper may be incorrect.

Given that electrodynamics renormalized according to the Gell-Mann-Low specifications differs only by a renormalization from conventionally renormalized electrodynamics,¹³ it is possible to express all the amplitudes of the Gell-Mann-Low theory in terms of the conventional theory. The formulae connecting the two theories will now be obtained. Let the conventionally renormalized electromagnetic potential be $A_{c\mu}(x)$ and the conventionally renormalized electron field be $\psi_c(x)$. Let the corresponding fields of the Gell-Mann-Low theory be $A_{\lambda\mu}(x)$ and $\psi_\lambda(x)$. Then one requires that

$$A_{\lambda\mu}(x) = (z_{3\lambda})^{1/2} A_{c\mu}(x) \quad (\text{II. 22})$$

$$\psi_\lambda(x) = (z_{2\lambda})^{1/2} \psi_c(x) \quad (\text{II. 23})$$

The Gell-Mann-Low theory is parameterized differently from the conventional theory; the coupling constant in the Gell-Mann-Low theory is denoted e_λ . The mass parameterization is in terms of the physical electron mass m in both theories. The restrictions (II. 18) - (II. 21) imposed on the Gell-Mann-Low theory are sufficient to determine $z_{2\lambda}$, $z_{3\lambda}$, and e_λ in terms of conventionally renormalized amplitudes; when these are known all amplitudes of the Gell-Mann-Low theory are determined through Eqs. (II. 22) and (II. 23).

It is convenient to rewrite two of the Gell-Mann-Low restrictions in terms of the complete propagators. Let $S_\lambda(p)$ be the electron propagator. One has (in the Feynman gauge for the exact propagator¹⁴)

$$D_{\lambda\mu\nu}^{-1}(k) = -g_{\mu\nu} k^2 - g_{\mu\nu} k^2 \Pi_\lambda(k^2) \quad (\text{II. 24})$$

$$S_\lambda^{-1}(p) = \not{p} - m - \not{p} A_\lambda(p^2) - B_\lambda(p^2) \quad (\text{II. 25})$$

It is convenient to introduce invariant functions related to $D_{\lambda\mu\nu}(k)$ and $S_\lambda(p)$, and indicate explicitly their dependence on m , λ , and e_λ as well as momentum:

$$d(k^2/\lambda^2, m^2/\lambda^2, e_\lambda^2) = \{1 + \Pi_\lambda(k^2)\}^{-1} \quad (\text{II. 26})$$

$$s(p^2/\lambda^2, m^2/\lambda^2, e_\lambda^2) = \{1 - A_\lambda(p^2)\}^{-1} \quad (\text{II. 27})$$

Then

$$D_{\lambda\mu\nu}(k) = -g_{\mu\nu} (k^2)^{-1} d(k^2/\lambda^2, m^2/\lambda^2, e_\lambda^2) \quad (\text{II. 28})$$

There is a second invariant function for the electron propagator which will be defined later (Section IV). In the zero mass limit one has γ_5 invariance, which means $B_\lambda(p^2)$ is zero. In this limit one has

$$S_\lambda(p) = (\not{p})^{-1} s(p^2/\lambda^2, 0, e_\lambda^2) \quad (\text{II. 29})$$

The functions d and s are dimensionless¹⁵ which is why they are functions only of ratios of the variables p^2 , m^2 , and λ^2 . The Gell-Mann-Low restrictions (II. 18)

and (II.20) become

$$d(-1, m^2/\lambda^2, e_\lambda^2) = 1 \quad (\text{II.30})$$

$$s(-1, m^2/\lambda^2, e_\lambda^2) = 1 \quad (\text{II.31})$$

There are corresponding functions $d_c(k^2/m^2, e^2)$ and $s_c(p^2/m^2, e^2)$ for the conventionally renormalized theory; d_c satisfies

$$d_c(0, e^2) = 1 \quad (\text{II.32})$$

The conditions on $\Sigma_c(p)$ cannot be expressed in terms of s_c alone.

From the relations (II.22) and (II.23) one must have

$$D_{\lambda\mu\nu}(k) = z_{3\lambda} D_{c\mu\nu}(k) \quad (\text{II.33})$$

$$S_\lambda(p) = z_{2\lambda} S_c(p) \quad (\text{II.34})$$

which in turn means that

$$d(k^2/\lambda^2, m^2/\lambda^2, e_\lambda^2) = z_{3\lambda} d_c(k^2/m^2, e^2) \quad (\text{II.35})$$

$$s(p^2/\lambda^2, m^2/\lambda^2, e_\lambda^2) = z_{2\lambda} s_c(p^2/m^2, e^2) \quad (\text{II.36})$$

Putting k^2 and p^2 equal to $-\lambda^2$ and using the conditions (II.30) and (II.31) one gets

$$(z_{3\lambda})^{-1} = d_c(-\lambda^2/m^2, e^2) \quad (\text{II.37})$$

$$(z_{2\lambda})^{-1} = s_c(-\lambda^2/m^2, e^2) \quad (\text{II.38})$$

In order to determine e_λ in terms of e one must know how $\Gamma_{\lambda\mu}(p, q, k)$ and $\Gamma_{c\mu}(p, q, k)$ are related to vacuum expectation values of the fields. The formulae are as follows:

$$\Gamma_{\lambda\mu}(p, q, k) = (e_\lambda)^{-1} S_\lambda^{-1}(p) D_{\lambda\mu\nu}^{-1}(k) F_\lambda^\nu(p, q, k) S_\lambda^{-1}(q) \quad (\text{II.39})$$

$$F_{\lambda\nu}(p, q, k) = - \iint e^{ip \cdot x} e^{-iq \cdot y} \langle \Omega | T \psi_\lambda(x) \bar{\psi}_\lambda(y) A_{\lambda\nu}(0) | \Omega \rangle d^4x d^4y \quad (\text{II.40})$$

The factor e_λ^{-1} occurs in the definition of $\Gamma_{\lambda\mu}(p, q, k)$ along with the inverse propagators to ensure that $\Gamma_{\lambda\mu}(p, q, k)$ in lowest order is γ_μ (not $e_\lambda \gamma_\mu$); using the inverse of the exact propagators in Eq. (II.39) is necessary to remove self-energy insertions from external lines of vertex graphs.

Analogous formulae with e replacing e_λ and S_c^{-1} , $D_{c\mu\nu}^{-1}$ etc. replacing S_λ^{-1} , $D_{\lambda\mu\nu}^{-1}$, etc., give the vertex $\Gamma_{c\mu}(p, q, k)$. It follows from these formulae and Eqs. (II.22), (II.23), (II.33), and (II.34), that

$$e_\lambda \Gamma_{\lambda\mu}(p, q, k) = (z_{2\lambda})^{-1} (z_{3\lambda})^{-1/2} e \Gamma_{c\mu}(p, q, k). \quad (\text{II.41})$$

A particular consequence of this equation is that

$$e_\lambda \Gamma_{\lambda 1}(p^2) = (z_{2\lambda})^{-1} (z_{3\lambda})^{-1/2} e \Gamma_{c 1}(p^2) \quad (\text{II.42})$$

Using the Ward identities (II.9) and (II.13) and the definitions of s and s_c , this becomes

$$e_\lambda \left[s(p^2/\lambda^2, m^2/\lambda^2, e_\lambda^2) \right]^{-1} = (z_{2\lambda})^{-1} (z_{3\lambda})^{-1/2} e \left[s_c(p^2/m^2, e^2) \right]^{-1} \quad (\text{II.43})$$

Comparing with Eq. (II.36), one must have

$$e_\lambda = (z_{3\lambda})^{-1/2} e \quad (\text{II.44})$$

Using Eq. (II.37), this is

$$e_\lambda^2 = e^2 d_c(-\lambda^2/m^2, e^2) \quad (\text{II.45})$$

(This is Eq. (I.1) of the introduction.)

One can now completely reconstruct the Gell-Mann-Low form of renormalized quantum electrodynamics if the conventional renormalized form is known. To construct $d(k^2/\lambda^2, m^2/\lambda^2, e_\lambda^2)$, for example, one uses Eqs. (II.35) and (II.37) to give

$$d(k^2/\lambda^2, m^2/\lambda^2, e_\lambda^2) = d_c(k^2/m^2, e^2) / d_c(-\lambda^2/m^2, e^2) \quad (\text{II.46})$$

To reparameterize the ratio $d_c(k^2/m^2, e^2) / d_c(-\lambda^2/m^2, e^2)$ in terms of e_λ^2 one must first solve Eq. (II.45) to give e^2 as a function e_λ^2 . In perturbation theory this is a tedious but straightforward process and gives e^2 as a power series in e_λ^2 .

To derive the renormalization group equations one looks at the formulas connecting the Gell-Mann-Low renormalized fields for two different values of λ , say λ and λ' . The two sets of fields are connected by renormalization constants:

$$A_{\lambda\mu}(x) = z_{3\lambda\lambda'} A_{\lambda'\mu}(x) \quad (\text{II. 47})$$

$$\psi_{\lambda}(x) = z_{2\lambda\lambda'} \psi_{\lambda'}(x) \quad (\text{II. 48})$$

where

$$z_{3\lambda\lambda'} = z_{3\lambda} / z_{3\lambda'} \quad (\text{II. 49})$$

$$z_{2\lambda\lambda'} = z_{2\lambda} / z_{2\lambda'} \quad (\text{II. 50})$$

In consequence, one has

$$d(k^2/\lambda^2, m^2/\lambda^2, e_{\lambda}^2) = z_{3\lambda\lambda'} d(k^2/\lambda'^2, m^2/\lambda'^2, e_{\lambda'}^2) \quad (\text{II. 51})$$

$$s(p^2/\lambda^2, m^2/\lambda^2, e_{\lambda}^2) = z_{2\lambda\lambda'} s(p^2/\lambda'^2, m^2/\lambda'^2, e_{\lambda'}^2) \quad (\text{II. 52})$$

Also, from Eq. (II. 44) one has

$$e_{\lambda}^2 = (z_{3\lambda\lambda'})^{-1} e_{\lambda'}^2 \quad (\text{II. 53})$$

It follows from putting $p^2 = k^2 = -\lambda^2$ that

$$(z_{3\lambda\lambda'})^{-1} = d(-\lambda^2/\lambda'^2, m^2/\lambda'^2, e_{\lambda'}^2) \quad (\text{II. 54})$$

$$(z_{2\lambda\lambda'})^{-1} = s(-\lambda^2/\lambda'^2, m^2/\lambda'^2, e_{\lambda'}^2) \quad (\text{II. 55})$$

$$e_{\lambda}^2 = e_{\lambda'}^2 d(-\lambda^2/\lambda'^2, m^2/\lambda'^2, e_{\lambda'}^2) \quad (\text{II. 56})$$

The advantage of these equations is that they remain finite when $m \rightarrow 0$. It should also be noted that the function $s(p^2/\lambda^2, m^2/\lambda^2, e_{\lambda}^2)$ does not have infrared divergences for finite mass m while the function $s_c(p^2/m^2, e^2)$ does. This means $z_{2\lambda}$ is infrared divergent but $z_{2\lambda\lambda'}$ is not.

In the limit $m \rightarrow 0$, Eq. (II.56) becomes (with λ and λ' interchanged, for no good reason)

$$e_{\lambda'}^2 = e_{\lambda}^2 d(-\lambda'^2/\lambda^2, 0, e_{\lambda}^2) \quad (\text{II.57})$$

This is an equation which is true for any λ' and λ provided both are nonzero. As a result it defines a rather simple transformation group on a single variable x .

Think of $x = e_{\lambda}^2$ as a point on the positive real line; let T_s be the transformation which takes x into the point $x' = e_{\lambda'}^2$, i. e.,

$$x' = x d(-s, 0, x) \quad (\text{II.58})$$

with $s = \lambda'^2/\lambda^2$. So T_s is a transformation on a space of coupling constants. The transformations T_s have the group property: if

$$s_2 = s_1 s \quad (\text{II.59})$$

then

$$T_{s_2} = T_{s_1} T_s \quad (\text{II.60})$$

To prove this one first defines

$$(\lambda'')^2 = s_2(\lambda^2), \text{ i. e., } s_2 = \lambda''^2/\lambda^2 \quad (\text{II.61})$$

Then

$$s_1 = s_2/s = (\lambda''/\lambda')^2 \quad (\text{II.62})$$

Consider any point $x > 0$. Choose $e_{\lambda}^2 = x$. Since Eq. (II.57) holds for any λ' , one has in addition

$$e_{\lambda''}^2 = e_{\lambda}^2 d(-\lambda''^2/\lambda^2, 0, e_{\lambda}^2) \quad (\text{II.63})$$

Since λ is arbitrary one also has

$$e_{\lambda''}^2 = e_{\lambda'}^2 d(\lambda''^2/\lambda'^2, 0, e_{\lambda'}^2) \quad (\text{II.64})$$

Let x' be $e_{\lambda'}^2$ and x'' be $e_{\lambda''}^2$. Then T_s takes x to x' and T_{s_1} takes x' to x'' ; but T_{s_2} (by Eq. (II.63)) takes x directly to x'' . This proves the group property. In other

words, if one is given the coupling constant e_λ^2 for subtraction momentum λ , the corresponding coupling constant $e_{\lambda'}^2$ for subtraction momentum λ' is given by Eq. (II.57); the coupling constant $e_{\lambda''}^2$ for subtraction momentum λ'' is given by Eq. (II.63). By "corresponding" is meant that e_λ^2 , $e_{\lambda'}^2$, and $e_{\lambda''}^2$ all lead to the same physics but in different parameterizations. Now since $e_{\lambda'}^2$ and $e_{\lambda''}^2$ lead to the same physics they too must be related, namely by Eq. (II.64); this is what gives the group property. The group has an identity transformation, namely the transformation T_1 ; this is the identity due to the normalization condition (II.30). The transformation inverse to T_s is $T_{s^{-1}}$.

The group of transformations T_s will be called here the renormalization group. This is a less ambitious definition than is given by Bogoliubov and Shirkov⁴ or Petermann and Stueckelberg;³ however, it is easier to work with the transformations T_s than to consider the more complex transformations which make up the Petermann-Stueckelberg renormalization group.

The transformations T_s are nonlinear, because $x d(-s, 0, x)$ is nonlinear in x . So the renormalization group is different from the symmetry groups of quantum mechanics which are groups of linear transformations. Instead the renormalization group is analogous to the group of time translations in classical mechanics. In classical mechanics with time-independent potentials the equations of motion define infinitesimal transformations on phase space; the corresponding finite transformations are also nonlinear and also form a one parameter group. The analogy to classical mechanics will become important in the next section.

In practice one works mostly with the infinitesimal transformation of the renormalization group. The infinitesimal transformation determines the derivative $de_\lambda^2/d\lambda^2$. It is convenient to derive an equation for $de_\lambda^2/d\lambda^2$ which is valid for finite mass. This is obtained by differentiating Eq. (II.56) with respect to λ and then

setting $\lambda' = \lambda$. The result is of the form

$$d(e_\lambda^2)/d(\lambda^2) = \lambda^{-2} \psi(m^2/\lambda^2, e_\lambda^2) \quad (\text{II. 65})$$

where

$$\psi(m^2/\lambda^2, e_\lambda^2) = e_\lambda^2 \left[\partial d(-s, m^2/\lambda^2, e_\lambda^2) / \partial s \right]_{s=1} \quad (\text{II. 66})$$

The function ψ has a limit for $m \rightarrow 0$ because d does, and the function $\psi(0, x)$ gives the infinitesimal transformation of the renormalization group. In practice one can replace $\psi(m^2/\lambda^2, e_\lambda^2)$ by $\psi(0, e_\lambda^2)$ when $\lambda^2 \gg m^2$; hence solutions of the equation

$$de_\lambda^2/d\lambda^2 = \lambda^{-2} \psi(0, e_\lambda^2) \quad (\text{II. 67})$$

give the asymptotic behavior of e_λ for $\lambda \gg m$. The function $\psi(0, x)$ is explicitly known to order x^3 when x is small:⁴

$$\psi(0, x) = (12 \pi^2)^{-1} x^2 + (3/16 \pi^2) x^3 + \dots \quad (\text{II. 68})$$

One can also derive differential equations for $z_{2\lambda}$ and $z_{3\lambda}$. However $z_{3\lambda}$ is related to e_λ^2 by Eq. (II. 44) so we only give the equation for $z_{2\lambda}$. First differentiate Eq. (II. 55) with respect to λ^2 and then put $\lambda'^2 = \lambda^2$ (and note that $z_{2\lambda\lambda} = 1$):

$$\left. \partial z_{2\lambda\lambda'} / \partial \lambda^2 \right|_{\lambda'=\lambda} = \lambda^{-2} \sigma(m^2/\lambda^2, e_\lambda^2) \quad (\text{II. 69})$$

where

$$\sigma(m^2/\lambda^2, e_\lambda^2) = \left[\partial s(y, m^2/\lambda^2, e_\lambda^2) / \partial y \right]_{y=-1} \quad (\text{II. 70})$$

Since $z_{2\lambda} = (z_{2\lambda\lambda'}) (z_{2\lambda'})$ one has

$$dz_{2\lambda}/d(\lambda^2) = \left(\partial z_{2\lambda\lambda'} / \partial \lambda^2 \right) z_{2\lambda'} \quad (\text{II. 71})$$

Putting $\lambda'^2 = \lambda^2$ gives

$$dz_{2\lambda}/d\lambda^2 = (\lambda^2)^{-1} z_{2\lambda} \sigma(m^2/\lambda^2, e_\lambda^2) \quad (\text{II. 72})$$

The function σ has a zero mass limit because the function s does, so when $\lambda \gg m$ one has approximately

$$dz_{2\lambda}/d\lambda^2 = (\lambda^2)^{-1} z_{2\lambda} \sigma(0, e_\lambda^2) \quad (\text{II.73})$$

Bogoliubov and Shirkov⁴ also write down differential equations for the whole vertex function; they will not be needed here.

III. SOLUTIONS OF THE RENORMALIZATION GROUP EQUATIONS

A. Introduction

In the previous section the basic equation of the Gell-Mann-Low theory for electrodynamics was derived:

$$d(e_\lambda^2) / d(\ln \lambda^2) = \psi(0, e_\lambda^2) \quad (\text{III. 1})$$

valid for $\lambda \gg m$.

Imagine now that equations similar to this hold for other field theories besides electrodynamics, in particular for possible field theories of strong interactions (e.g., the gluon model or the σ model).¹⁶ Imagine also that the renormalization group equation is valid for large values of the coupling constant (e.g., large values of e_λ^2 in the case of electrodynamics).

To do a general analysis of all possible renormalization group equations would be a hopeless task. One has no information on the behavior of $\psi(0, e_\lambda^2)$ when e_λ^2 is of order one or larger, and in some theories of strong interactions the renormalization group equations involve coupled differential equations for several λ -dependent coupling constants (e.g., Eqs. (I. 6) and (I. 7) of the introduction). However, as pointed out in the introduction, the renormalization group equations are analogous to equations of motion in nonlinear mechanics; in this analogy $\ln \lambda^2$ is analogous to the time t and e_λ^2 is analogous to a coordinate x .

In classical nonlinear mechanics there are standard types of asymptotic behavior for large t which occur in many different kinds of systems. The simplest asymptotic behavior is a fixed (equilibrium) point. For Eq. (III. 1), a fixed point is a point x for which $\psi(0, x) = 0$. For Eqs. (I. 6) and (I. 7) a fixed point is a pair of values x, y for which $\psi_1(x, y)$ and $\psi_2(x, y)$ both vanish. It is quite common especially in electric circuit problems to have a system which starts off with a

transient (time dependent) behavior but which settles into a time-independent state (fixed point) as $t \rightarrow \infty$; this can occur no matter how many dependent variables (x , or x, y , or etc.) are needed to describe the system. A limit cycle is another type of asymptotic behavior which also occurs in many systems. A limit cycle is a periodic solution of a nonlinear set of equations of motion; because of the non-linearity the amplitude of the solution is fixed as well as the period.¹⁷ Other types of asymptotic behavior are also possible but are not so easily characterized.

The purpose of this section is to discuss the possibility that the asymptotic solution of the renormalization group equations is either a fixed point or limit cycle. These are not the only possibilities but other possibilities are more difficult to analyze. Studying the consequences of a fixed point or a limit cycle makes clear the importance of the renormalization group for field theory.

The asymptotic behavior of the solution of the renormalization group equations is important for several reasons. For example, there are two experimental quantities which should reflect directly the qualitative behavior of the coupling constants for large λ . One is the total cross section for $e^+ - e^-$ annihilation into hadrons discussed in the introduction; the other is the Callan-Gross integral over deep inelastic electron scattering cross sections.¹⁸ The latter involves a more complicated analysis and will not be discussed further here. The consequence of a fixed point or limit cycle behavior for $e^+ - e^-$ annihilation will be discussed later in this section. Contrasting fixed point behavior with limit cycle behavior suggests experimental tests which distinguish fixed points from limit cycles and probably from other types of behavior as well. (For example, if the cross section $\sigma_{TOT}(q^2)$ for annihilation of $e^+ + e^-$ to hadrons behaves as $1/q^2$ for large q^2 then the asymptotic solution of the renormalization group for strong interactions is probably a fixed point. See Section III.H), hence one may learn from experiment whether one must study asymptotic behaviors other than the fixed point.

In order to discuss the possibility of a fixed point one must study functions $\psi(0, x)$ which have fixed points. It is convenient to discuss a function $\psi(0, x)$ which has at least three positive roots x ; this allows one to consider various types of behavior connected with fixed points. Only the case of one coupling constant, as in Eq. (III. 1), will be discussed explicitly; the discussion is easily generalized to the case of two or more coupling constants.

To discuss a limit cycle one must have at least two coupling constants so the limit cycle will be discussed using Eqs. (I. 6) and (I. 7).

In applying the renormalization group to strong interactions one must distinguish two situations. The first alternative is that strong interactions remain distinguishable from weak or electromagnetic or other interactions for arbitrarily large momenta, and that there is a theory of strong interactions valid for all momenta which neglects these other interactions. Given this alternative the discussion of the renormalization group for strong interactions, assuming fixed point asymptotic behavior, is similar to Gell-Mann and Low's discussion of electrodynamics.¹ In particular one can predict the values of the bare coupling constants of strong interactions but the physical coupling constants are undetermined theoretically and can only be found experimentally. The second alternative is that at some cutoff Λ large compared to 1 GeV the electromagnetic or weak or other corrections to strong interactions become too large to be treated as a perturbation. In this case a theory of strong interactions in isolation is only valid for momenta small compared to the cutoff and one must allow for large corrections to the λ -dependent coupling constants when λ is of order Λ . The consequence of this (as will be shown below) is that the physical coupling constants are predictable theoretically if one knows the precise form of the renormalization group equations (and if the solution of these equations is a fixed point for $\lambda \gg 1$ GeV but $\lambda \ll \Lambda$). Both

alternatives will be explained in detail when the fixed point hypothesis is applied to strong interactions.

The renormalization group equations can be derived in detail only after a field theory has been solved since the function $\psi(0, x)$ is defined in terms of a propagator of the interacting theory (in theories other than electrodynamics, ψ depends on vertex functions as well), and the propagator is not known unless the theory has been solved. Any property of the field theory which one deduces by solving the renormalization group equations ought in principle to be discernable directly from the solution of the theory, making a discussion of the renormalization group equation unnecessary. However, one does not have solutions to field theories except in perturbation theory; the analysis of the renormalization group equations described here is one way of groping towards the nature of strongly interacting field theories.

B. Integration of the Differential Equation

Now consider solutions of the renormalization group equations with fixed point asymptotic behavior. Equation (III. 1) for electrodynamics will be discussed explicitly but the analysis applies to other field theories as well. Suppose that the function $\psi(0, x)$ has at least three positive roots x_1 , x_2 , and x_3 with $0 < x_1 < x_2 < x_3$. Let these roots all be simple roots. From the perturbation theory formula (II. 68) $\psi(0, x)$ has a double root at $x = 0$ and is positive when x is positive and small. Suppose that $\psi(0, x)$ is bounded, continuous, and differentiable for all x . One consequence of these assumptions is that $\psi(0, x)$ is positive for $0 < x < x_1$, negative for $x_1 < x < x_2$ and positive again for $x_2 < x < x_3$. The range $x > x_3$ will not be discussed here. A function $\psi(0, x)$ with these properties is shown in Fig. 1. These assumptions are made so that the solutions of the renormalization group equation will illustrate several forms of fixed point asymptotic behavior; there is no way of

knowing whether these assumptions are true for quantum electrodynamics or any other given field theory.

For $\psi(0, x)$ to be negative contradicts the Källén-Lehmann representation¹⁹ for the photon propagator. The Källén-Lehmann representation gives

$$d_c(k^2/m^2, e^2) = 1 + k^2 \int_0^\infty \rho(k'^2/m^2, e^2) \frac{1}{k^2 - k'^2 - i\epsilon} dk'^2 \quad (\text{III. 2})$$

where ρ is a positive spectral function. This equation holds provided no subtractions are needed to make the integral converge. It follows from this formula and Eq. (II. 45) that

$$de_\lambda^2/d\lambda^2 = e^2 \int_0^\infty \rho(k'^2/m^2, e^2) \frac{k'^2}{(\lambda^2 + k'^2)^2} dk'^2 \quad (\text{III. 3})$$

which is positive but might go to zero when $\lambda \rightarrow \infty$.

The function $\psi(0, x)$ will be permitted to be negative anyway. The reason is that Eq. (III. 1) is regarded here only as a prototype for the renormalization group equations for arbitrary field theories; in most of which $de_\lambda^2/d\lambda^2$ involves vertex functions as well as propagators and can be negative. Also, it is known from the example of the Lee model²⁰ that a renormalized theory does not necessarily satisfy the requirement of a positive definite metric which is assumed in the proof that $\rho(k'^2/m^2, e^2)$ is positive. So it may be that $\psi(0, x)$ does go negative even in electrodynamics.

Now consider the renormalization group differential equation Eq. (III. 1). This is the zero mass equation; the effects of a finite mass will be discussed later. The zero mass equation can be integrated to give

$$\ln(\lambda^2) - \ln(\lambda'^2) = F(e_\lambda^2) - F(e_{\lambda'}^2) \quad (\text{III. 4})$$

where λ and λ' are arbitrary, and

$$F(x) = \int_c^x \psi(0, x')^{-1} dx' \quad (\text{III. 5})$$

and c is an arbitrary constant which cancels out in the difference

$F(e_{\lambda}^2) - F(e_{\lambda'}^2)$. It is convenient to distinguish three ranges of the variable x :

$0 < x < x_1$, $x_1 < x < x_2$, and $x_2 < x < x_3$. The region beyond x_3 will not be discussed

here. First suppose x is in the range $0 < x < x_1$. Let c also lie between 0 and x_1 ,

so the integration region in Eq. (III.5) does not cross any zeros of ψ . $F(x)$ has

the following properties in the range $0 < x < x_1$: it is increasing (since $\psi(0, x')$ is

positive for $0 < x' < x_1$), it goes to $+\infty$ logarithmically as $x \rightarrow x_1$ (due to the simple

zero in $\psi(0, x')$ for $x' \rightarrow x_1$), and it goes to $-\infty$ proportional to x^{-1} when $x \rightarrow 0$

due to the double zero in $\psi(0, x')$ as $x' \rightarrow 0$. The function $F(x)$ is plotted in Fig. 2.

Now look at Eq. (III.4). Let λ' be fixed, and let $e_{\lambda'}^2$ be chosen arbitrarily but in the range $0 < e_{\lambda'}^2 < x_1$. As λ increases, $\ln \lambda^2$ increases; therefore $F(e_{\lambda}^2)$ must

increase and this means e_{λ}^2 must increase. For $\lambda^2 \rightarrow \infty$, $F(e_{\lambda}^2)$ must go to ∞

also, which means $e_{\lambda}^2 \rightarrow x_1$. (This argument should be checked using Fig. 2.)

In the other limit $\lambda^2 \rightarrow 0$, $\ln \lambda^2$ goes to $-\infty$ and this forces e_{λ}^2 to go to 0. This is

true for any value of $e_{\lambda'}^2$ in the range $0 < e_{\lambda'}^2 < x_1$. In fact, if one lets $f(y)$ be the

function inverse to $F(x)$ (i. e., $x = f(y)$ if $y = F(x)$) then the solution to Eq. (III.4) is

$$e_{\lambda}^2 = f \left[\ln(\lambda^2) - \ln(\lambda'^2) + F(e_{\lambda'}^2) \right] \quad (\text{III.6})$$

which means a change in e_{λ}^2 is equivalent to translating the solution for $e_{\lambda'}^2$ by a

fixed distance in the "time" variable $\ln \lambda^2$. A set of solution curves for e_{λ}^2 as a

function of $\ln \lambda^2$ is shown in Fig. 3. All the curves which lie in the range $0 < e_{\lambda}^2 < x_1$

go to x_1 as $\lambda^2 \rightarrow \infty$; they all go to 0 as $\lambda^2 \rightarrow 0$.

Similar analyses can be performed for solution curves lying in the intervals

$x_1 < e_{\lambda}^2 < x_2$ and $x_2 < e_{\lambda}^2 < x_3$. The curves lying in the range $x_1 < e_{\lambda}^2 < x_2$ go to x_1

for $\lambda^2 \rightarrow \infty$ and go to x_2 for $\lambda^2 \rightarrow 0$; curves lying in the range $x_2 < e_{\lambda}^2 < x_3$ go to

x_3 for $\lambda^2 \rightarrow \infty$ and go to x_2 for $\lambda^2 \rightarrow 0$.

There are also special solutions of the differential equation, namely the fixed points: $e_\lambda^2 \equiv 0$, $e_\lambda^2 \equiv x_1$, $e_\lambda^2 \equiv x_2$, or $e_\lambda^2 \equiv x_3$.

What one sees from this analysis is that in the limits $\lambda^2 \rightarrow \infty$ or $\lambda^2 \rightarrow 0$ any solution of the zero mass differential equation Eq. (III. 1) must approach one of the fixed points 0, x_1 , x_2 , or x_3 . This is despite the fact that for a fixed and finite λ , e_λ^2 is continuously variable. Underlying this result is an amplification and deamplification effect in the solution of the differential equation. Suppose one considers two initial conditions x_A and x_B for x at $t = 0$ and then integrates Eq. (I. 5); one gets two solutions $x_A(t)$ and $x_B(t)$. Suppose that $x_B \approx x_A$. As t increases, the difference $x_B(t) - x_A(t)$ may increase, in which case the difference can become large when t is large: this is amplification. It occurs, for example, if x_B and x_A are both near x_2 ; it is clear from Fig. 3 that the solutions $x_B(t)$ and $x_A(t)$ will separate as t increases. (If $x_A(t)$ and $x_B(t)$ both approach the limit x_1 or both approach x_3 when $t \rightarrow \infty$ the separation will eventually reach a maximum and then decrease again.) It is also possible for the curves $x_A(t)$ and $x_B(t)$ to approach each other as t increases, becoming equal as $t \rightarrow \infty$; this is deamplification. Amplification and deamplification can be discussed quantitatively by defining solutions $x(t, x_A)$ of Eq. (I. 5) which depend on the initial value:

$$x(0, x_A) = x_A \quad (\text{III. 7})$$

and then looking at $\partial x(t, x_A) / \partial x_A$. If there is strong amplification between 0 and t then $\partial x(t, x_A) / \partial x_A$ will be much larger than 1. Strong deamplification means that $\partial x(t, x_A) / \partial x_A \ll 1$. Since the function $x(t, x_A)$ satisfies

$$\frac{\partial x}{\partial t}(t, x_A) = \psi[0, x(t, x_A)] \quad (\text{III. 8})$$

one gets by differentiation

$$\frac{\partial}{\partial t} \left[\frac{\partial x}{\partial x_A}(t, x_A) \right] = \frac{\partial x}{\partial x_A}(t, x_A) \frac{\partial \psi}{\partial x}[0, x(t, x_A)] \quad (\text{III. 9})$$

If one thinks of $\partial\psi(0, x(t, x_A))/\partial x$ as being a known function $u(t)$, this equation can be integrated. One needs a boundary condition: from the boundary condition (III. 7) one has

$$\frac{\partial x}{\partial x_A}(0, x_A) = 1 \quad (\text{III. 10})$$

The result is

$$\frac{\partial x}{\partial x_A}(t, x_A) = \exp\left\{\int_0^t u(t') dt'\right\} \quad (\text{III. 11})$$

If $\partial\psi/\partial x$ is positive, $\partial x(t, x_A)/\partial x_A$ increases with t ; if $\partial\psi/\partial x$ is negative $\partial x/\partial x_A$ decreases. To make $\partial x/\partial x_A$ be near zero or infinity requires that the integral in the exponent be large. With the form of $\psi(0, x)$ being postulated here $\partial\psi/\partial x$ is not itself large and a large integral can come only from a large value of t (near $+$ or $-\infty$).

Now the case of finite mass will be discussed briefly. The main qualitative effect of having finite mass is that large amplification and deamplification effects are confined to the region $\lambda \gg m$. What happens for $\lambda \ll m$ is that $\partial\psi/\partial x$ is proportional to λ^2/m^2 and is too small to cause much amplification and deamplification. The appearance of the factor λ^2 is best seen from Eq. (II. 45): differentiating it, one gets

$$\psi(m^2/\lambda^2, e_\lambda^2) = \lambda^2 d[e_\lambda^2]/d\lambda^2 = (\lambda^2/m^2) \left[\partial d_c(y, e^2)/\partial y \right]_{y = -\lambda^2/m^2} \quad (\text{III. 12})$$

The function d_c is well behaved for $\lambda \rightarrow 0$ when m is finite, hence $\partial d_c(-\lambda^2/m^2, e^2)/\partial \lambda^2$ is finite for $\lambda \rightarrow 0$ (or perhaps mildly singular due to the 3 photon contribution to the spectral function, which has a threshold at $k^2 = 0$). Expressing e^2 in terms of e_λ^2 causes no trouble, at least in perturbation theory, nor does differentiation with respect to e_λ^2 . So $\partial\psi(m^2/\lambda^2, x)/\partial x$ has a factor λ^2/m^2 when $\lambda \ll m$, which is $m^{-2} e^{2t}$ in terms of t , which makes the region $t \ll \ln m$ negligible. Thus large

amplification and deamplification effects which require $\partial\psi/\partial x$ to be finite over a large range of t occur only for $\lambda \gg m$. For qualitative purposes one can avoid integrating the finite mass equation, instead integrating the zero mass equation but stopping at $\lambda = m$ instead of going to $\lambda = 0$. The value of e_m^2 obtained this way should qualitatively be similar to e^2 obtained from integrating the exact equation; in particular $\partial(e_m^2)/\partial e^2$ should be neither very small nor very large.

C. Fixed Point Solutions and Scale Invariance

What is the significance of the solutions described above of the renormalization group differential equation? First the special fixed point solutions will be discussed. Consider for example the solution $e_\lambda^2 = x_1$ for all λ with m being zero. It will now be shown that this solution defines a scale invariant field theory.²¹ Consider a dimensional analysis of the fields $A_{\lambda\mu}(z)$ and $\psi_\lambda(z)$ with e_λ^2 being a constant independent of λ and m being zero. The only variables left are z and λ . In the free field limit the dimensions of $A_{\lambda\mu}(z)$ and $\psi_\lambda(z)$ are fixed by the canonical commutation rules giving $A_{\lambda\mu}(z)$ the dimensions of mass and $\psi_\lambda(z)$ the dimensions (mass)^{3/2}.

When one does perturbation theory with a finite cutoff the dimensions of the unrenormalized fields are still fixed by the canonical commutation rules. The renormalized fields differ from the unrenormalized fields by cutoff-dependent factors which in principle could carry dimensions;²² in practice the renormalization conditions (II.30) and (II.31) imposed on the renormalized fields ensure that they also have canonical dimensions²³ and that the renormalization constants are dimensionless. To be specific, setting $d(-1, m^2/\lambda^2, e_\lambda^2)$ and $s(-1, m^2/\lambda^2, e_\lambda^2)$ equal to 1 for all e_λ^2 means that d and s are dimensionless for all e_λ , not just $e_\lambda = 0$, and this means the dimensions of $A_{\lambda\mu}$ and ψ_λ are also independent of e_λ . One can apply dimensional analysis to an arbitrary vacuum expectation value of the fields, e. g.,

$$T_{\mu\nu}(z_1, z_2, z_3, z_4, \lambda) = \langle \Omega | T A_{\lambda\mu}(z_1) A_{\lambda\nu}(z_2) \psi_\lambda(z_3) \bar{\psi}_\lambda(z_4) | \Omega \rangle \quad (\text{III.13})$$

The product $\lambda^{-5} T_{\mu\nu}(z_1, z_2, z_3, z_4, \lambda)$ is dimensionless and can depend only on the products $z_1\lambda$, $z_2\lambda$, $z_3\lambda$, and $z_4\lambda$. Similar results apply to arbitrary vacuum expectation values. These results can be represented schematically by the equation

$$\lambda^{-1} A_{\lambda\mu}(z) = A_{1\mu}(\lambda z) \quad (\text{III. 14})$$

$$\lambda^{-3/2} \psi_{\lambda}(z) = \psi_1(\lambda z) \quad (\text{III. 15})$$

($A_{1\mu}(z)$ means $A_{\lambda'\mu}(z)$ with $\lambda' = 1$; λ' is put equal to 1 whenever an arbitrary fixed value of λ' is needed.) These equations are to mean that vacuum expectation values such as $T_{\mu\nu}(z_1, z_2, z_3, z_4, \lambda)$ satisfy the identities that result from making the substitutions (III. 14) and (III. 15) in the vacuum expectation value. The equations cannot be understood as operator relations since one might have instead

$$\lambda^{-1} A_{\lambda\mu}(z) = U_{\lambda}^+ A_{1\mu}(\lambda z) U_{\lambda} \quad (\text{III. 16})$$

$$\lambda^{-1} \psi_{\lambda}(z) = U_{\lambda}^+ \psi_1(\lambda z) U_{\lambda} \quad (\text{III. 17})$$

where U_{λ} is a unitary transformation; these equations will also reproduce the result of dimensional analysis on $T_{\mu\nu}(z_1, z_2, z_3, z_4, \lambda)$.

In addition to the dimensional relations there are the renormalization relations:

$$A_{\lambda\mu}(z) = (z_{3\lambda 1})^{1/2} A_{1\mu}(z) \quad (\text{III. 18})$$

$$\psi_{\lambda}(z) = (z_{2\lambda 1})^{1/2} \psi_1(z) \quad (\text{III. 19})$$

For $T_{\mu\nu}$ these relations give

$$T_{\mu\nu}(z_1, z_2, z_3, z_4, \lambda) = z_{3\lambda 1} z_{2\lambda 1} T_{\mu\nu}(z_1, z_2, z_3, z_4, 1) \quad (\text{III. 20})$$

Using both Eqs. (III. 14) - (III. 15) and (III. 18) - (III. 19) one can eliminate the fields $A_{\lambda\mu}(z)$ and $\psi_{\lambda}(z)$, leaving formulae relating $A_{1\mu}(z)$ and $\psi_1(z)$ to $A_{1\mu}(\lambda z)$ and $\psi_1(\lambda z)$ i. e., scaling relations. First we need to determine $z_{3\lambda 1}$ and $z_{2\lambda 1}$.

The constant $z_{3\lambda 1}$ has the form (from Eq. (II. 53))

$$z_{3\lambda 1} = e_1^2 / e_\lambda^2 \quad (\text{III. 21})$$

Since e_λ^2 is constant, $e_\lambda^2 = e_1^2$ and $z_{3\lambda 1}$ is 1. To obtain $z_{2\lambda 1}$ one solves Eq. (II. 72) for $z_{2\lambda}$; since $z_{2\lambda 1}$ is $z_{2\lambda} / z_{21}$, one obtains

$$z_{2\lambda 1} = \lambda^{2\sigma} \quad (\text{III. 22})$$

with

$$\sigma = \sigma(0, x_1) \quad (\text{III. 23})$$

Thus one obtains the relation

$$\lambda^5 T_{\mu\nu}(\lambda z_1, \lambda z_2, \lambda z_3, \lambda z_4, 1) = T_{\mu\nu}(z_1, z_2, z_3, z_4, \lambda) = \lambda^{2\sigma} T_{\mu\nu}(z_1, z_2, z_3, z_4, 1) \quad (\text{III. 24})$$

This equation gives a scaling law for $T_{\mu\nu}(z_1, z_2, z_3, z_4, 1)$ namely it is equal to $\lambda^{5-2\sigma} T_{\mu\nu}(\lambda z_1, \lambda z_2, \lambda z_3, \lambda z_4, 1)$. The scaling law for arbitrary vacuum expectation values is represented schematically by the equations

$$A_{1\mu}(z) = \lambda A_{1\mu}(\lambda z) \quad (\text{III. 25})$$

$$\psi_1(z) = \lambda^{3/2-\sigma} \psi_1(\lambda z) \quad (\text{III. 26})$$

If Eqs. (III. 16) and (III. 17) hold then the actual operator relations are

$$U_\lambda A_{1\mu}(z) U_\lambda^\dagger = \lambda A_{1\mu}(\lambda z) \quad (\text{III. 27})$$

$$U_\lambda \psi_1(z) U_\lambda^\dagger = \lambda^{3/2-\sigma} \psi_1(\lambda z) \quad (\text{III. 28})$$

As a result of these equations the unitary transformations U_λ generate scale transformations of the fields $A_{1\mu}$ and ψ_1 , so the theory is scale invariant. I cannot justify Eqs. (III. 27) and (III. 28), but in any case all vacuum expectation values of $A_{1\mu}(z)$ and $\psi_1(z)$ scale as if the theory were scale invariant, which presumably means the theory is scale invariant.

The quantity $3/2 - \sigma$ is the "scale dimension" of the field $\psi_1(z)$. Unless σ is 0, the scale dimension of $\psi_1(z)$ is different from its dimension in mass units. The field $\lambda^{-\sigma} \psi_\lambda(z)$ is (from Eqs. (III.19) and (III.22)) equal to $\psi_1(z)$ and therefore independent of λ ; this field has both scale and mass dimensions $3/2 - \sigma$. The fact that interacting fields can have different scale dimensions from free fields has been noted in many circumstances.²⁴⁻²⁶ The scale dimension of $A_{1\mu}(z)$ is 1, from Eq. (III.27) and is not changed by interaction. This is presumably because of the field equation which makes $\square A_{\lambda\mu}(z)$ proportional to the electromagnetic current $j_{\lambda\mu}(z)$. The scale dimension of the current has to be 3 in order that the equal time commutator

$$[j_{10}(z, z_0), \psi_1(y, z_0)] = \delta^3(z - y) \psi_1(y, z_0) \quad (\text{III.29})$$

be scale invariant. Hence $\square A_{1\mu}(z)$ must have scale dimension 3 and $A_{1\mu}(z)$ itself must have scale dimension 1.

D. Renormalization Group, Fixed Points, and Strong Interactions in Isolation

Suppose now that Eq. (III.1) is the renormalization group equation for strong interactions. This would mean that there is only one renormalized coupling constant in strong interactions, as in the gluon model. The coupling constant e is interpreted in this section to be the renormalized coupling constant of strong interactions, and m will be assumed to be a typical strong interaction mass (about 1 GeV). Assume that strong interactions can be isolated from other interactions approximately for all momenta (the opposite assumption will be discussed later). In this case the renormalized coupling constant e is an arbitrary parameter in the theory and the finite mass renormalization group equation should be solved with the boundary condition $e_\lambda \rightarrow e$ as $\lambda \rightarrow 0$. In practice it is more convenient to discuss the zero mass renormalization group equation; the solution e_λ of the finite

mass equation satisfies the zero mass equation approximately when λ is much larger than m . One can now discuss a solution e'_λ of the zero mass equation such that $e'_\lambda \rightarrow e_\lambda$ as $\lambda \rightarrow \infty$, the difference being some power of m .

The quantity e'_m should depend in a reasonable way on e , that is, one does not expect large amplification or deamplification effects in the dependence of e'_m on e . So in the following e'_m will be taken to be the input parameter instead of e , and we shall discuss only the zero mass renormalization group equation. In the following we shall write e_λ for e'_λ . The function ψ will be assumed to have the form discussed in Section III. B. Suppose for convenience that e_m^2 lies between 0 and x_2 . If this is so e_λ^2 will approach x_1 as $\lambda \rightarrow \infty$. It is not necessary for λ to be enormous before e_λ^2 is close to x_1 : Since $\psi(0, x)$ is of order 1 when x is of order 1 (except when x is near x_1, x_2 , etc.), the solution curves change reasonably rapidly with λ and soon approach x_1 . The only exception is if e_m^2 is close to x_2 or 0; but if e_m^2 is an arbitrary parameter fixed by experiment it would seem unlikely that it would be close to any preassigned number such as x_2 . (It is also assumed that e_m is not small.)

One can determine the rate of approach of e_λ^2 to x_1 as $\lambda \rightarrow \infty$. Let $x(t)$ be near x_1 . Then one can expand $\psi(0, x)$ about $x = x_1$:

$$\psi(0, x) \simeq \psi(0, x_1) + [x(t) - x_1] \partial\psi(0, x_1)/\partial x \quad (\text{III. 30})$$

It is convenient to introduce the constants

$$a_n = \partial\psi(0, x_n)/\partial x \quad (\text{III. 31})$$

for each fixed point. Then, approximately

$$dx/dt \simeq a_1 [x(t) - x_1] \quad (\text{III. 32})$$

The solution of this equation is

$$x(t) = x_1 + c_1 e^{a_1 t} \quad (\text{III. 33})$$

where c_1 is an arbitrary constant. It is evident from Fig. 1 that a_1 is negative, so $x(t) \rightarrow x_1$ as $t \rightarrow \infty$ exponentially. Translated in terms of e_λ^2 , this gives

$$e_\lambda^2 \simeq x_1 + c_1 \lambda^{2a_1} \quad (\text{III. 34})$$

The constant c_1 will depend on the initial condition e_m^2 . If $e_m^2 - x_1$ is small also then c_1 will be linear in $e_m^2 - x_1$, but if $e_m^2 - x_1$ is large the linearized Eq. (III. 32) will be incorrect for t near $\ln m^2$ and the relation of c_1 to $e_m^2 - x_1$ will be nonlinear. In any case e_λ^2 approaches its limiting value x_1 as an inverse power of λ .

When e_λ was introduced in Section II it arose as an alternative definition of the renormalized coupling constant; in the Gell-Mann-Low renormalization program all Feynman amplitudes are considered as functions of e_λ not e . This raises a problem. If one has a function such as $d(k^2/\lambda^2, m^2/\lambda^2, e_\lambda^2)$ which depends on e_λ^2 , one would suppose that a small change in e_λ^2 implies a small change in d . But if λ is very large a small change in e_λ^2 away from x_1 is amplified to give a large and nonlinear change in e_m^2 ; so e_m^2 and any quantity which depends directly on e_m^2 will not change just a little when e_λ^2 changes a little. This means it is not very appropriate to use e_λ^2 for a large value of λ as one's renormalized coupling constant for amplitudes which are best parameterized in terms of e_m^2 . Another way to say this is that physically the qualitative nature of a given amplitude should be determined by a qualitative knowledge of the physical couplings which determine that amplitude. If one has to specify a coupling constant to 1% accuracy in order to determine the amplitude to 50% accuracy, there is something wrong. The response to this problem is, I think, to consider the coupling constant e_λ^2 for different ranges of the momentum λ to be physically distinct coupling constants governing distinct physical processes. The natural thing to expect is that e_λ with $\lambda \sim m$ governs amplitudes with momenta of order m or less; e_λ with $\lambda \sim 10 m$ covers amplitudes whose momenta are of order $10 m$, and so forth. Within a particular order of

magnitude range of momenta one can specify arbitrarily the exact value of λ for which e_λ is the coupling constant.

If λ is much larger than m the amplitudes for which e_λ^2 is the relevant coupling constant are off-mass-shell amplitudes with virtual masses of order λ , such as the photon propagator with $k^2 \sim \lambda^2$. A scattering amplitude with large energies and momentum transfers, but for particles on the mass shell, cannot be said to involve only e_λ^2 for λ large; the mass shell condition suggests that the amplitude will also be strongly affected by the value of e_m^2 . The behavior of scattering amplitudes at large energies and momentum transfers is outside the scope of this paper.

The way this picture shows up in perturbation theory is that a function such as $d(k^2/\lambda^2, m^2/\lambda^2, e_\lambda^2)$ has a perturbation expansion in e_λ^2 whose coefficients are of order 1 (apart from factors of 2π which are considered to be of order 1 in this paper) when k^2 is of order λ^2 ; but if k^2/λ^2 or λ^2/k^2 is large the coefficient of $(e_\lambda^2)^n$ contains the large factor²⁷ $[\ln(k^2/\lambda^2)]^n$ which makes d much more sensitive to changes in e_λ^2 . So perturbation theory confirms the hypothesis that e_λ^2 is the appropriate coupling constant for momenta of order λ and not so appropriate for momenta much larger or smaller than λ .

In summary one defines a sequence of coupling constants, say e_m, e_{10m}, e_{100m} , etc. All of these have to be specified in order to determine qualitatively the physics at arbitrary momenta, due to the large amplification or deamplification effects that can occur in the relation between different coupling constants in the sequence.

For large momentum λ , e_λ^2 is close to x_1 and m/λ is small; this means propagators and other amplitudes at large momenta will be close to the scale invariant amplitudes for $e_\lambda^2 = x_1$ and $m = 0$. So scale invariance is a broken

symmetry of the theory. Furthermore, as seen earlier in the case of the field ψ one must expect scale dimensions of local fields to differ from free field dimensions as was assumed in Ref. 24. One consequence of broken scale invariance is that $\sigma_{\text{TOT}}(q^2)$, the $e^+ - e^-$ annihilation cross section into hadrons, behaves as $(q^2)^{-1}$ when q^2 is large.²⁵

E. Strong Interactions with a Cutoff²⁸

Suppose now that there is a cutoff momentum Λ beyond which one cannot distinguish strong interactions from other interactions. It will be assumed that $\Lambda \gg m$, i. e., Λ is much larger than 1 GeV. The effect of the cutoff is to shift the boundary condition on the renormalization group equation from $\lambda = m$ to $\lambda = \Lambda$. The reason is this. With a cutoff, the strong interaction theory that one develops for laboratory energies is no more than an infrared approximation to a more complicated theory combining strong and other interactions. The more complicated theory covers energies of order Λ or less; strong interaction theory is valid only for momenta much less than Λ . The coupling constant e_λ for λ of order Λ will now be determined by properties of the more complicated theory rather than by strong interactions alone. Under these circumstances, the chances are negligible that e_λ with $\lambda \sim \Lambda$ will be close to the fixed point x_1 , since the location of x_1 is a property of the isolated strong interaction theory and not of the combined theory. Another way of saying this is that if weak interactions, say, become strong at momenta of order Λ , then there will be large corrections to e_λ when λ is of order Λ due to weak interactions, making it unlikely that e_λ^2 will be near x_1 . Suppose therefore that e_λ^2 has a random value, but for convenience assume that it lies between x_1 and x_3 . (The case that e_λ^2 is less than x_1 will be discussed later: see Section III. G.) Now if one solves the renormalization group equation, one sees from Fig. 3 that e_λ^2 rapidly approaches x_2 as λ decreases. In fact, one sees that

for $\lambda \ll \Lambda$, e_λ^2 has the form

$$e_\lambda^2 \simeq x_2 + c_2(\lambda/\Lambda)^{2a_2} \quad (\text{III. 35})$$

where a_2 is given by Eq. (III.31) and c_2 is a constant of order 1 since e_Λ^2 differs from x_2 by of order 1. In particular e_m^2 has the form

$$e_m^2 \simeq x_2 + c_2(m/\Lambda)^{2a_2} \quad (\text{III. 36})$$

This means e_m^2 is very little affected by the exact value of e_Λ^2 ; the dependence of e_m^2 on e_Λ^2 is through the constant c_2 and this constant is multiplied by the small coefficient $(m/\Lambda)^{2a_2}$. This in turn means that the ordinary renormalized coupling constant e^2 is also only slightly dependent on the value of e_Λ^2 . This assumes that e_Λ^2 is restricted to the range $x_1 < e_\Lambda^2 < x_3$; if e_Λ^2 lies outside this range e_m^2 will be very different.

The consequence of introducing the cutoff Λ and requiring e_Λ^2 to be arbitrary is that e_m^2 and therefore e^2 is fixed precisely except for corrections which are small when m/Λ is small. In other words there is a bootstrap condition for the renormalized coupling constant, apart from small cutoff dependent effects. The bootstrap condition is not a consistency condition: the field theory of strong interactions in isolation exists and is (by assumption) well behaved for any value of e ; it is the influence of other interactions at the cutoff Λ that forces e to be the solution of the bootstrap.

F. Precise Formulation of the Bootstrap Condition²⁸

The bootstrap condition will now be discussed in more detail. Consider the exact equation for e_λ^2 , Eq. (II. 65), with finite mass. The equation is to be solved with the value of e_Λ^2 being the boundary condition. It will be assumed that e_Λ^2 lies somewhere between x_1 and x_3 ; what happens when e_Λ^2 is outside this range is discussed later in this section. It is also assumed that all cutoff-dependent effects

are absorbed into the constant e_Λ^2 and do not, for example, change the function $\psi(m^2/\lambda^2, e_\lambda^2)$. This assumption is surely an oversimplification, but the essential features of the bootstrap are not changed by making this simplification. It will also be assumed that Λ/m is sufficiently large so that over a sizable range of λ , m/λ and λ/Λ are both small. In the range $\lambda < \Lambda$ but $\lambda \gg m$ the exact function $\psi(m^2/\lambda^2, e_\lambda^2)$ is approximately $\psi(0, e_\lambda^2)$; therefore the solution e_λ^2 behaves as shown in Fig. 3 and is close to x_2 for $\lambda \ll \Lambda$. This suggests that one define a solution of the finite mass equation which is exactly x_2 when $\lambda \gg m$, and compare this solution with the exact solution. So one has two functions, say e_λ and $e_{\lambda 1}$, where

$$de_\lambda^2/d(\ln\lambda^2) = \psi(m^2/\lambda^2, e_\lambda^2) \quad (\text{III. 37})$$

$$de_{\lambda 1}^2/d(\ln\lambda^2) = \psi(m^2/\lambda^2, e_{\lambda 1}^2) \quad (\text{III. 38})$$

and $e_{\lambda 1}^2$ satisfies the boundary condition

$$\lim_{\lambda \rightarrow \infty} e_{\lambda 1}^2 = x_2 \quad (\text{III. 39})$$

while the boundary condition for e_λ^2 is the value of e_Λ^2 . The functions e_λ^2 and $e_{\lambda 1}^2$ are illustrated in Fig. 4.

The functions e_λ^2 and $e_{\lambda 1}^2$ are both close to x_2 in the range $\lambda \gg m$ but $\lambda \ll \Lambda$. This means that e_λ^2 is close to $e_{\lambda 1}^2$ for λ of order m also because the differential equation deamplifies the difference between e_λ^2 and $e_{\lambda 1}^2$ as λ decreases (see below). The advantage of defining $e_{\lambda 1}$ is that it has no cutoff dependence. It is also uniquely defined since $e_{\lambda 1}$ must satisfy the zero mass renormalization group equation for large λ , and only one solution of the zero mass equation goes to x_2 when $\lambda \rightarrow \infty$. This is a situation reminiscent of the one dimensional Schrödinger equation in the bound state region; the Schrödinger equation has one solution which goes to zero at infinity, while it has many solutions which blow up at infinity. Since $e_{\lambda 1}$ can depend only on λ and m and is dimensionless it must have the form

$$e_{\lambda 1}^2 = \xi(\lambda^2/m^2) \quad (\text{III. 40})$$

for some function ξ . In particular the value of $e_{\lambda 1}^2$ for $\lambda = 0$ is $\xi(0)$ which is a fixed number independent of the mass m .

As long as e_{λ}^2 is near $e_{\lambda 1}^2$ one can write a linearized equation for the difference $e_{\lambda}^2 - e_{\lambda 1}^2$:

$$d[e_{\lambda}^2 - e_{\lambda 1}^2]/d(\ln \lambda^2) = \partial\psi/\partial x(m^2/\lambda^2, e_{\lambda 1}^2)[e_{\lambda}^2 - e_{\lambda 1}^2] \quad (\text{III.41})$$

When $\lambda \gg m$ but $\lambda \ll \Lambda$ one has $e_{\lambda 1}^2 \simeq x_2$ and $m^2/\lambda^2 \ll 1$, giving $\partial\psi/\partial x \simeq a_2$. Since a_2 is positive, the difference $e_{\lambda}^2 - e_{\lambda 1}^2$ decreases as λ decreases, except that for $\lambda \sim m$, $\partial\psi/\partial x$ might change sign due to finite mass effects. However large amplification cannot occur for $\lambda \lesssim m$ (see Section III. B) so even if $\partial\psi/\partial x$ does change sign for $\lambda \sim m$, the difference $e_{\lambda}^2 - e_{\lambda 1}^2$ cannot become large.

More precisely, for $m \ll \lambda \ll \Lambda$ one has

$$e_{\lambda}^2 - e_{\lambda 1}^2 \simeq c_2 (\lambda/\Lambda)^{2a_2} \quad (\lambda \gg m, \lambda \ll \Lambda) \quad (\text{III.42})$$

for some constant c_2 .

Using this as a boundary condition, the solution of Eq. (III.41) is

$$e_{\lambda}^2 = e_{\lambda 1}^2 + c_2 \exp \left\{ - \int_{\lambda^2}^{\Lambda^2} u(\lambda'^2/m^2)(\lambda'^2)^{-1} d(\lambda'^2) \right\} \quad (\text{III.43})$$

with

$$u(\lambda'^2/m^2) = \partial\psi/\partial x(m^2/\lambda'^2, \xi[\lambda'^2/m^2]) \quad (\text{III.44})$$

When $\lambda^2 \gg m^2$, $\xi(\lambda^2/m^2)$ is approximately x_2 , $u(\lambda^2/m^2)$ is approximately a_2 and Eq. (III.43) reduces to Eq. (III.42). When λ^2 is 0 one has

$$e^2 = e_{01}^2 + c_2 \exp \left\{ - \int_0^{\Lambda^2} (\lambda'^2)^{-1} u(\lambda'^2/m^2) d\lambda'^2 \right\} \quad (\text{III.45})$$

The large part of the integral in the exponent comes from the region of integration $m \ll \lambda' \ll \Lambda$. The result is

$$e^2 \simeq e_{01}^2 + c_2 (m^2/\Lambda^2)^{a_2} \quad (\text{III.46})$$

where w is a constant, namely

$$w = \exp \left\{ - \int_0^{m^2} (\lambda'^2)^{-1} u(\lambda'^2/m^2) d\lambda'^2 - \int_{m^2}^{\Lambda^2} \lambda'^2 [u(\lambda'^2/m^2) - a_2] d\lambda'^2 \right\} \quad (\text{III. 47})$$

The function $u(\lambda'^2/m^2) - a_2$ goes to zero as $\lambda' \rightarrow \infty$; it is assumed here that it goes to zero fast enough so that the region $\lambda'^2 > \Lambda^2$ is negligible. The quantity e_{01}^2 is $\xi(0)$ and is the bootstrap value of the renormalized coupling constant; the actual value for e^2 differs from the bootstrap value by an amount of order $(m^2/\Lambda^2)^{a_2}$.

Note that the bootstrap value of the renormalized coupling constant is computed by a complex procedure. First one must locate the fixed point x_2 of the zero mass renormalization group equation. This means solving the equation $\psi(0, x) = 0$, which is the bootstrap condition. Having found the solution x_2 , one must then solve the differential equation for $e_{\lambda 1}^2$ with the boundary condition $e_{\lambda 1}^2 \rightarrow x_2$ as $\lambda \rightarrow \infty$; the value of $e_{\lambda 1}^2$ for $\lambda = 0$ is the bootstrap value of the renormalized coupling constant. The bootstrap condition is a condition on a zero mass theory, a theory which very likely has no S matrix due to infrared divergences.²⁵ The bootstrap condition is determined by the function ψ which is in turn defined in terms of a propagator well off the mass shell; this is a very different bootstrap from those that have been proposed in the context of S-matrix theory.

For momenta between m and Λ the theory is approximately scale invariant, since e_{λ}^2 is close to the fixed point x_2 and the mass can be neglected. However the scale invariant theory differs from the scale invariant theory associated with the fixed point x_1 ; for example the scale dimension of the field ψ is different in the two cases since $\sigma(0, x_2)$ need not be the same as $\sigma(0, x_1)$.

Whether or not there is a cutoff Λ , the short distance behavior of strong interactions is determined by a fixed point of the renormalization group. However a different fixed point is involved if there is a cutoff. What is the difference?

The fixed point x_2 has the property that solutions of the zero mass renormalization group equation in the vicinity of x_2 approach x_2 as $\lambda \rightarrow 0$. In contrast, solutions in the vicinity of x_1 or x_3 for finite λ go away from x_1 or x_3 as $\lambda \rightarrow 0$. One can say that x_2 is "infrared-stable" while x_1 and x_3 are "infrared instable." With a cutoff, one has to allow e_Λ^2 to be arbitrary which means it is unlikely to be equal to x_1 or x_3 , or to any infrared-unstable fixed point. Hence the solution of the renormalization group equation will approach one of the infrared stable fixed points of $\psi(0, x)$. The infrared-stable fixed points are $x = 0$, $x = x_2$, plus possibly roots beyond x_3 . If e_Λ^2 lies between x_1 and x_3 then x_2 is the relevant fixed point; if e_Λ^2 is less than x_1 then $x = 0$ is the relevant fixed point. The significance of $x = 0$ as a fixed point will be discussed in Section III. G. What happens if $e_\Lambda^2 > x_3$ will not be discussed here. Without a cutoff it is the renormalized coupling constant e^2 that is arbitrary; one then follows solutions of the differential equation out to $\lambda = \infty$, and it is the fixed points which are stable in this limit that become the possible asymptotic solutions of e_λ^2 for large λ . The possible fixed points are x_1 and x_3 ; these might be called "ultraviolet-stable" fixed points.

The bootstrap condition resulting from the presence of a cutoff is, precisely, that $\psi(0, x)$ be zero and that x be an infrared-stable root. Any infrared-stable root is acceptable, so if there is more than one the correct one has to be determined experimentally.

G. Electrodynamics with a Cutoff

To conclude the discussion of fixed points the case of electrodynamics will be considered. Now e , e_λ , and ψ refer to electrodynamics rather than strong interactions. The same form is assumed for ψ as in Section III. B. While the renormalized coupling constant e^2 of electrodynamics is small, one sees from Fig. 3 that $e_\lambda^2 \rightarrow x_1$ for $\lambda \rightarrow \infty$. The constant x_1 is fixed independent of e^2 and so cannot

be arbitrarily small. This suggests that all particles will couple strongly to photons at sufficiently high momenta; but this would mean that electrodynamics and strong interactions would mix strongly, suggesting that pure electrodynamics is valid only below a cutoff momentum Λ . Suppose this is the case and that e_Λ^2 is therefore arbitrary as discussed earlier, but happens to be smaller than x_1 . Then for $\lambda \ll \Lambda$, e_λ^2 is close to the fixed point zero. One can as before define a bootstrap value for the renormalized coupling constant by solving the renormalization group equation for a function $e_{\lambda 1}^2$ which goes to zero as $\lambda \rightarrow \infty$. The solution is $e_{\lambda 1}^2 = 0$. The departure of e^2 from 0 is therefore a cutoff dependent effect, as discussed earlier. Because of the special nature of the fixed point $x = 0$, e^2 does not vary as a power of the cutoff. The function $\psi(0, x)$ has no term linear in x for x near zero so one cannot find the dependence of e^2 on Λ from a linearized equation. Instead one must keep the quadratic term in ψ , giving the approximate equation for e_λ^2 small:

$$de_\lambda^2/d(\ln \lambda^2) = (12\pi^2)^{-1} e_\lambda^4 \quad (\text{III. 48})$$

(This equation neglects all contributions to vacuum polarization except for electrons; other particles will increase the factor $(12\pi^2)^{-1}$ by a presently unknown factor. The boundary condition is that e_Λ^2 be arbitrary which presumably means of order 1.

The solution is

$$\left(e_\lambda^2\right)^{-1} - \left(e_\Lambda^2\right)^{-1} = (12\pi^2)^{-1} \ln(\Lambda^2/\lambda^2) \quad (\text{III. 49})$$

or

$$e_\lambda^2 = e_\Lambda^2 \left\{ 1 + (12\pi^2)^{-1} e_\Lambda^2 \ln(\Lambda^2/\lambda^2) \right\}^{-1} \quad (\text{III. 50})$$

For $\lambda \ll \Lambda$, for which $e^2 \ln(\Lambda^2/\lambda^2)$ is large, one gets the approximate form

$$e_\lambda^2 \approx (12\pi^2) [\ln(\Lambda^2/\lambda^2)]^{-1} \quad (\text{III. 51})$$

So e_λ^2 decreases as an inverse logarithm of λ^2 for $\lambda \ll \Lambda$. One has to have an astronomical value of Λ/m to make the renormalized coupling constant e^2 be as small as $1/137$.

In the case of electrodynamics the small cutoff dependence of the coupling constant is very noticeable since it is only the cutoff dependent term that makes e^2 nonzero. For strong interactions, where the bootstrap value of the renormalized coupling constant should be of order 1, a small cutoff dependence would be much harder to detect experimentally.

H. Limit Cycles

If there are at least two renormalized coupling constants in strong interactions, as in pseudoscalar meson theory, there is an intriguing alternative to a fixed point, namely a limit cycle.^{8, 17} To illustrate the hypothesis of a limit cycle, suppose there are two renormalized coupling constants; then the zero mass renormalization group equations have the form

$$dx/dt = \psi_1(x, y) \quad (\text{III. 52})$$

$$dy/dt = \psi_2(x, y) \quad (\text{III. 53})$$

where the functions ψ_1 and ψ_2 are analogous to the function $\psi(0, x)$, t is $\ln \lambda^2$, and x and y are the momentum-dependent coupling constants, say $x = g_\lambda$, $y = h_\lambda$. The functions ψ_1 and ψ_2 will be mass dependent when λ is of order the masses in the theory, but this dependence will not be exhibited explicitly. It is assumed that the mass dependence can be neglected for large enough λ .

The solutions of Eqs. (III. 52) and (III. 53) will define trajectories in a two dimensional space with coordinates x and y .

A limit cycle is a special trajectory which is a closed orbit, namely a solution $\{x(t), y(t)\}$ which satisfies

$$x(t + \tau) = x(t) \quad (\text{III. 54})$$

and

$$y(t + \tau) = y(t) \quad (\text{III. 55})$$

where τ is a constant giving the period of the limit cycle. Generally the trajectories in the neighborhood of a limit cycle are not closed. Instead they either approach the cycle as $t \rightarrow \infty$, or they move away from the cycle as t increases, in neither case closing on themselves. There are general conditions on the functions ψ_1 and ψ_2 which ensure the existence of a limit cycle without determining τ or the exact form of the cycle.⁸ For instance if one can find an annulus with the property that trajectories can go into the annulus but not out of it (that is, not cross out of the outer ring nor cross inside the inner ring of the annulus) then there is a limit cycle contained in the annulus. The condition that trajectories only go into the annulus is easily checked since this means that the velocity vector $\langle \psi_1(x, y), \psi_2(x, y) \rangle$ must point into the annulus for all points $\{x, y\}$ on the inner and outer rings of the annulus. For this theorem to hold, there must not be any fixed points inside the annulus.

A detailed discussion of limit cycles will not be given here. There is one important observation to make. If the coupling constants g_λ and h_λ approach a limit cycle as $\lambda \rightarrow \infty$ instead of a fixed point, there is a chance that this will be experimentally observable. Consider the total cross section $\sigma_{\text{TOT}}(q^2)$ for $e^+ - e^-$ annihilation into hadrons: q is the momentum transfer. Assuming electrodynamics is treated to lowest order, $\sigma_{\text{TOT}}(q^2)$ is the absorptive part of the propagator for the electromagnetic current of hadrons, times known factors. By dimensional analysis $\sigma_{\text{TOT}}(q^2)$ has the form

$$\sigma_{\text{TOT}}(q^2) = \alpha(q^2)^{-1} f(q^2/\lambda^2, m/\lambda, g_\lambda, h_\lambda) \quad (\text{III. 56})$$

where m stands for all possible mass parameters in the theory and α is the fine structure constant. The mass dependence should be negligible for q^2 and λ^2 large.

The formula is valid for any value of λ . In particular, one can set $\lambda^2 = q^2$; if q^2 is large m^2/q^2 is negligible and

$$\sigma_{\text{TOT}}(q^2) = \alpha(q^2)^{-1} f(1, 0, g_q, h_q) \quad (\text{III.57})$$

It is unlikely that f is independent of g_λ and h_λ , since vacuum polarization in electrodynamics is also the current-current propagator in electrodynamics and it is coupling-constant dependent. So as g_q and h_q change with q^2 , so will $q^2 \sigma_{\text{TOT}}(q^2)$ and one will see perpetual oscillations in the $e^+ - e^-$ total hadronic cross section in the limit of large q^2 .

There are other forms of asymptotic behavior besides fixed points and limit cycles. There will be no discussion of further alternatives here. The example of the limit cycle suggests that any asymptotic behavior other than a fixed point will mean that $q^2 \sigma_{\text{TOT}}(q^2)$ will not approach a constant in the limit of large q^2 .

J. Conclusions and Remarks

There are three basic results of this section. First, if the asymptotic solution of the renormalization group equations for strong interactions is a fixed point, then strong interactions will have broken scale invariance as a symmetry. Second, if in addition there is a large but finite cutoff Λ above which strong interactions cannot be isolated from other interactions, the fixed point must be infrared-stable and there is a bootstrap condition which determines renormalized coupling constants of strong interactions. Third, if the asymptotic solution of the renormalization group equations is not a fixed point then $q^2 \sigma_{\text{TOT}}(q^2)$ for $e^+ - e^-$ annihilation will not be constant for large q^2 ; if there is a limit cycle in particular, then $q^2 \sigma_{\text{TOT}}(q^2)$ will oscillate perpetually for large q^2 , with a fixed period if plotted versus $\ln q^2$.

All these results are crucially dependent on the assumption that functions such as $\psi(m^2/\lambda^2, x)$ have a nontrivial limit for $m \rightarrow 0$. If instead $\psi(m^2/\lambda^2, x)$ were to

go to zero as $m \rightarrow 0$ for all x , the analysis given above would collapse and the asymptotic limit of e_λ^2 for $\lambda \rightarrow \infty$ would be a straightforward function of e^2 . If in perturbation theory $\psi(m^2/\lambda^2, x)$ were to contain logarithms of m^2/λ^2 it would be easy for the sum of the perturbation series to approach zero as $m \rightarrow 0$, for example if the sum of the series gave something like

$$\psi\left(m^2/\lambda^2, e_\lambda^2\right) = e_\lambda^4 \exp\left[-e_\lambda^2 \ln(\lambda^2/m^2)\right] \quad (\text{III. 58})$$

(a more complicated example is needed if one is to fit the known term of order e_λ^6). This does not mean that proving the existence of the zero mass limit for $\psi(m^2/\lambda^2, x)$ to all orders in x is the crucial problem. The crucial problem is to determine whether the physics of a strongly coupled field theory is such that the results obtained above are reasonable.

IV. BREAKING OF SCALE INVARIANCE

In the previous section it was found that the fixed point solutions of the renormalization group equations for zero mass define scale invariant field theories. For the specific type of function $\psi(0, x)$ discussed in the previous section there are at least three distinct scale invariant theories defined by the fixed points x_1 , x_2 , and x_3 . The purpose of this section is to study the nature of scale breaking. One can break scale invariance in two ways. One way is to let the mass be finite instead of zero. The other way is to choose a solution e_λ^2 of the zero mass renormalization group equations which is not equal to a fixed point. Then the variation of e_λ^2 with λ breaks scale invariance, as will be seen below. Both these forms of scale breaking will be investigated in this section. In particular we shall study amplitudes at high momenta for which the scale breaking is small and develop a perturbation method for computing the scale breaking terms.

A theory of scale breaking has been proposed elsewhere.²⁴ In this theory the nature of scale breaking corrections at large momenta is determined by the scale dimensions of the terms in the Lagrangian density which break scale invariance. The interaction Lagrangian density associated with the renormalized charge e_λ should be a renormalized form of the local product $\bar{\psi}_\lambda(x) \gamma_\mu \psi_\lambda(x) A_\lambda^\mu(x)$; the interaction Lagrangian density associated with the renormalized mass should be a renormalized form of the product $\bar{\psi}_\lambda(x) \psi_\lambda(x)$. The problem of renormalizing composite fields such as $\bar{\psi}_\lambda(x) \gamma_\mu \psi_\lambda(x) A_\lambda^\mu(x)$ and showing that these fields are connected with the renormalized coupling constant and mass will not be discussed here. It will simply be assumed that the part of the Lagrangian density which determines the charge e_λ and the mass m involves two local interactions denoted $\mathcal{L}_{e\lambda}(x)$ and $\mathcal{L}_{m\lambda}(x)$ respectively.

Explicit calculations (based on the renormalization group equations) of scale breaking corrections will be compared with the general theory of Ref. 24. One result will be that the scale dimension of $\mathcal{L}_{e\lambda}(x)$ is $4 + 2a_n$ in the scale invariant theory defined by the fixed point x_n . The constant a_1 is negative, the constant a_2 is positive so the dimension of $\mathcal{L}_{e\lambda}(x)$ can be either less than 4 or greater than 4 depending on which fixed point one uses. It was pointed out in Ref. 24 that interactions with dimension >4 act like nonrenormalizable interactions while interactions with dimension <4 act like superrenormalizable interactions or mass terms. (Interactions with dimension <4 were called generalized mass terms in Ref. 24.) So the interaction $\mathcal{L}_{e\lambda}(x)$ changes its character considerably when one switches from the fixed point x_1 to the fixed point x_2 .

To discuss the dimension of $\mathcal{L}_{m\lambda}(x)$ will require an extension of the renormalization group equations to include a λ -dependent mass parameter.²⁹ The result will be to conclude that $\mathcal{L}_{m\lambda}(x)$ could be either a generalized mass term or a nonrenormalizable interaction at either x_1 or x_2 , the choice being determined by detailed dynamics which are unknown at present.

A. Scale Breaking Through Nonconstant e_λ

To start with scale breaking due to nonconstant e_λ will be considered. The mass m is taken to be zero. Suppose that over some range of λ , e_λ^2 is close to the fixed point x_n . As shown in Section III. D., the approximate form for e_λ^2 as long as e_λ^2 is close to x_n is

$$e_\lambda^2 = x_n + c_n \lambda^{2a_n} \quad (IV.1)$$

where c_n is an arbitrary small constant and a_n is given by Eq. (III.31). Equation (IV.1) neglects terms quadratic in the difference $(e_\lambda^2 - x_n)$, i. e., terms of order c_n^2 . Expansions will now be sought for $z_{3\lambda\lambda}$, $z_{2\lambda\lambda}$, and vacuum expectation values,

also valid to first order in c_n . Given these expansions, one can construct scaling laws for first order scale breaking terms in the vacuum expectation values.

Since $z_{3\lambda\lambda'} = e_{\lambda'}^2/e_\lambda^2$ (Eq. (II.53)) one has

$$z_{3\lambda\lambda'} \simeq \left[x_n + c_n (\lambda')^{2a_n} \right] \left[x_n + c_n \lambda^{2a_n} \right]^{-1} \simeq 1 + (x_n)^{-1} c_n \left[(\lambda')^{2a_n} - \lambda^{2a_n} \right] \quad (\text{IV.2})$$

The equation for $z_{2\lambda\lambda'}$ is (cf. Eqs. (II.73) and (II.50)):

$$\partial z_{2\lambda\lambda'} / \partial (\ln \lambda^2) = z_{2\lambda\lambda'} \sigma(0, e_\lambda^2) \quad (\text{IV.3})$$

with the boundary condition

$$z_{2\lambda'\lambda'} = 1 \quad (\text{IV.4})$$

The solution of this equation is

$$z_{2\lambda\lambda'} = \exp \left\{ \int_{\lambda'}^{\lambda} \sigma(0, e_{\lambda''}^2) (\lambda'')^{-1} 2 d\lambda'' \right\} \quad (\text{IV.5})$$

If

$$\sigma_n = \tau(0, x_n) \quad (\text{IV.6})$$

$$\tau_n = \partial \sigma(0, x_n) / \partial x \quad (\text{IV.7})$$

then to first order in c_n

$$\begin{aligned} z_{2\lambda\lambda'} &\simeq \exp \left\{ \sigma_n \ln(\lambda^2/\lambda'^2) + (\tau_n c_n / a_n) \left[\lambda^{2a_n} - (\lambda')^{2a_n} \right] \right\} \\ &\simeq (\lambda/\lambda')^{2\sigma_n} \left\{ 1 + (\tau_n c_n / a_n) \left[\lambda^{2a_n} - (\lambda')^{2a_n} \right] \right\} \end{aligned} \quad (\text{IV.8})$$

Consider now a typical vacuum expectation value, e. g. ,

$$T_{\mu\nu} \left(z_1^\lambda, z_2^\lambda, z_3^\lambda, z_4^\lambda, m/\lambda, e_\lambda^2 \right) = \lambda^{-5} \langle \Omega | T A_{\lambda\mu}(z_1) A_{\lambda\nu}(z_2) \psi_\lambda(z_3) \bar{\psi}_\lambda(z_4) | \Omega \rangle \quad (\text{IV.9})$$

($T_{\mu\nu}$ is dimensionless so depends only on dimensionless variables, as indicated.)

Putting $m = 0$ and using the renormalization relations one has³⁰

$$T_{\mu\nu} \left(z_1^\lambda, z_2^\lambda, z_3^\lambda, z_4^\lambda, 0, e_\lambda^2 \right) = \lambda^{-5} z_{3\lambda 1} z_{2\lambda 1} T_{\mu\nu} \left(z_1, z_2, z_3, z_4, 0, e_1^2 \right) \quad (\text{IV.10})$$

Write

$$T_{\mu\nu}(z_1, z_2, z_3, z_4) = T_{\mu\nu}(z_1, z_2, z_3, z_4, 0, x_n) \quad (\text{IV. 11})$$

$$U_{\mu\nu}(z_1, z_2, z_3, z_4) = \left. \circ T_{\mu\nu} / \partial x(z_1, z_2, z_3, z_4, 0, x) \right|_{x=x_n} \quad (\text{IV. 12})$$

Then to first order in c_n

$$T_{\mu\nu}(z_1, z_2, z_3, z_4, 0, e_1^2) \simeq T_{\mu\nu}(z_1, z_2, z_3, z_4) + c_n U_{\mu\nu}(z_1, z_2, z_3, z_4) \quad (\text{IV. 13})$$

and similarly for $T_{\mu\nu}(z_1^\lambda, \dots, z_4^\lambda, 0, e_\lambda^2)$. Equation (IV. 10) to first order in c_n , is

$$\begin{aligned} & T_{\mu\nu}(z_1^\lambda, \dots, z_4^\lambda) + c_n \lambda^{2a_n} U_{\mu\nu}(z_1^\lambda, \dots, z_4^\lambda) \\ &= \lambda^{2\sigma_n - 5} \left\{ 1 + c_n \left[\tau_n a_n^{-1} - x_n^{-1} \right] \left[\lambda^{2a_n} - 1 \right] \right\} T_{\mu\nu}(z_1, \dots, z_4) + \lambda^{2\sigma_n - 5} c_n U_{\mu\nu}(z_1, \dots, z_4) \end{aligned} \quad (\text{IV. 14})$$

For $c_n = 0$ this gives the scaling law found in Section III. C:

$$T_{\mu\nu}(\lambda z_1, \dots, \lambda z_4) = \lambda^{2\sigma_n - 5} T_{\mu\nu}(z_1, \dots, z_4) \quad (\text{IV. 15})$$

The terms proportional to c_n in Eq. (IV. 14) give the following result. Let

$$U_{\mu\nu}(z_1, \dots, z_4) = U'_{\mu\nu}(z_1, \dots, z_4) + \left[\tau_n a_n^{-1} - x_n^{-1} \right] T_{\mu\nu}(z_1, \dots, z_4) \quad (\text{IV. 16})$$

Then

$$U'_{\mu\nu}(\lambda z_1, \dots, \lambda z_4) = \lambda^{2\sigma_n - 2a_n - 5} U'_{\mu\nu}(z_1, \dots, z_4) \quad (\text{IV. 17})$$

One can now derive to order c_n the scaling law for $T_{\mu\nu}(z_1, \dots, z_4, 0, e_\lambda^2)$ with λ held fixed at $\lambda = 1$, i. e., e_λ^2 fixed at e_1^2 :

$$\begin{aligned} T_{\mu\nu}(s z_1, \dots, s z_4, 0, e_1^2) &= s^{2\sigma_n - 5} \left\{ 1 + c_n \left[\tau_n a_n^{-1} - x_n^{-1} \right] \right\} T_{\mu\nu}(z_1, \dots, z_4) \\ &+ c_n s^{2\sigma_n - 2a_n - 5} U'_{\mu\nu}(z_1, \dots, z_4) \end{aligned} \quad (\text{IV. 18})$$

Equation (IV.18) shows that the first order change in $T_{\mu\nu}(z_1, \dots, z_4, 0, e_1^2)$ when $e_1^2 \neq x_n$ is to renormalize the scale invariant term $T_{\mu\nu n}(z_1, \dots, z_4)$ by the factor $\left\{1 + c_n \left[\tau_n a_n^{-1} - x_n^{-1} \right] \right\}$ and to add the term $c_n U'_{\mu\nu n}(z_1, \dots, z_4)$ which obeys a separate scaling law. It is easily seen that when the above analysis is generalized to an arbitrary vacuum expectation value, there is a renormalization which is equivalent to a renormalization of the fields:

$$A_{1\mu}(z) \longrightarrow \left[1 - 1/2 c_n x_n^{-1} \right] A_{1\mu}(z) \quad (IV.19)$$

$$\psi_1(z) \longrightarrow \left[1 + 1/2 c_n \tau_n a_n^{-1} \right] \psi_1(z) \quad (IV.20)$$

and there is an extra term which always scales by an extra factor s^{-2a_n} relative to the scale invariant term.

Now consider the Lagrangian description of scale breaking.³¹ For convenience the subtraction momentum λ will be set equal to 1. Assume that changing the coupling constant from x_n to e_1^2 is equivalent to adding a term $(e_1^2 - x_n) \mathcal{L}_{e1}(z)$ to the Lagrangian density, where $\mathcal{L}_{e1}(z)$ is a finite local field. Then the term of first order in c_n in $T_{\mu\nu}$ can be obtained from lowest order perturbation theory:

$$T_{\mu\nu 1}(z_1, \dots, z_4, 0, e_1^2) \simeq T_{\mu\nu n}(z_1, \dots, z_4) + c_n \int \langle \Omega | T A_{1\mu}(z_1) A_{1\nu}(z_2) \psi_1(z_3) \bar{\psi}_1(z_4) \mathcal{L}_{e1}(y) | \Omega \rangle d^4 y \quad (IV.21)$$

where (from Eq. (IV.1)) $e_1^2 - x_n$ has been replaced by c_n and the vacuum expectation value multiplying c_n is computed in the unperturbed (scale invariant) theory.

Ultraviolet divergences could arise in the integral due to singularities when $y = z_1, z_2, z_3,$ or z_4 ; an infrared divergence could occur for $y \rightarrow \infty$. It will be assumed here that these divergences are absent or unimportant. If so the scaling properties of the integral are determined by scale invariance. Write

$$W_{\mu\nu}(z_1, \dots, z_4, y) = \langle \Omega | T A_{1\mu}(z_1) A_{1\nu}(z_2) \psi_1(z_3) \bar{\psi}_1(z_4) \mathcal{L}_{e1}(y) | \Omega \rangle \quad (IV.22)$$

The scale dimension of $A_{1\mu}$ is 1; the scale dimension of ψ_1 is $3/2 - \sigma_n$ (cf., Section III. C); let the scale dimension of \mathcal{L}_{e1} be d_e . Then from scale invariance

$$W_{\mu\nu}(sz_1, \dots, sy) = s^{2\sigma_n - 5 - d_e} W_{\mu\nu}(z_1, \dots, y) \quad (\text{IV.23})$$

Hence the scaling behavior of the integral in Eq. (IV.21) is

$$\begin{aligned} \int d^4y W_{\mu\nu}(sz_1, \dots, sz_4, y) &= s^4 \int d^4y' W_{\mu\nu}(sz_1, \dots, sz_4, sy') \\ &= s^{2\sigma_n - 5} s^{4 - d_e} \int d^4y' W_{\mu\nu}(z_1, \dots, z_4, y') \end{aligned} \quad (\text{IV.24})$$

So the prediction of the Lagrangian theory is that the term of order c_n in $T_{\mu\nu}$ scales with an extra factor s^{4-d_e} relative to the scale invariant term. There is no term of order c_n in Eq. (IV.21) which renormalizes the scale invariant term. However, we are using the unsubtracted form of perturbation theory which means one has no freedom to specify a normalization for the perturbed fields. In the renormalization group calculation a normalization is specified for the perturbed fields, and to achieve this normalization one must expect to add a renormalization term to Eq. (IV.21). Hence it is fair to interpret the explicit c_n term in Eq. (IV.21) as corresponding to $c_n U'_{\mu\nu n}(z_1, \dots, z_4)$. Comparing the scaling law for the c_n term of Eq. (IV.21) with the scaling law for $U'_{\mu\nu n}$, one must have

$$4 - d_e = -2a_n \quad (\text{IV.25})$$

i. e.,

$$d_e = 4 + 2a_n \quad (\text{IV.26})$$

In Ref. 24, the theory of scale breaking was stated in the form of a simple rule. Applied to this first order calculation the rule is that if the coefficient c_n of \mathcal{L}_{e1} is assigned the dimensions $(\text{mass})^{4-d_e}$, then the term of first order in c_n will have the same dimensions as the scale invariant term. The calculation performed here confirms this rule. In Ref. 24 this rule was hedged in that the terms

of first order (or higher) in c_n might involve logarithms that would spoil a strict scaling law. Logarithms might occur if the integral in Eq. (IV.21) required subtractions. The renormalization group calculation indicates that no logarithms occur in this case.

For the fixed point x_1 , a_1 is negative and e_λ^2 goes away from x_1 as λ decreases. In this case the scale breaking should become important for low momenta, i. e., large distances. It is clear from the scaling law (IV.18) that the scale breaking term does increase relative to the scale invariant term as the scale length s increases. The dimension of $\mathcal{L}_{e_1}(z)$ in this case is less than four, which means it is a generalized mass term in the language of Ref. 24. In contrast, for the fixed point x_2 , a_2 is positive and the departure from scale invariance increases as one goes to large momenta or short distances. This is what one expects of a nonrenormalizable interaction. In this case $\mathcal{L}_{e_1}(z)$ has dimension greater than four which means it is a nonrenormalizable interaction in the notation of Ref. 24.

If $\mathcal{L}_{e_1}(z)$ is the interaction associated with the constant e_1 in the Lagrangian, a term proportional to $\mathcal{L}_{e_1}(z)$ should be present in the Lagrangian even for $e_1 = x_n$. But if $\mathcal{L}_{e_1}(z)$ is a scale breaking interaction it obviously cannot be present in the Lagrangian of a scale invariant theory. Evidently, the part of the Lagrangian which describes the scale invariant theory must be distinguished from the term proportional to $\mathcal{L}_{e_1}(z)$ which describes departures of e_1^2 from x_n . This distinction must somehow arise in the process of defining the renormalized field $\mathcal{L}_{e_1}(z)$. It is difficult to study this problem in the context of this paper; it will not be discussed further.

In all the discussion of this paper it is assumed that none of the a_n are zero. For a_n to be zero means $\psi(0, x)$ has a double root (at least) at $x = x_n$ and the discussion of scale breaking is more complicated. Since the root of $\psi(0, x)$ at $x=0$

is a double root one has some experience with the scale breaking accompanying a double root, from ordinary perturbation theory. For general field theories a double root for $x_n \neq 0$ seems unlikely and will not be discussed further.

B. Scale Breaking Through Finite Mass

The next problem to be discussed is scale breaking due to nonzero mass. To analyze this problem it seems to be necessary to extend the renormalization group by defining a λ -dependent mass m_λ and obtaining a differential equation for m_λ . A method for doing this has been suggested by Ericksson.²⁹ The idea is to replace the subtraction condition on the mass shell for $\Sigma_\lambda(p)$ (Eq. (II.19) by a second condition at momentum equal to λ . This will mean essentially that the propagator $S_\lambda(p)$ reduces to $(\not{p} - m_\lambda)^{-1}$ when $p^2 = \lambda^2$ so the mass parameter m_λ is defined in terms the behavior of $S_\lambda(p)$ for $p \sim \lambda$. The parameter m_λ is then used instead of m to parameterize the mass dependence of amplitudes renormalized at momentum λ . The precise form of the new subtraction condition for $\Sigma_\lambda(p)$ has been chosen arbitrarily from many possibilities. The new subtraction condition is

$$B_\lambda(-\lambda^2) = 0 \quad (\text{IV.27})$$

($B_\lambda(p^2)$ was defined in Eq. (II.4)). This condition replaces Eq. (II.19).

Because of the new condition for Σ_λ , the fields $A_{\lambda\mu}$ and ψ_λ discussed below are different from the fields defined in Section II; also they and the functions d , s , etc., are functions of m_λ^2/λ^2 instead of m^2/λ^2 . In addition to the function $s(p^2/\lambda^2, m_\lambda^2/\lambda^2, e_\lambda^2)$, it is convenient to define a second function

$$s_M(p^2/\lambda^2, m_\lambda^2/\lambda^2, e_\lambda^2) = \{1 - A_\lambda(p^2)\}^{-1} \{1 + (m_\lambda)^{-1} B_\lambda(p^2)\} \quad (\text{IV.28})$$

The relation of $S_\lambda(p)$ to Σ_λ is

$$S_\lambda^{-1}(p) = \not{p} - m_\lambda - \Sigma_\lambda(p) \quad (\text{IV.29})$$

From this and the definitions of s and s_M one gets

$$S_\lambda(p) = s\left(p^2/\lambda^2, m_\lambda^2/\lambda^2, e_\lambda^2\right) \left\{ p - m_\lambda s_M\left(p^2/\lambda^2, m_\lambda^2/\lambda^2, e_\lambda^2\right) \right\}^{-1} \quad (\text{IV.30})$$

From the renormalization conditions (II.20) and (IV.27) one gets

$$s_M\left(-1, m_\lambda^2/\lambda^2, e_\lambda^2\right) = 1 \quad (\text{IV.31})$$

(The normalization conditions for $s\left(-1, m_\lambda^2/\lambda^2, e_\lambda^2\right)$ and $d\left(-1, m_\lambda^2/\lambda^2, e_\lambda^2\right)$ are that both equal 1, as before.)

Given the values of e_λ and m_λ and the functional form of s_M one can compute the physical mass m from the condition that $S_\lambda(p)$ have a pole for $p = m$, namely one must have

$$m = m_\lambda s_M\left(m^2/\lambda^2, m_\lambda^2/\lambda^2, e_\lambda^2\right) \quad (\text{IV.32})$$

This is in general an implicit equation for m ; in perturbation theory it has a unique solution for m in the form

$$m = m_\lambda + (\text{power series in } e_\lambda^2) \quad (\text{IV.33})$$

where the power series begins with a term of order e_λ^2 .

The renormalization group differential equations can now be derived as in Section II. The equations for e_λ^2 , $z_{3\lambda}$, and $z_{2\lambda}$ are unchanged in form. However, ψ and σ will be different functions than the functions in Section II because of the different renormalization conditions; also they depend on m_λ^2/λ^2 instead of m^2/λ^2 . In addition to these equations there is an equation for m_λ . Since $S_\lambda^{-1}(p)$ must be $(z_{2\lambda\lambda'})^{-1}$ times $S_{\lambda'}^{-1}(p)$ one has in particular

$$\begin{aligned} & \left\{ s\left(p^2/\lambda^2, m_\lambda^2/\lambda^2, e_\lambda^2\right) \right\}^{-1} m_\lambda s_M\left(p^2/\lambda^2, m_\lambda^2/\lambda^2, e_\lambda^2\right) \\ &= (z_{2\lambda\lambda'})^{-1} \left\{ s\left(p^2/\lambda'^2, m_{\lambda'}^2/\lambda'^2, e_{\lambda'}^2\right) \right\}^{-1} m_{\lambda'} s_M\left(p^2/\lambda'^2, m_{\lambda'}^2/\lambda'^2, e_{\lambda'}^2\right) \end{aligned} \quad (\text{IV.34})$$

Using Eq. (II. 52) for $z_{2\lambda\lambda'}$, setting $p^2 = -\lambda^2$, and using Eq. (IV. 31), one gets

$$m_\lambda = m_{\lambda'}, s_M(-\lambda^2/\lambda'^2, m_{\lambda'}^2/\lambda'^2, e_{\lambda'}^2) \quad (IV. 35)$$

Differentiate with respect to λ^2 , then put $\lambda'^2 = \lambda^2$, and perform some further manipulation, and one has

$$d(m_\lambda/\lambda)/d(\ln\lambda)^2 = (m_\lambda/\lambda) \phi_M(m_\lambda^2/\lambda^2, e_\lambda^2) \quad (IV. 36)$$

where

$$\phi_M(m_\lambda^2/\lambda^2, e_\lambda^2) = -.5 + \left[\partial s_M(-s, m_\lambda^2/\lambda^2, e_\lambda^2) / \partial s \right]_{s=1} \quad (IV. 37)$$

The first term $-.5$ in ϕ_M comes from differentiating the factor λ^{-1} in m_λ/λ . The second term is of order e_λ^2 since s_M has the form

$$s_M(-s, m_\lambda^2/\lambda^2, e_\lambda^2) = 1 + \text{order}(e_\lambda^2) \quad (IV. 38)$$

The function $s_M(-s, m_\lambda^2/\lambda^2, e_\lambda^2)$ is finite for $m_\lambda \rightarrow 0$. The factor $m_\lambda^{-1} B_\lambda(p^2)$ in the definition of s_M does not cause trouble in this limit, at least in perturbation theory, because every graph contributing to $B_\lambda(p^2)$ contains a factor m_λ . The reason for this is that if m_λ is zero, the theory is γ_5 invariant and in a γ_5 -invariant theory $\Sigma_\lambda(p)$ can only contain terms proportional to \not{p} . Since s_M is finite for $m_\lambda \rightarrow 0$ the function ϕ_M is also.

The basic equations of the renormalization group now consist of two coupled equations. If one writes x for e_λ^2 , y for m_λ/λ , and t for $\ln\lambda^2$, the coupled equations are

$$dx/dt = \psi_M(y^2, x) \quad (IV. 39)$$

$$dy/dt = y \phi_M(y^2, x) \quad (IV. 40)$$

The function ψ_M has the subscript to distinguish it from the function ψ of Sections II and III. It is a different function because of the new subtraction method used in its definition. However, for zero mass the two subtraction methods should

define the same theory so it will be assumed that $\psi_M(0, x) = \psi(0, x)$. The differential equations for x and y have the same form as the renormalization group equations for two coupling constant theories in the absence of mass. However, because of the factor y multiplying ϕ_M , any root x_n of $\psi_M(0, x) = 0$ automatically defines a fixed point $x = x_n$, $y = 0$ of the coupled equations. The coupled equations will be discussed here only in the neighborhood of these fixed points; a general discussion is beyond the scope of this paper. Near the fixed point $x = x_n$, $y = 0$, Eqs. (IV.39) and (IV.40) can be linearized, giving

$$dx/dt = a_n(x - x_n) \quad (IV.41)$$

$$dy/dt = b_n y \quad (IV.42)$$

where a_n is given by Eq. (III.31) and

$$b_n = \phi_M(0, x_n) \quad (IV.43)$$

The general solution of the linearized equations is

$$(x - x_n) = c_n e^{a_n t} \quad (IV.44)$$

$$y = d_n e^{b_n t} \quad (IV.45)$$

where c_n and d_n are arbitrary constants. Translated back in terms of e_λ^2 , etc., these equations read

$$e_\lambda^2 = x_n + c_n \lambda^{2a_n} \quad (IV.46)$$

$$m_\lambda = d_n \lambda^{1+2b_n} \quad (IV.47)$$

The scale breaking corrections to vacuum expectation values are easily determined by the method used previously. To order c_n and d_n the amplitude $T_{\mu\nu}$

is found to satisfy the scaling law

$$\begin{aligned} T_{\mu\nu}(sz_1, \dots, sz_4, m_1, e_1^2) &= s^{2\sigma_n-5} \left\{ 1 + c_n \left[\tau a_n^{-1} - x_n^{-1} \right] \right\} T_{\mu\nu n}(z_1, \dots, z_4) \\ &+ c_n s^{2\sigma_n-2a_n-5} U_{\mu\nu n}(z_1, \dots, z_4) + d_n s^{2\sigma_n-2b_n-5} V_{\mu\nu n}(z_1, \dots, z_4) \end{aligned} \quad (\text{IV.48})$$

where

$$V_{\mu\nu n}(z_1, \dots, z_4) = \left. \partial T_{\mu\nu} / \partial y(z_1, \dots, z_4, y, x_n) \right|_{y=0} \quad (\text{IV.49})$$

There are corresponding formulae for any vacuum expectation value. If there is a term $m_1 \mathcal{L}_{m_1}(z)$ in the Lagrangian density corresponding to the mass parameter m_1 , then $\mathcal{L}_{m_1}(z)$ has scale dimension $4 + 2b_n$; this follows from the same argument that gave the dimension $4 + 2a_n$ for $\mathcal{L}_{e_1}(z)$.

The constant b_n is known only for the fixed point $x = y = 0$ (call this $n = 0$) for which $b_0 = -.5$; in this case the interaction $\mathcal{S}_{m_1}(z)$ is a generalized mass term. For the nonzero fixed points such as x_1 and x_2 the sign of b_n is not known so $\mathcal{S}_{m_1}(z)$ could be either a generalized mass term or a nonrenormalizable interaction.³²

A peculiar situation arises if $\mathcal{L}_{m_1}(z)$ is a nonrenormalizable interaction, i. e., if $b_n > 0$. First one notes that if $\lambda = m$ then m_λ is of order m . The reason is this. The normalization condition on s_M (Eq. (IV.31)) puts $s_M(-1, m_\lambda^2/\lambda^2, e_\lambda^2) = 1$. Therefore, barring exceptional circumstances, $s_M(1, m_\lambda^2/\lambda^2, e_\lambda^2)$ should be of order 1. If $\lambda = m$ the mass condition is

$$m = m_m s_M(1, m_m^2/m^2, e_m^2) \quad (\text{IV.50})$$

With s_M of order 1, this equation requires m_m to be of order m , i. e., m_λ/λ is of order 1 for $\lambda = m$. For large λ , m_λ/λ must be small if finite mass corrections are to be small for large λ . This requires that m_λ/λ decrease as λ increases. But in the vicinity of the fixed point ($e_\lambda^2 \simeq x_n$, $m_\lambda/\lambda \simeq 0$), m_λ/λ decreases only

if b_n is negative. More generally, whether or not e_λ^2 is near a fixed point, m_λ^2/λ^2 decreases as λ increases only if $\phi_M(m_\lambda^2/\lambda^2, e_\lambda^2)$ is negative. There is no guarantee that $\phi_M(m_\lambda^2/\lambda^2, e_\lambda^2)$ is negative except when e_λ^2 is small. If m_λ^2/λ^2 does not decrease as λ increases then the asymptotic solution of Eqs. (IV.39) and (IV.40) for large t (i. e., large $\ln \lambda^2$) will not be one of the fixed points with $y=0$; instead it would be a fixed point with $y \neq 0$ or a limit cycle or some other type of behavior. However, if the physical mass is zero, then there is a special solution of the renormalization group equations with $m_\lambda = 0$ for all λ and the analysis of previous sections applies to this special solution.

A fixed point x_n with $b_n < 0$ will be called "mass-stable" while a fixed point with $b_n > 0$ will be called "mass-unstable." What we have shown is that only the mass-stable fixed points among the x_n will be relevant to finite mass theories. This conclusion can be restated; the conclusion is that for finite mass theories the interaction $\mathcal{L}_{m1}(z)$ must be a generalized mass term; if $\mathcal{L}_{m1}(z)$ is a nonrenormalizable interaction then only the zero mass theory exists. This confirms the assumptions of Ref. 24. More generally if there is a variable parameter in the low energy behavior of the field theory (either a mass or a renormalized coupling constant) the corresponding interaction Lagrangian density must be a generalized mass term, not a nonrenormalizable interaction. The case of an ordinary renormalizable interaction (one with dimension four exactly) hopefully does not occur for nontrivial fixed points such as x_1 , x_2 , or x_3 , since it would seem unlikely that the constants a_n or b_n would be exactly zero.

The analysis of this section shows that the question of mass-independence for large momenta is more complicated than the perturbation theory calculations indicate. As long as e_λ^2 is small, so that perturbation theory is valid, $\phi_M(m_\lambda^2/\lambda^2, e_\lambda^2)$ is approximately $-.5$, which means m_λ/λ decreases with λ and is small when

$\lambda \gg m$. But when e_λ^2 is of order 1, ϕ_M may be positive in which case m_λ/λ increases with λ and if so no mass independence is possible for large λ .

V. RENORMALIZATION GROUP AND STRONG INTERACTIONS

The purpose of this section is to discuss what form the renormalization group should take for strong interactions. It is assumed here that a renormalization group exists for strong interactions. The discussion is based on the work of Sections III and IV. However, no particular model (such as the gluon model⁵ or the σ -model⁶) is assumed here.

Analysis of the renormalization group for electrodynamics (see Section III. G) shows that the λ -dependent charge e_λ increases with λ , eventually becoming of order 1.^{1, 4} By this is meant that no matter how small the renormalized charge e is, e_λ becomes of order some fixed number independent of e if λ is large enough. This suggests that there is a cutoff Λ beyond which radiative corrections to strong interactions are too large to be treated as a perturbation. So it will be assumed here that the theory of strong interactions in isolation is valid only below the cutoff Λ . For purposes of discussion it will be assumed that electrodynamics rather than weak interactions or some other interaction is the cause of the cutoff.

It is evident from Section IV that including the mass parameter m_λ in the renormalization group equations makes the renormalization group method more powerful; so it will be assumed here that the renormalization group of strong interactions includes mass parameters as well as coupling constants. Furthermore, the equations of the renormalization group have the same form for mass parameters as for coupling constants provided one replaces mass parameters such as m_λ by dimensionless parameters such as m_λ/λ . The parameters which are distinguished in the renormalization group equations are those which break an internal symmetry. For example, m_λ/λ is a symmetry breaking parameter in electrodynamics (it breaks γ_5 symmetry), and as a result if m_λ/λ is zero for one value of λ , it is zero for all values of λ . More generally, the renormalization group respects the

possible internal symmetries of a field theory in the sense that if the parameters which break the symmetry are zero for one value of λ , then these parameters will remain zero for all values of λ .

A solution of the renormalization group equations for strong interactions should consist of a set of symmetry preserving coupling constants $g_{1\lambda}, g_{2\lambda}, \dots, g_{n\lambda}$ and a set of symmetry violating constants $h_{1\lambda}, h_{2\lambda}, \dots, h_{k\lambda}$. For the purposes of this discussion mass parameters are divided by λ and included among these "coupling constants". All coupling constants are to be dimensionless. (Coupling constants of any superrenormalizable interactions, such as a ϕ^3 interaction of a scalar field, are also divided by λ to make them dimensionless, and then included in the list.) For purposes of discussion the symmetries of strong interactions will be assumed to be P, C, T, and $U(3) \times U(3)$.³³ Experiment indicates that the symmetry violating part of strong interactions break $U(3) \times U(3)$ leaving P, C, T, isospin, and strangeness intact; the theoretical discussion given here will allow for more arbitrary types of symmetry breaking. The number n of symmetry conserving constants and the number k of symmetry violating constants will not be specified. To write the renormalization group equations it is convenient to introduce an abstract notation. Let P_λ be the point in an n-dimensional space (S_1) with coordinates $(g_{1\lambda}, g_{2\lambda}, \dots, g_{n\lambda})$, and let Q_λ be the point in a k-dimensional space (S_2) with coordinates $(h_{1\lambda}, \dots, h_{k\lambda})$. Then the general form of the renormalization group equations is

$$dP_\lambda/d(\ln \lambda^2) = T_1(P_\lambda, Q_\lambda) \quad (V.1)$$

$$dQ_\lambda/d(\ln \lambda^2) = T_2(P_\lambda, Q_\lambda) \quad (V.2)$$

where $T_1(P_\lambda, Q_\lambda)$ is itself a point in S_1 and $T_2(P_\lambda, Q_\lambda)$ is a point in S_2 , i. e., T_1 has n components and T_2 has k components. The point $T_2(P_\lambda, Q_\lambda)$ is zero when $Q_\lambda = 0$; T_1 is completely unknown.

The renormalization group will be discussed assuming the points P_λ, Q_λ go to a fixed point of the group when λ becomes large compared to a typical strong interaction mass (i. e., for $\lambda \gg 1$ GeV). As explained in Section III. H, experiments on $e^+ - e^-$ annihilation at large momentum transfers can probably distinguish between fixed point asymptotic behavior and other types of asymptotic behavior (such as a limit cycle). Until further experimental or theoretical information is available it seems more sensible to discuss the fixed point than to try to discuss more general asymptotic behavior; for example, one doesn't even have a classification of the possible asymptotic forms for solutions of more than two simultaneous nonlinear equations. (See however Section VI.) It is commonly assumed that $U(3) \times U(3)$ becomes an exact symmetry at small distances, i. e., large λ . Hence it will be assumed that the fixed point is of the form

$$P = P_f \quad (V.3)$$

$$Q = 0 \quad (V.4)$$

For P_f to be a fixed point one must have

$$T_1(P_f, 0) = 0 \quad (V.5)$$

According to the discussion of Section III. E, the fixed point should be infrared stable due to the presence of the cutoff Λ . The argument was that $P_\lambda - P_f$ and Q_λ are likely to be of order 1 when $\lambda \sim \Lambda$ due to large radiative corrections. Therefore $P_\lambda - P_f$ and Q_λ must decrease as λ decreases in order that $P_\lambda \simeq P_f$ and $Q_\lambda \simeq 0$ for $(1 \text{ GeV}) \ll \lambda \ll \Lambda$. But this is unlikely if the linearized equations for $P_\lambda - P_f$ and Q_λ have solutions which increase as λ decreases. Unfortunately this analysis leads to a nonsensical result. If Q_λ decreases as λ decreases, and is small for $(1 \text{ GeV}) \ll \lambda$, then Q_λ will be extremely small for $\lambda \sim 1$ GeV or less. But this would mean that $U(3) \times U(3)$ breaking would be small at laboratory energies,

whereas in fact $U(3) \times U(3)$ breaking is large at these energies. It is therefore necessary to presume that there are some $U(3) \times U(3)$ breaking parameters which increase as λ decreases. In particular one expects there to be two parameters, say $h_{1\lambda}$ and $h_{2\lambda}$, which break $SU(3) \times SU(3)$ according to the Glashow-Weinberg (Gell-Mann, Oakes, and Renner) theory.³⁴ We shall also assume there is a third parameter $h_{3\lambda}$, which preserves $SU(3) \times SU(3)$ but breaks $U(3) \times U(3)$.³⁵ These three parameters should increase as λ decreases, becoming of order 1 between 100 MeV and 1 GeV where $SU(3) \times SU(3)$ and $U(3) \times U(3)$ are strongly broken.

Since $h_{1\lambda}$, $h_{2\lambda}$, and $h_{3\lambda}$ are small for $\lambda \gg 1$ GeV and decreasing as λ increases they will be very small indeed when λ is of order Λ . This is possible only if there are no large radiative corrections to $h_{1\lambda}$, $h_{2\lambda}$, or $h_{3\lambda}$ when $\lambda \sim \Lambda$. It is hard to see how this can come about unless these coupling constants also break an electrodynamic symmetry. If they do break a symmetry of electrodynamics then electrodynamic corrections to $h_{1\lambda}$, etc., will be of order $h_{1\lambda} e_\lambda^2$, etc. instead of e_λ^2 and will not be a problem. This means there must be a symmetry common to electrodynamics and strong interactions which is broken by the couplings $h_{1\lambda}$, $h_{2\lambda}$, and $h_{3\lambda}$; a logical choice is axial baryon number since the usual electrodynamic Lagrangian for strong interactions preserves axial baryon number. This probably must be a symmetry of weak interactions also in order that weak corrections to $h_{1\lambda}$, etc., at large momenta not be large.

While the fixed point has to be infrared unstable with respect to the couplings $h_{1\lambda}$, $h_{2\lambda}$ and $h_{3\lambda}$, it must be infrared stable to symmetry breaking parameters which do get large radiative corrections for $\lambda \sim \Lambda$. For example $h_{4\lambda}$ might break $SU(3) \times SU(3)$ without breaking axial baryon number or other (electrodynamic + strong) symmetries; then $h_{4\lambda}$ will be large for $\lambda \sim \Lambda$ and must decrease as λ decreases. Also, it seems

likely that all coupling constants that preserve the symmetries of strong interactions, namely the constants P_λ , will have large radiative corrections and therefore P_f must be infrared stable to perturbations of P_λ about P_f . Since the theory defined by the coupling constants $(P_f, 0)$ is scale invariant (by the analysis of Section III. C), this means that the breaking of scale invariance at low momenta is due entirely to couplings which also break internal symmetries; in particular all generalized mass terms must break an internal symmetry. A generalized mass term is any coupling which causes particles to have finite mass rather than zero mass. It is interesting to note that there are no weakly coupled scalar particles in nature; scalar particles are the only kind of free particles whose mass term does not break either an internal or a gauge symmetry.

This discussion can be summarized by saying that mass or symmetry breaking terms must be "protected" from large corrections at large momenta due to various interactions (electromagnetic, weak, or strong). A symmetry breaking term, such as $h_{1\lambda}$, $h_{2\lambda}$ or $h_{3\lambda}$, is protected if, in the renormalization group equation for $h_{1\lambda}$, $h_{2\lambda}$, or $h_{3\lambda}$, the right hand side is proportional to $h_{1\lambda}$, $h_{2\lambda}$, $h_{3\lambda}$ or other small coupling constants even when high order strong, electromagnetic or weak corrections are taken into account. The mass terms for the electron and muon and the weak boson, if any, must also be protected. This requirement means that weak interactions cannot be mediated by scalar particles.³⁶

One basic mystery remains from this analysis, namely why is the breaking of axial baryon number small when $\lambda \sim \Lambda$; even if the mixing of electrodynamics with strong interactions does not force the breaking to be large it is strange that it is small without being zero.

According to the analysis of Section III. E, all the renormalized coupling constants of strong interactions could be computed by solving the renormalization

group equations. This is no longer true. There is no argument that can determine $h_{1\lambda}$, $h_{2\lambda}$, or $h_{3\lambda}$ for $\lambda \sim 1$ GeV and the values of these constants for one value of λ must be determined from experiment. The renormalization group can then be used to fix the values of $h_{1\lambda}$, $h_{2\lambda}$, and $h_{3\lambda}$ for other values of λ . If there are other coupling constants which increase as λ decreases, these coupling constants must also be determined from experiment. Such coupling constants will be small for $\lambda \sim \Lambda$ and therefore must also be protected. This presumably means these constants are also symmetry breaking terms. Hopefully, the dominant symmetry breaking terms are $h_{1\lambda}$, $h_{2\lambda}$, and $h_{3\lambda}$; then other symmetry breaking terms, while surely present, are small for $\lambda \sim 1$ GeV, and cannot increase further for $\lambda \ll 1$ GeV because amplification ceases for λ less than the hadron masses (see Section III. B).

The renormalization group for strong interactions contains mass terms and coupling constants for any superrenormalizable interactions. Should it include nonrenormalizable interactions? The answer is Yes, for several reasons. It was shown in Section IV that the interaction Lagrangian density $\mathcal{L}_{e1}(x)$ is a nonrenormalizable interaction in the neighborhood of an infrared-stable fixed point. This will also be true of the interactions associated with the nonsymmetry breaking couplings $g_{1\lambda} \dots g_{n\lambda}$ of strong interactions, since the fixed point P_f must be infrared-stable except for symmetry breaking. So in effect some nonrenormalizable interactions are already present in the renormalization group. Conversely, there is no reason to suppose that a symmetry breaking interaction which is nonrenormalizable in perturbation theory will stay nonrenormalizable near a fixed point with large coupling constants: for example the $U(3) \times U(3)$ breaking constant $h_{3\lambda}$ might correspond to a nonrenormalizable interaction in perturbation theory (especially in the gluon model where in perturbation theory there are no renormalizable interactions or mass terms which break $U(3) \times U(3)$ without breaking

$SU(3) \times SU(3)$ also). So it may be essential to include interactions which are nonrenormalizable in perturbation theory to find all the generalized mass terms near a strongly interacting fixed point. Furthermore, there has never been any fundamental physical distinction between nonrenormalizable interactions and renormalizable ones so one would like to treat them on an equal footing. Finally there is a model with a renormalization group which can be solved rigorously in strong coupling which necessarily includes nonrenormalizable interactions.³⁷

If the renormalization group for strong interactions includes nonrenormalizable couplings it will be difficult to construct it as a simple extension of the Gell-Mann-Low group, requiring instead that one start from scratch. It will also be considerably more complicated than the Gell-Mann-Low group since there are an infinite number of nonrenormalizable interactions. Whether the conclusions of this paper actually apply to such a group remains to be seen; but surely these conclusions are an indication of the kind of physics that can come out of such a group. The inclusion of nonrenormalizable interactions in the renormalization group equations does not change the conclusion that scale breaking in strong interactions is due only to generalized mass terms, provided that the asymptotic solution of the renormalization group is a fixed point.

VI. The $\Delta I = 1/2$ RULE

It has been shown elsewhere²⁴ that one can understand the $\Delta I = 1/2$ rule in nonleptonic weak interactions given that strong interactions are scale invariant at short distances and that dimensions of local fields in strong interactions are not as predicted from canonical field theory. The purpose of this section is to show that if the strong interactions have a renormalization group then the $\Delta I = 1/2$ rule can be understood without assuming broken scale invariance. In other words, the $\Delta I = 1/2$ rule can be understood regardless of what kind of asymptotic solution the renormalization group equations have (fixed point, limit cycle, or otherwise).

The first part of the analysis of Ref. 24 will be assumed here. According to this analysis, the current-current Lagrangian for nonleptonic weak interactions can be approximated by the form

$$\mathcal{L}_w(x) \simeq \sum_n G_n O_n(x) \quad (\text{VI. 1})$$

where the fields $O_n(x)$ are a set of local fields and the G_n are constants. This form assumes that the weak boson mass or weak interaction cutoff is of order M where $M \gg (1 \text{ GeV})$. The fields $O_n(x)$ are assumed to belong to irreducible representations of $SU(3) \times SU(3)$.

Before discussing the consequences of the renormalization group for $\mathcal{L}_w(x)$, we shall try to explain what is meant by "understanding" the $\Delta I = 1/2$ rule. What will be argued in the following is that $\mathcal{L}_w(x)$ is dominated by a single field out of the set $\{O_n(x)\}$, say $O_m(x)$. The argument below gives no clue as to which field dominates. This means that $\mathcal{L}_w(x)$, to a good approximation, belongs to a single irreducible representation of $SU(3) \times SU(3)$, despite the fact that the current-current product contains several representations of $SU(3) \times SU(3)$. However, one cannot determine theoretically which representation of $SU(3) \times SU(3)$ will dominate. The current-current product contains the following representations

of $SU(3) \times SU(3)$: $(1, 1)$; $(8, 1) \oplus (1, 8)$; $(27, 1) \oplus (1, 27)$; $(10, 1) \oplus (1, 10)$; $(\bar{10}, 1) \oplus (1, \bar{10})$; and $(8, 8)$. In addition, since $SU(3) \times SU(3)$ is not an exact symmetry there is some leakage into other representations as well.

The theoretical analysis given below predicts that one $SU(3) \times SU(3)$ representation will dominate in $\mathcal{L}_W(x)$, and the dominance is by a power of M over all other representations. One has to look to experiment to find out which $SU(3) \times SU(3)$ representation dominates. It is obvious from the K-decay amplitudes that the dominant $SU(3) \times SU(3)$ representation contains no $\Delta I = 3/2$ term. The factor 20 that separates the $\Delta I = 1/2$ amplitude in K-decay from the $\Delta I = 3/2$ amplitude is too large to be accounted for credibly without having $\Delta I = 1/2$ dominance in the effective Lagrangian $\mathcal{L}_W(x)$. This limits the possible dominant $SU(3) \times SU(3)$ representations to 2: $(8, 1) \oplus (1, 8)$ and $(3, \bar{3}) \oplus (\bar{3}, 3)$ (The representation $(8, 1) \oplus (1, 8)$ seems more likely since it does not require leakage in order to occur.)

One uses K-decay to choose the dominant $SU(3) \times SU(3)$ representation. The theoretical argument then predicts the following otherwise mysterious facts: 1) The large dominance of $\Delta I = 1/2$ over $\Delta I = 3/2$ in K-decay; this is possible because the dominance is by a power of M , and this factor should be large providing that M is large compared to strong interaction masses (i. e., $M \gg (1 \text{ GeV})$); 2) Large $\Delta I = 1/2$ dominance in baryon nonleptonic decays, i. e., universal $\Delta I = 1/2$ dominance.

The theoretical analysis is the following. In a renormalization group analysis of \mathcal{L}_W , it is characterized by a set of λ -dependent coupling constants $G_{n\lambda}$. When λ is of order M these coupling constants are the same size for different n . In particular the coupling constants corresponding to different $SU(3) \times SU(3)$ representations have the same order of magnitude. This is because there is nothing peculiar about two currents separated by a distance M^{-1} if it is in a matrix element where all fields are separated by of order M^{-1} . But the constants $G_{n\lambda}$ for $\lambda \sim M$

measure the size of $\mathcal{L}_w(x)$ when sandwiched between fields separated by $\sim M^{-1}$. Thus the various $SU(3) \times SU(3)$ representations should have roughly the same strength in $\mathcal{L}_w(x)$ when $\lambda \sim M$. The coupling constants which determine decay rates are the $G_{n\lambda}$ with $\lambda \lesssim 1$ GeV. These must be determined by solving the renormalization group equations between $\lambda = 1$ GeV and $\lambda = M$. But as shown in Section III. B there can be large amplification or deamplification effects in this interval (assuming $M \gg 1$ GeV), and these effects can be different for different $SU(3) \times SU(3)$ representations.

To be more precise, consider the form of the renormalization group equations for $G_{n\lambda}$. Assume for convenience that each n refers to a different $SU(3) \times SU(3)$ representation. The equations can be linearized with respect to the $G_{n\lambda}$; the symmetry violating constants $h_{n\lambda}$ will be neglected (in the region above 1 GeV which is important for producing amplification or deamplification the $h_{n\lambda}$ are small and will not change the analysis appreciably). Then the renormalization group equations have the form

$$dG_{n\lambda}/d(\ln \lambda^2) = G_{n\lambda} U_n [P_\lambda] \quad (VI.2)$$

where P_λ is the set of symmetry conserving coupling constants for momentum λ . The U_n are unknown functions of P_λ . There are no cross terms relating $dG_{n\lambda}/d(\ln \lambda^2)$ to $G_{\ell\lambda}$ with $\ell \neq n$ because of symmetry requirements. The solution of these equations is

$$G_{n\lambda} = G_{nM} \exp \left\{ \int_M^\lambda 2 U_n [P_{\lambda'}]^{-1} d\lambda' \right\} \quad (VI.3)$$

Equivalently, one can write

$$G_{n\lambda} = (\lambda^2/M^2)^{\alpha_n(\lambda)} G_{nM} \quad (VI.4)$$

where $\alpha_n(\lambda)$ is

$$\alpha_n(\lambda) = [\ln \lambda^2 - \ln M^2]^{-1} \int_M^\lambda 2U_n(P_{\lambda'}) (\lambda')^{-1} d\lambda' \quad (\text{VI.5})$$

If U_n were a constant independent of λ' , then $\alpha_n(\lambda)$ would be equal to U_n . The constant $\alpha_n(\lambda)$ is presumably of order 1, in the absence of more precise information, but should be different for different n . If the asymptotic solution of the renormalization group equations is a fixed point then U_n would be constant for $\lambda' \gg 1$ GeV where $P_{\lambda'}$ is near the fixed point. In this case α_n would be related to the dimension of the field O_n ; as long as the fields O_n have different dimensions, α_n would also be different for different n . Even if the renormalization group does not have fixed point asymptotic behavior there is an exponent $\alpha_n(1 \text{ GeV})$ which defines an "effective dimension" for the fields O_n over the interval $1 \text{ GeV} < \lambda' < M$. The exponents $\alpha_n(1 \text{ GeV})$ determine the amount of amplification or deamplification that results from going from G_{nM} to $G_{n\lambda}$ with $\lambda \sim 1 \text{ GeV}$. As long as the α_n are different for different n , the low energy coupling constants $G_{n\lambda}$ with $\lambda \sim 1 \text{ GeV}$ will differ by powers of M for different n . This ensures the dominance of one $SU(3) \times SU(3)$ representation in the phenomenological Lagrangian for nonleptonic decays.

The existence of the $\Delta I = 1/2$ rule experimentally is encouragement to believe that a renormalization group does exist for strong interactions, since all other explanations of the $\Delta I = 1/2$ rule are unsatisfactory for one reason or another.^{24, 25}

VII. FINAL REMARKS

The application of the renormalization group to strong interactions leads to profound results, for example, the possibility that the short distance behavior of strong interactions is described by a limit cycle. It is disturbing, therefore, that the renormalization group results derive from a not very profound property of perturbation theory, namely that renormalized amplitudes renormalized by the

Gell-Mann-Low method have a zero mass limit. Furthermore, the renormalization group is not involved in the standard procedures for solving field theory, yet the renormalization group analysis makes important predictions about the nature of the solutions of field theory. One would be happier about these predictions if the renormalization group itself were a more essential part of the structure of field theory than it appears to be from perturbation theory.

There is a model field theory described elsewhere³⁷ which suggests that the renormalization group is an essential part of understanding a strongly coupled field theory. The model is a truncated version of the charged scalar theory of pions coupled to a fixed source. In the truncated version of this theory the pi mesons are restricted to discrete wave packet states centered on the source; the n th state has mean momentum $m\Lambda^n$, where m is the pion mass and Λ is a large number. The renormalization group for the model is defined as part of solving the model as an expansion in Λ^{-1} . The renormalization group of the model has the following properties:

- 1) the renormalization group is defined before the solution of the theory is known,
- 2) the renormalization group equations of the model involve coupling constants of all possible nonrenormalizable interactions in the model,
- 3) there is no way (known to the author, at least) of obtaining a complete solution of the model, except by solving the renormalization group equations.

The author suspects these three properties will also be true of strongly interacting relativistic fields. This is because the reason for the importance of the renormalization group in the model is that it has an infinite number of disparate scales of energy, namely the energy scales m , $m\Lambda$, $m\Lambda^2$, etc. The function of the renormalization group transformation is to solve the part of the Hamiltonian

involving energies of one scale, say $m\Lambda^n$, assuming parts of the Hamiltonian with energies $\gg m\Lambda^n$ have already been solved. In order to solve the infinite number of energy scales which exceed energies of practical interest one must iterate the renormalization group transformation an infinite number of times thus making asymptotic behavior of this transformation of crucial importance in solving the model. (For details, see Ref. 37). All relativistic field theories also have this infinite sequence of energy scales; they arise due to the possibility of creating and particles of any energy from m to ∞ . The problem in relativistic theory is that one does not have only the discrete energy scales m , $m\Lambda$, etc.; one has all energies in between these values, so a perturbation expansion in Λ^{-1} is impossible for relativistic theories. This does not mean one does not have disparate energy scales present. In the model one can only solve one order of magnitude of energy at a time; it is hard to see how one can do more than this in a relativistic theory. So in a relativistic theory one should also look for a renormalization group transformation which solves one order of magnitude of energies in the Hamiltonian.³⁸

This discussion will not be pursued here; its purpose is only to emphasize that the next step after reading this paper is to study the model of Ref. 37.

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APPENDIX:

REVIEW OF THE GELL-MANN-LOW THEORY

There are many reasons why the Gell-Mann-Low theory¹ is hard to understand. But perhaps the most puzzling feature of it is that the reasoning and assumptions which led Gell-Mann and Low to formulate the differential equations of the renormalization group seem totally irrelevant to the conclusions one derives from the differential equations. In Gell-Mann and Low's paper the motivation for setting up the renormalization group was the observation that while the renormalized photon propagator $d_c(k^2/m^2, e^2)$ is logarithmically divergent when $m^2 \rightarrow 0$, the cutoff but unrenormalized Feynman graphs for the propagator are not divergent in this limit (at least in low orders). Gell-Mann and Low therefore propose a generalization of the usual renormalization procedure which is not divergent in low orders when $m^2 \rightarrow 0$; they then discuss properties of their renormalized theory which leads them to derive the differential equation (I.2). One's reaction to going through this analysis is that one is looking at rather trivial and technical aspects of renormalized perturbation theory which cannot be the basis for any very basic results. In the author's opinion the analysis of Gell-Mann and Low is in fact dealing with technical aspects of field theory, but by doing so they have stumbled on an equation which embodies very fundamental properties of quantum field theory. New ideas are often discovered for irrelevant reasons, so this opinion is not unreasonable. Also, perturbation theory is the one approximation method which can be computed without any understanding whatsoever of the equations one is solving by perturbation theory. What this means is that while qualitative features of the original equations will be reflected in qualitative features of its perturbation expansion, to understand these qualitative features one has to study the original

equations and not just their perturbation expansion. So perhaps one should expect that a derivation of the renormalization group equations from perturbation theory will not be very illuminating.

The Gell-Mann-Low theory will now be reviewed, much in the manner that Gell-Mann and Low presented it. The only major change is not to use Ward's method of renormalization³⁹ as a basis for modifying the usual renormalization program. The author has also benefited from the review of the theory in Bogoliubov and Shirkov;⁴ however, they provide very little motivation for the calculations they describe.

Consider lowest order vacuum polarization. If one computes this graph and renormalizes it in the conventional fashion it gives a contribution to vacuum polarization, denoted $\Pi_{c\mu\nu}(k)$, (k is the photon momentum) which is:

$$\Pi_{c\mu\nu}(k) = (e^2/4\pi^2) (g_{\mu\nu} k^2 - k_\mu k_\nu) I(k^2) \quad (\text{A.1})$$

The exact form of $I(k^2)$ is given in Bogoliubov and Shirkov.⁴⁰ If $m^2 < k^2$, $I(k^2)$ is approximately

$$I(k^2) \simeq (1/3) \ln |k^2/m^2| \quad (\text{A.2})$$

In the limit $m \rightarrow 0$, $I(k^2)$ is logarithmically divergent. Gell-Mann and Low point out that if one looks instead at the unrenormalized but cutoff vacuum polarization diagram, it does not diverge as $m \rightarrow 0$. It is worth showing this in a way that is generalized easily to higher order diagrams. A cutoff version of the graph gives a function $\Pi_{\Lambda\mu\nu}(k)$:

$$\Pi_{\Lambda\mu\nu}(k) = -ie^2 \int_p \text{Trace} \left\{ \gamma_\mu \frac{(\not{p} + m) \gamma_\nu (\not{p} - \not{k} + m)}{(p^2 - m^2 + i\epsilon)[(p-k)^2 - m^2 + i\epsilon]} \right\} \frac{\Lambda^4}{(p^2 - \Lambda^2 + i\epsilon)[(p-k)^2 - \Lambda^2 + i\epsilon]} \quad (\text{A.3})$$

where \int_p is shorthand for $(2\pi)^{-4} \int d^4p$ and Λ is the cutoff.⁴¹

It is convenient to consider spacelike k and choose a Lorentz frame in which the time component k_0 is zero. In this frame one can rotate the contour of integration in p_0 from the real axis to the imaginary axis (no singularities of the integrand are crossed if the rotation is counterclockwise). This means one replaces p_0 by ip_4 where p_4 is real. The denominator of the integrand now has the form $(p^2 + m^2)[(p - k)^2 + m^2](p^2 + \Lambda^2)[(p - k)^2 + \Lambda^2]$ where p^2 now means $p_4^2 + p_4^2$ and likewise for $(p - k)^2$, (i. e., one has a Euclidean metric instead of a Lorentz metric).

With cutoffs present the integral has no ultraviolet divergences, so the only way it can diverge is through vanishing denominators. If m^2 is zero the denominators p^2 and $(p - k)^2$ can vanish. Since the metric is Euclidean, p^2 can vanish only if all four components of p vanish. Hence the integral is divergent only if the integrand is as singular as p^{-4} for $p \rightarrow 0$ or as $(p - k)^{-4}$ when $(p - k) \rightarrow 0$, which is not the case.

Even if k is zero, so the denominator behaves as $(p^2)^2$ for $p \rightarrow 0$, the integral does not diverge because there are two powers of p from the numerator when k and m are both zero.

Why then does the renormalized graph diverge for $m \rightarrow 0$? The reason lies in the way one defines the renormalized function $\Pi_{c\mu\nu}(k)$. The function $\Pi_{\Lambda\mu\nu}(k)$ contains terms proportional to $g_{\mu\nu}$, $g_{\mu\nu}k^2$, and $k_\mu k_\nu$, which are cutoff dependent and must be subtracted in order to give a finite result. The customary procedure is to subtract the expansion in k of $\Pi_{\Lambda\mu\nu}(k)$ to order k^2 , namely to define

$$\Pi_{c\mu\nu}(k) = \Pi_{\Lambda\mu\nu}(k) - \Pi_{\Lambda\mu\nu}(0) - k^\pi k^\sigma \Pi_{\Lambda\mu\nu, \pi\sigma}(0) \quad (\text{A. 4})$$

where

$$\Pi_{\Lambda\mu\nu, \pi\sigma}(k) = \partial^2 \Pi_{\Lambda\mu\nu}(k) / \partial k^\pi \partial k^\sigma \quad (\text{A. 5})$$

With this form of subtraction, $\Pi_{c\mu\nu}(k)$ is of order k^4 when $k \rightarrow 0$; this means that the radiative correction to the photon propagator, which is proportional to $(k^2)^{-2}\Pi_{c\mu\nu}(k)$, is finite as $k \rightarrow 0$. Hence the pole of the exact photon propagator is the same as the pole term in the free propagator.

The divergence for $m \rightarrow 0$ in the renormalized vacuum polarization comes from the subtraction $\Pi_{\Lambda\mu\nu, k\sigma}(k)$. One can see that differentiating $\Pi_{\Lambda\mu\nu}(k)$ twice with respect to k and then setting $k = 0$ makes the integral in Eq. (A.3) diverge: each differentiation makes the integrand more singular by one power of p at $p = 0$, and hence the integrand for $\Pi_{\Lambda\mu\nu, k\sigma}(0)$ behaves as p^{-4} when $m = 0$ causing a divergence.

Low order diagrams for the electron propagator and the vertex function show a similar feature, namely that the unrenormalized but cutoff Feynman integrals are finite for zero mass. One shows these results using the same method described above for the vacuum polarization graph. To get a divergence requires that several denominators vanish simultaneously and for low order graphs it is trivial to check that this cannot happen. In the case of the vertex function, one needs some non-zero external momenta to prevent too many denominators from vanishing simultaneously (and likewise for four point functions, five point functions, etc.). In other words the external momenta provide an infrared cutoff for vertex function diagrams. This was noted by Gell-Mann and Low. Also one must treat specially graphs with self-energy corrections on internal lines. Otherwise one can get a string of propagators depending on a single momentum p say and all diverging when $p \rightarrow 0$. One can sum up the self-energy corrections to internal lines to give exact propagators; an exact propagator has at worst a simple pole when $p^2 \rightarrow 0$. However, the condition for an exact propagator to be singular at momentum zero is that the physical mass of the electron be zero,

not that the bare mass of the electron be zero. This is why the problem of infrared divergences arises when the physical electron mass is zero and not necessarily when the bare mass of the electron is zero.

It is not trivial to test all higher orders graphs for divergences at zero mass because there are so many diagrams to be checked.

A similar analysis can be made for the electron propagator with finite mass. There are infrared divergences due to the photon mass being zero in the renormalized electron propagator. However unrenormalized but cutoff diagrams of low order do not show this divergence either. One shows this by considering the electron propagator with time-like momentum p in the rest frame of p . If $|p_0| \leq m$ one can again rotate contours of integration so that internal momenta are in a Euclidean metric. Consider an internal electron line in an electron self-energy graph carrying momentum $p-k$ where k is an internal momentum. After the rotation the propagator for the line has the form

$$[\gamma_0(p_0 - ik_4) + \underline{\gamma} \cdot \underline{k} + m] \left[-2ip_0k_4 - k_4^2 - \underline{k}^2 - (m-p_0)(m+p_0) \right]^{-1}$$

The propagator diverges if and only if $p_0 = m$ and all four components of k vanish. To produce a divergent integral requires several denominators to vanish simultaneously; to produce such a divergence in an electron self-energy graph one must differentiate at least once with respect to p_0 and then put $p_0 = m$. This is in fact done as part of the conventional renormalization procedure for the electron propagator (see below), but if one does not renormalize the low order graphs for the electron self energy, they are finite for zero photon-mass.

Since the zero mass singularities of amplitudes seem to come in from the subtractions of the conventional renormalization program, Gell-Mann and Low propose that one set up an alternative method of renormalization which will not introduce such singularities. The basic idea can be illustrated with lowest order

vacuum polarization. Suppose the unrenormalized but cutoff vacuum polarization $\Pi_{\Lambda\mu\nu}(k)$ is defined using Pauli-Villars regularization;⁴² then $\Pi_{\Lambda\mu\nu}(k)$ has the form

$$\Pi_{\Lambda\mu\nu}(k) = (e^2/4\pi^2) (g_{\mu\nu} k^2 - k_\mu k_\nu) I(k^2, \Lambda^2) \quad (\text{A.6})$$

where $I(k^2, \Lambda^2)$ will not be written down explicitly here. The cutoff dependence in $I(k^2, \Lambda^2)$ is in a constant independent of k^2 , and by custom it is removed by defining the renormalized function $I(k^2)$ as

$$I(k^2) = I(k^2, \Lambda^2) - I(0, \Lambda^2) \quad (\text{A.7})$$

The singularity can also be removed by subtracting $I(-\lambda^2, \Lambda^2)$ from $I(k^2, \Lambda^2)$, where λ is arbitrary. (The subtraction is made at a space-like momentum because one wants the subtraction to be real: see below.) So one can define a whole set of possible renormalized functions $I_\lambda(k^2)$ by

$$I_\lambda(k^2) = I(k^2, \Lambda^2) - I(-\lambda^2, \Lambda^2) \quad (\text{A.8})$$

The function $I(k^2, \Lambda^2)$ is finite for $m \rightarrow 0$ provided k^2 is nonzero. If $k^2 = 0$ then I is proportional to a second derivative of $\Pi_{\Lambda\mu\nu}(k)$ with respect to k , which diverges logarithmically for $m \rightarrow 0$. This means that $I_\lambda(k^2)$ is finite for $m \rightarrow 0$ if λ and k^2 are held fixed and neither is zero. So in summary $I(k^2)$ diverges for $m \rightarrow 0$ for any value of k^2 (except $k^2 = 0$), whereas $I_\lambda(k^2)|_{m=0}$ is finite for $k^2 \neq 0$ but diverges for the special value $k^2 = 0$.

In general the proposal of Gell-Mann and Low is that when making subtractions to remove divergences from the unrenormalized theory, these subtractions should be made at a subtraction momentum λ , rather than at momentum 0 or at the electron mass m . As a result one gets, in low orders at least, amplitudes which are finite at zero electron mass except for special values of the external momenta. The photon propagator of the zero mass theory will not have just a simple pole singularity for $k = 0$: it will have logarithmic singularities multiplying the pole. The same

is true of the electron propagator on the mass shell. These results are no surprise because the same is true of the electron propagator of the finite mass theory, due to photon infrared divergences.

The renormalization procedure with subtractions made at a momentum λ must obey the same restrictions as conventional renormalization. That is, the renormalized theory is allowed to differ only in the following respects from the unrenormalized cutoff theory:

- a) The renormalized electron and photon fields can differ from the unrenormalized fields by renormalization constants.
- b) The renormalized theory can be reparameterized in terms of a phenomenologically defined coupling constant and mass in place of the bare coupling constant and bare mass.
- c) One can make gauge transformations at will; a gauge transformation is accomplished by adding a term proportional to $k_\mu k_\nu$ to the free photon propagator.

The substitution of $I_\lambda(k^2)$ for $I(k^2)$ is equivalent to a renormalization and a change of gauge. To see this one must compare the photon propagators for the two cases. The standard renormalized propagator to order e^2 is⁴³

$$D_{\text{c}\mu\nu}(k) = (k^2)^{-1} \left\{ -g_{\mu\nu} + (e^2/4\pi)(k^2)^{-1} \left(k_\mu k_\nu - g_{\mu\nu} k^2 \right) I(k^2) \right\} \quad (\text{A. 9})$$

The function $I_\lambda(k^2)$ may be written

$$I_\lambda(k^2) = I(k^2) - I(-\lambda^2) \quad (\text{A. 10})$$

Hence the renormalized propagator $D_{\lambda\mu\nu}(k)$ of Gell-Mann and Low is

$$D_{\lambda\mu\nu}(k) = D_{\text{c}\mu\nu}(k) - (e^2/4\pi)(k^2)^{-2} \left(k_\mu k_\nu - g_{\mu\nu} k^2 \right) I(-\lambda^2) \quad (\text{A. 11})$$

One can replace $e^2 g_{\mu\nu} (k^2)^{-1}$ by $-e^2 D_{\text{c}\mu\nu}(k)$ since the difference of these two expressions is of order e^4 which is being neglected anyways. So one has,

neglecting order e^4 ,

$$D_{\lambda\mu\nu}(k) = z_{3\lambda} D_{c\mu\nu}(k) - (e^2/4\pi) (k^2)^{-2} k_\mu k_\nu I(-\lambda^2) \quad (\text{A.12})$$

where

$$z_{3\lambda} = 1 - (e^2/4\pi) I(-\lambda^2) \quad (\text{A.13})$$

The remaining $k_\mu k_\nu$ term can be absorbed into a change of gauge of the conventionally renormalized propagator. If one adds $-(e^2/4\pi) (k^2)^{-2} k_\mu k_\nu I(-\lambda^2)$ to the free propagator of the conventional theory, then the complete propagator of the conventional theory is $D'_{c\mu\nu}(k)$ with

$$D'_{c\mu\nu}(k) = (k^2)^{-1} \left\{ -g_{\mu\nu} - (e^2/4\pi) g_{\mu\nu} I(k^2) + (e^2/4\pi) k_\mu k_\nu (k^2)^{-1} \left[I(k^2) - I(-\lambda^2) \right] \right\} \quad (\text{A.14})$$

and

$$D_{\lambda\mu\nu}(k) = z_{3\lambda} D'_{c\mu\nu}(k) \quad (\text{A.15})$$

neglecting terms of order e^4 .

Having to make a change of gauge when comparing the Gell-Mann-Low propagator with the conventional propagator is a nuisance. To simplify this problem, Gell-Mann and Low choose the gauge of the free propagator such that the exact propagator is in the Feynman gauge no matter which renormalization procedure is used. The Gell-Mann-Low prescription means that the exact propagator $D_{c\mu\nu}(k)$ to order e^2 is

$$D_{c\mu\nu}(k) = (k^2)^{-1} \left\{ -g_{\mu\nu} - (e^2/4\pi^2) g_{\mu\nu} I(k^2) \right\} \quad (\text{A.16})$$

and the corresponding free propagator is

$$(k^2)^{-1} \left\{ -g_{\mu\nu} - (e^2/4\pi^2) (k^2)^{-1} k_\mu k_\nu I(k^2) \right\}$$

The propagator $D_{\lambda\mu\nu}(k)$ is $z_{3\lambda} D_{c\mu\nu}(k)$ and the corresponding free propagator is

$$(k^2)^{-1} \left\{ -g_{\mu\nu} - (e^2/4\pi^2) (k^2)^{-1} k_\mu k_\nu L_\lambda(k^2) \right\}$$

Gell-Mann and Low calculate all Feynman graphs using the exact propagator for internal lines instead of the free propagator. As a result it is of no importance in practice that the free propagator undergoes a gauge change when the renormalization method is changed. Furthermore, to compute the exact propagator one computes only the $g_{\mu\nu}$ term in vacuum polarization. In contrast, Bogoliubov and Shirkov do not allow the free propagator to depend on e^2 ; instead they work mainly in the Landau gauge, or else make a change of gauge as the renormalization method is changed. Bogoliubov and Shirkov claim that the Gell-Mann-Low treatment is wrong; I see nothing wrong with it and will use the Gell-Mann-Low approach in the following.

The subtraction momentum λ can be chosen arbitrarily. However the renormalized theory will have an apparently nontrivial dependence on λ . Nevertheless, the physical consequences of the theory must be independent of λ . The transformations which connect the renormalized theories with different values of λ are the renormalization group transformations. They are discussed in Section II.

The above discussion should make clear the ideas involved in generalizing the usual renormalization procedure such that subtractions are made at a momentum λ rather than on the photon or electron mass shell.

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16. The renormalization group equations for pseudoscalar meson theory are formu-
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does not know what kind of solution of the renormalization group equations cor-
responds to a theory with scale invariance but a noninvariant vacuum.

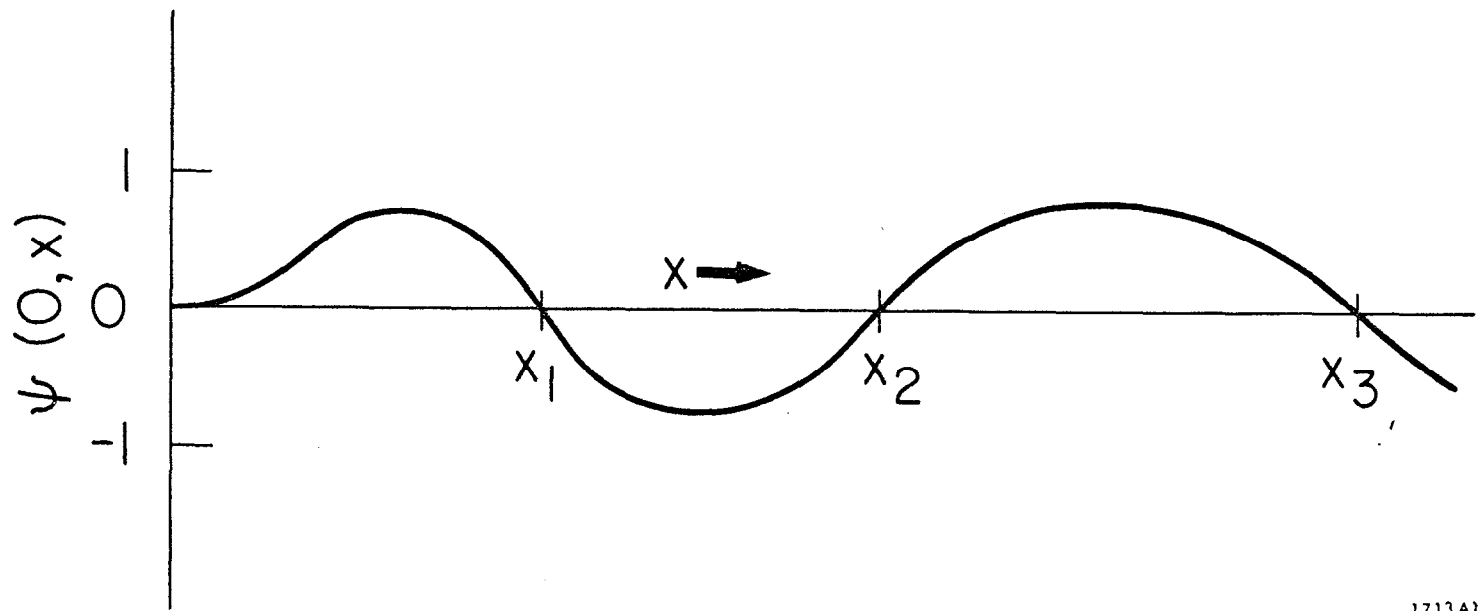
22. Consider for example the constant Z_2 relating the unrenormalized electron field to the renormalized electron field. The only absolute requirement on Z_2 is that it contain all the cutoff dependence of the unrenormalized field. In second order Z_2 will have the form $1 + \eta e^2 [\ln \Lambda^2 + \eta']$ where η is a calculable constant and η' is arbitrary. It is not necessary for η' to be a logarithm of a mass so that $[\ln \Lambda^2 + \eta'] = \ln(\Lambda^2/M^2)$, making the argument of the logarithm dimensionless. Any η' will ensure that Z_2 includes all cutoff dependence of the unrenormalized field. One can put $\eta' = 0$, for example. In this case the renormalized field does not have a well defined dimension. If Z_2 to all orders has the special form $\exp\{\eta e^2 \ln \Lambda^2\}$ (when parameters like η' are set equal to zero) then $Z_2 = (\Lambda^2)^{\eta e^2}$ and the renormalized field does have dimensions, but not the same as the unrenormalized field.
23. One is talking here about "mass dimensions," namely the dimensions fields have in dimensional analysis. Later in this section "scale dimensions" will be defined which give the dimensions of fields under operator scale transformations. The two dimensions can be distinct because in a scale transformation of dimensional analysis, λ is scaled, while an operator scale transformation does not change λ .
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27. If $k^2 \ll m^2$ the factor is $[\ln(m^2/\lambda^2)]^n$ (and similarly if $\lambda^2 \ll m^2$).
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29. This extension was suggested by K. E. Eriksson, *Nuovo Cimento* 30, 1423 (1963).
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31. What follows is a nonrigorous argument deriving the scale breaking theory of
Ref. 24 to first order.
32. There is one example known where a mass term in perturbation theory becomes
nonrenormalizable in strong coupling. The example is the Thirring model: a
spinor field in one space and one time dimension interacting via the Fermi
interaction. The dimension of the mass term in the Thirring model is
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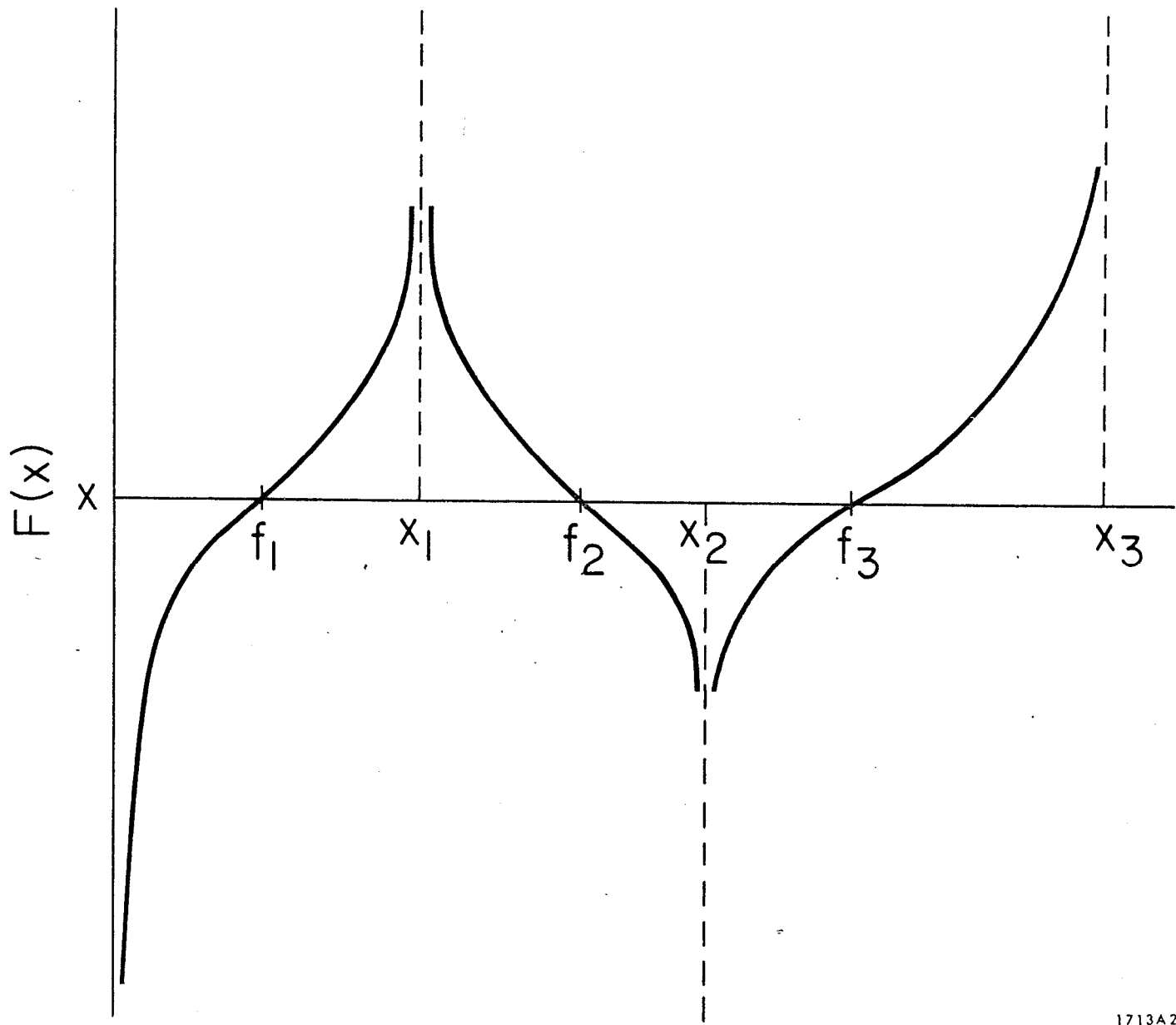
FIGURE CAPTIONS

1. An example of a function $\psi(0, x)$ which has a double root at $x = 0$ and single roots at $x = x_1$, x_2 , and x_3 .
2. Plot of the function $F(x)$ assuming $\psi(0, x)$ is the function shown in Fig. 1. The constant of integration c is also shown; $c = f_1$ if $0 < x < x_1$; $c = f_2$ if $x_1 < x < x_2$; $c = f_3$ if $x_2 < x < x_3$. The constants f_1 , f_2 , and f_3 are chosen arbitrarily.
3. Solutions of the zero mass renormalization group equations for e_λ^2 plotted vs $\ln \lambda^2$.
4. Solutions e_λ^2 and $e_{\lambda_1}^2$ of the finite mass renormalization group equation, with boundary conditions $e_\lambda^2 \rightarrow \Lambda^2$ for e_λ^2 and $e_{\lambda_1}^2 \rightarrow x_2$ for $\lambda \rightarrow \infty$.



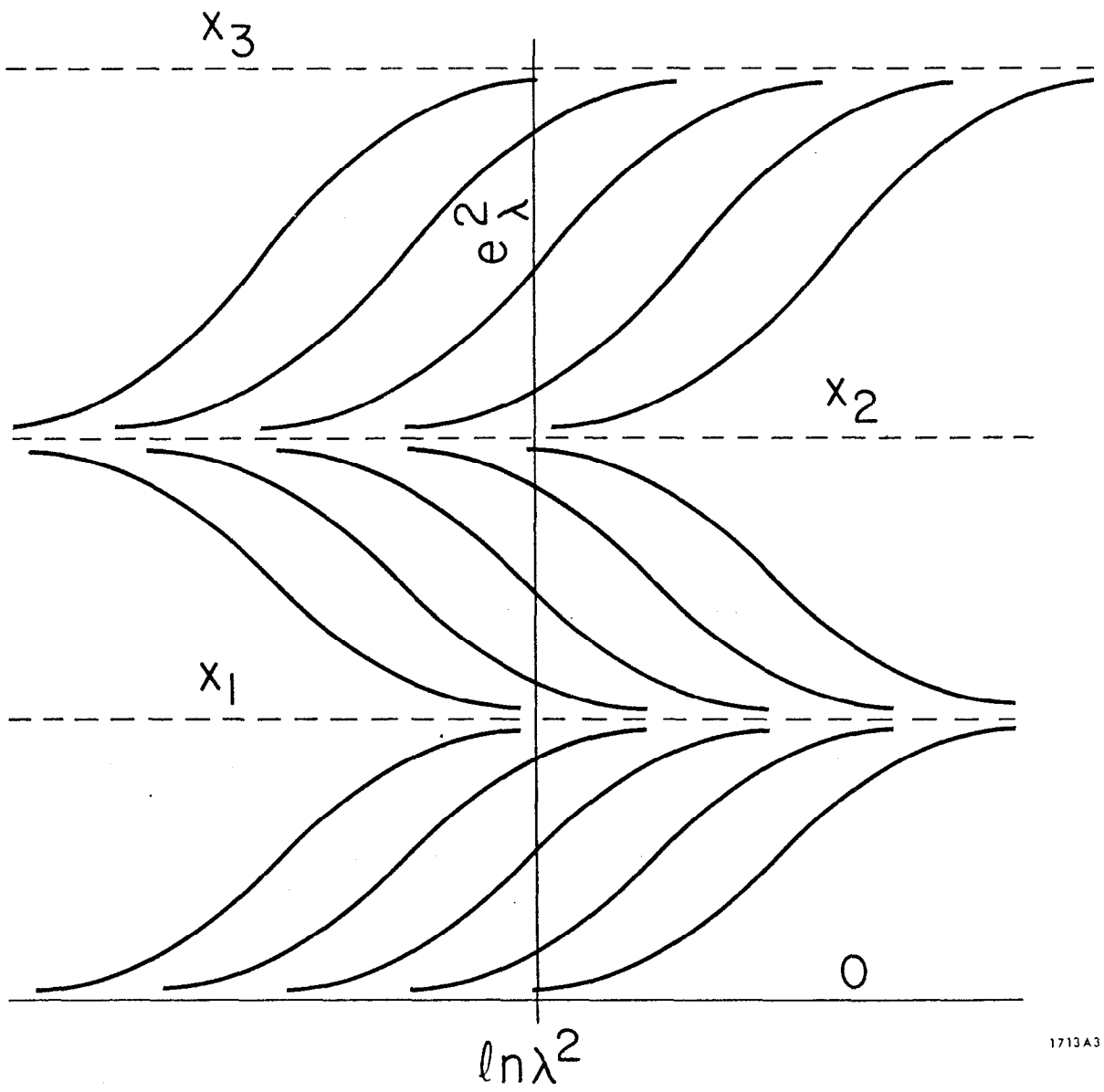
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Fig. 1



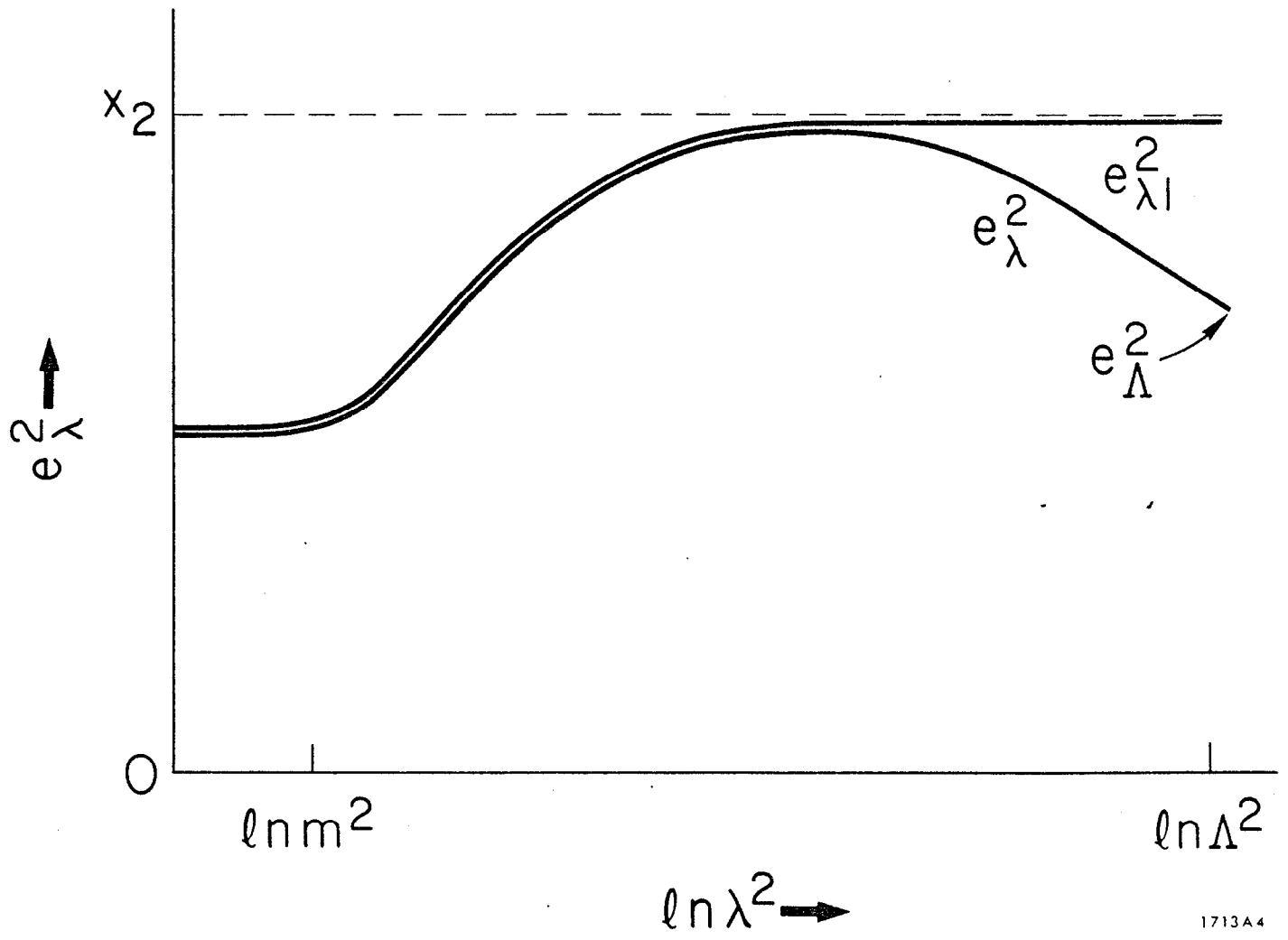
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Fig. 2



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Fig. 3



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Fig. 4