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ACAUSAL DISPERSION RELATIONS AND A FUNDAMENTAL LENGTH †

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ABSTRACT

We study the use of dispersion relations, modified to violate causality, as a tool to limit a fundamental acausal length. We find that unless the usual dispersion relations are found to be violated, acausal dispersion relations give no new information. This means that the only presently believable limit on an acausal length is given by dimensional analysis; since dispersion relations have been tested to incident energies of ~ 20 BeV, any fundamental acausal length is probably less than $\hbar c/20\ \text{BeV} \approx 10^{-15}$cm.

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I. INTRODUCTION

The principle of microcausality, that fields commute at spacelike separations, enters crucially into the proofs of dispersion relations[1]. The strongest evidence for the validity of microcausality lies in the agreement of the forward pion-nucleon dispersion relations with experiment[2]. However, to understand exactly to what degree microcausality is valid, it is necessary to explore the consequences of acausal models. Ideally one would like to show that these models make predictions, which may be in the form of modified dispersion relations, that disagree with experiment.

Acausal models usually postulate some functional dependence for the commutator in spacelike regions; e.g. an exponential fall off in $|\vec{x}|$ or in $\sqrt{|\vec{x}|^2 - x_0^2}$. Generally these models contain a parameter measuring the distance within which violations of causality are appreciable. This parameter is usually referred to as a "fundamental length". It may enter as the decay constant in an exponential fall off of the commutator for spacelike distances or it may measure the size of some spacelike hyperboloid outside of which the commutator vanishes.

Given a particular acausal model, one can follow through the derivation of forward pion-nucleon dispersion relations to investigate the analytic properties of the amplitude. This has been discussed in some detail by Oehme[3]. Generally one finds that the acausal amplitude has singularities in the energy plane beyond those dictated by unitarity. Using Cauchy's theorem one can relate the real and imaginary parts of the physical amplitude in a similar manner to the usual dispersion relations but with modifications due to the additional singularities. The fundamental length parameter controls the size of these modifications.

Our objective in this paper is to study what can be said about the bounds on a fundamental length from comparison of the experimentally measured forward pion-nucleon amplitudes and the predictions of a particular simple model. This subject has been investigated extensively by Lindenbaum and his co-workers using an acausal model in which the amplitude develops an essential singularity at infinity[4]. They claim disagreement between the predictions of this model and experiment for the symmetric pion-nucleon forward amplitude when the fundamental length is $\geq 10^{-16}$ cm. For reasons to be discussed below, we conclude that this disagreement relies heavily on unphysical assumptions about the precise form of the asymptotic amplitude and does not in fact imply an upper bound on such a fundamental length.[5] The important result is that the usual causal dispersion relations are presently satisfied within experimental errors by existing data[2]. We will indicate below the difficulties involved in trying to limit a fundamental length in this model.

We find that particular acausal models do not yield limits on a fundamental acausal length as long as the usual causal dispersion relations are satisfied within experimental error. This means that it is important to test the usual dispersion relations, and not acausal relations, until a violation is found. We finally conclude that the only presently valid estimate of a limit for a fundamental length is given by dimensional analysis; since dispersion relations seem to work to an energy of 20 BeV, it is likely that a fundamental length is smaller than $\hbar c/20$ BeV or 10^{-15} cm.

The specific model we are discussing postulates that the field commutators vanish identically in the region outside of a spacelike hypersurface, $x_0^2 - x^2 \leq -\ell^2$, rather than outside the light cone, $x_0^2 - x^2 \leq 0$. Actually this sort of breakdown in microcausality is not possible in the framework of axiomatic field theory, where it has been shown that the vanishing of commutators within any open spacelike region necessitates their vanishing for all spacelike separations[6]. Nevertheless, we feel this model is worth discussing as it is mathematically simple and furthermore is the model upon which previous assertions, with which we disagree, concerning a fundamental length have been based[4]. In addition, a breakdown in microcausality might be accompanied by further violation of the principles of axiomatic field theory, such as strict Lorentz invariance at short distances.

II. A DISPERSION RELATION FOR THE ACAUSAL CASE

The amplitude we consider throughout this paper is the symmetric pion-nucleon amplitude in the forward direction:

$$f^{+}(\omega) = \frac{1}{2}\left[f_{\pi^{+}p}(\omega) + f_{\pi^{-}p}(\omega)\right] . \tag{1}$$

Here ω is the laboratory energy of the pion,and the normalization of the amplitude the usual with

$$| f_{\pi^{\pm} p}(\omega) |^{2} = \left(\frac{d\sigma_{\pi^{\pm}p}}{d\Omega}\right)_{LAB} \Bigg|_{\theta = 0^{\circ}} . \tag{2}$$

In the forward direction the nucleon cannot change helicity; thus, there is only one amplitude. The amplitude is divided into real and imaginary parts by

$$f^{+}(\omega) = D^{+}(\omega) + i\, A^{+}(\omega) . \tag{3}$$

The usual causal, once-subtracted dispersion relation is written[1]

$$D^+(\omega) = D^+(\mu) + \frac{f^2 k^2}{M\left(1-\left(\frac{\mu}{2M}\right)^2\right)\left(\omega^2+\left(\frac{\mu^2}{2M}\right)^2\right)} + \frac{2k^2}{\pi} P\int_\mu^\infty \frac{\omega' d\omega' A^+(\omega')}{(\omega'^2-\mu^2)(\omega'^2-\omega^2)} \tag{4}$$

where k is the magnitude of the pion laboratory momentum, μ is the pion mass, and M is the nucleon mass. The residue of the nucleon pole and the subtraction constant are given experimentally by (in natural units: $\hbar = c = \mu = 1$) [7]

$$\begin{aligned} f^2 &= 0.081 \\ D^+(\mu) &= -0.002 \ . \end{aligned} \tag{5}$$

Finally we note that $A^+(\omega)$ is related to the total pion-nucleon cross sections by the optical theorem:

$$A^+(\omega) = \frac{k}{8\pi}\left(\sigma^{\pi^+ p}_{TOT}(\omega) + \sigma^{\pi^- p}_{TOT}(\omega)\right) \ . \tag{6}$$

In the acausal case we assume

$$< p \,|\, [j(x), j^+(0)] \,|\, p > \; = \; 0 \quad \text{for } x^2 = x_0^2 - \vec{x}^2 \leq \ell^2. \tag{7}$$

Here $j(x)$ is the pion current, and $| p >$ represents a single nucleon state. It can easily be shown[3,8] that, with this structure for the commutator, the amplitude has an exponential singularity at infinity in the upper half ω plane which is no worse than $e^{-i\omega\ell}$. This singularity arises from the sharp spatial cut off imposed on the commutator. It is interesting to note that in this model no singularities can appear at finite ω in the upper half plane.

Let us now define a new function $f_0^+(\omega)$ from the acausal amplitude by

$$f_0^+(\omega) = e^{i\omega\ell} f^+(\omega) \quad . \tag{8}$$

Clearly $f_0^+(\omega)$ is still analytic in the upper half ω plane. Since $f^+(\omega)$ is no more singular than $e^{-i\omega\ell}$ at infinity, $f_0^+(\omega)$ is polynomially bounded at infinity. Still following Oehme[3,8], we can write a dispersion relation for $f_0^+(\omega)$. Writing this equation in terms of $A^+(\omega)$ and $D^+(\omega)$ with the assumption of a single subtraction gives

$$D^+(\omega)\cos\omega\ell - A^+(\omega)\sin\omega\ell = D^+(\mu)\cos\mu\ell + \frac{f^2 k^2 \cos\frac{\mu^2\ell}{2M}}{M\left(1-\left(\frac{\mu}{2M}\right)^2\right)\left(\omega^2-\left(\frac{\mu^2}{2M}\right)^2\right)}$$

$$+ \frac{2k^2}{\pi} \int \frac{\omega' d\omega' \left(A^+(\omega')\cos\omega'\ell + D^+(\omega')\sin\omega'\ell\right)}{(\omega'^2-\omega^2)(\omega'^2-\mu^2)} \tag{9}$$

One would like to use the optical theorem to determine $A^+(\omega)$ where the pion-nucleon total cross sections are known. However, it might be that a breakdown of causality is associated with a breakdown of the optical theorem as well. Nevertheless, as Oehme has pointed out[3], such a deviation from the usual unitarity conditions is not required in an acausal theory such as that considered here. Thus we will consider the optical theorem derived from the usual unitarity condition as still valid and use it to determine $A^+(\omega)$.

Let us temporarily assume that the total cross sections, and therefore $A^+(\omega)$, are well known at all energies. Then Eq. (9) becomes an integral equation for $D^+(\omega)$. We can immediately establish two important properties of this equation. First we note that $D^+(\omega)$ obtained from the usual dispersion relation, Eq. (4), is also a solution to Eq. (9). This can be checked either by direct substitution or simply by noting that $f(\omega)$ as obtained from Eq. (4) is a polynomially bounded function analytic in the upper half ω plane, and such an analytic function multiplied by $e^{i\omega\ell}$ retains these properties, allowing us to write Eq. (9). Second, it is clear that for a given $A^+(\omega)$, the solution to Eq. (9) is not unique. For example, any solution for $D^+(\omega)$ can have terms like $(\cos\omega m - \cos\mu m)$ when $0 \leq m \leq \ell$ arbitrarily added to it to give other solutions.

This arbitrariness in the solution is physically reasonable. In order to obtain Eq. (9), one multiplies the amplitude by $e^{i\omega\ell}$ to cancel the effects of acausal behavior at Lorentz distances up to ℓ. In so doing, one also cancels possible acausal behavior at smaller distances $m \leq \ell$. Hence any amplitude with this form of acausal behavior over a distance $m \leq \ell$, and in particular the causal amplitude with $m = 0$, satisfy Eq. (9). We note that the arbitrary terms which one can add to the solution to Eq. (9) are of the same order as the acausal effects one is looking for. According to Eq. (8), one expects to see evidence of acausal behavior in $f^+(\omega)$ to first order in $\omega\ell$, but the arbitrariness of the solutions to Eq. (9) appears in terms of order ωm for any $m \leq \ell$. Clearly the two effects can be comparable.

We are confronted with a serious difficulty in making use of Eq. (9). One would like to use this equation to predict the real part of the amplitude for comparison with experiment. However, one must first impose additional constraints on the function $D^+(\omega)$ in order to specify it uniquely. The exact solution which is singled out depends sensitively on the constraints imposed and its physical significance depends on the physical basis of the constraints. In the next section we shall discuss the manner in which Lindenbaum and his collaborators[4] choose a solution to this equation.

III. A PARTICULAR ACAUSAL SOLUTION

In order to solve Eq. (9) for $D^+(\omega)$, Lindenbaum and his co-workers[4] do not specify $A^+(\omega)$ for all energies but rather specify $A^+(\omega)$ for ω less than some experimental cutoff, Ω, and then specify $\mathrm{Im}\, f_0^+(\omega) = \cos\omega\ell\, A^+(\omega) + \sin\omega\ell\, D^+(\omega)$ for energies above Ω. Specifically they assume $\mathrm{Im}\, f_0^+(\omega)$ is a smooth function of form

$$\mathrm{Im}\, f_0^+(\omega) = \frac{k}{8\pi}\left(\alpha + \frac{\beta}{\omega^p}\right) \tag{10}$$

where α, β, and p are parameters chosen to fit the experimental cross section when $\omega\ell \ll 1$ and $\mathrm{Im}\, f_0^+(\omega) \approx \mathrm{Im}\, f(\omega)$. They pick $1/\ell \gg \Omega \gg \mu$ so the assumed behavior is smooth at Ω. With these assumptions Eq. (9) was solved iteratively for $D^+(\omega)$.

The strong assumption made in Eq. (10) is that $\mathrm{Im}\, e^{i\omega\ell} f^+(\omega)$ is a smooth function of ω for a particular value of ℓ. Behavior parametrized by distances other than ℓ is specifically excluded. This assumption has the serious difficulty that by the optical theorem it does not have a positive definite cross section. For example the cross section $\sigma^{\pi^+ p}_{TOT}(\omega) + \sigma^{\pi^- p}_{TOT}(\omega)$ is given at high energies by

$$\sigma^{\pi^+ p}_{TOT}(\omega) + \sigma^{\pi^- p}_{TOT}(\omega) \xrightarrow[\omega \to \infty]{} \alpha \cos\omega\ell - \frac{8\pi}{\omega}\mathrm{Re}\, f_0^+(\omega)\sin\omega\ell\ . \tag{11}$$

This expression clearly changes sign between $\omega\ell = n\pi$ and $\omega\ell = (n+1)\pi$. This problem of a negative cross section is unavoidable if one requires $\mathrm{Im}\, e^{i\omega\ell} f^+(\omega)$ to be asymptotically smooth. This assumption rules out the physically interesting possibility of the total cross section oscillating about a positive constant. Let us note here that the original model does not require an oscillating asymptotic cross section at all; all oscillations in the amplitude might appear only in the real part.

The solution obtained by Lindenbaum[4] with $\ell = 10^{-16}$ cm. disagrees with the experimentally measured $D^+(\omega)$ over a wide range of energies. The question is whether the disagreement with the data reflects on the assumption of a fundamental length or rather on the inappropriateness of the constraint imposed to obtain the solution. For several reasons we believe the latter interpretation. First of all, we know that $D^+(\omega)$ as predicted by the usual dispersion relation agrees with the data and is also a solution to the modified relation in Eq. (9). Secondly, by making $\mathrm{Im}\, f_0^+(\omega)$ asymptotically smooth, a very particular oscillatory behavior was assumed for the total cross section, which actually became negative for certain values of the energy. Finally, we note that the same $A^+(\omega)$ was always assumed for ω less than Ω; so, the difference between the Lindenbaum solution to Eq. (9) and the prediction of the usual dispersion relation must entirely lie in assumptions on the amplitude for ω greater than Ω, where there are no measurements as yet. We have done detailed calculations with different asymptotic assumptions to verify this last point. Thus the disagreement of this special solution with the data does not originate in any experimentally measured quantity. It is instead only the result of a specific assumption about the very high energy behavior of the amplitude, an assumption which requires an oscillatory behavior of the total cross section which need not appear in the model and which violates unitarity rather violently. Since it is this calculation on which estimates of a fundamental length are based, we conclude that the validity of these estimates is in serious doubt.

IV. CONCLUSION

It remains to discuss in what ways acausal theories, such as that incorporated into Eq. (9), might be of use in putting limits on a fundamental length. The outlook unfortunately is not good. Having established the arbitrariness of the solution to Eq. (9) for a given $A^{+}(\omega)$, we are at a loss as to how to specify which solution is physically meaningful. Extrapolations such as that employed by Lindenbaum's group[4], being unsupported by the data, cannot yield physically meaningful results.

Other models, such as those discussed by Oehme[3], in which the amplitude develops additional singularities in the finite energy plane have the handicap that one cannot measure the amplitude along these new singularities. This introduces ambiguities in the amplitude analogous to those in the above model. Thus, since the usual causal dispersion relations are satisfied within experimental limits, these relations derived from acausal models also add no further information.

We do not wish to imply here that further effort should not be expended on testing the usual dispersion relations. It would indeed be interesting if the usual analyticity conditions were violated. If further experiments do yield discrepancies in the usual dispersion relations, then a search should be made for an acausal relation to fit the data. In other words, Eq. (9) might be useful if the measured amplitude did not satisfy the usual dispersion relation within experimental error. Then if Eq. (9) were satisfied only for ℓ larger than some ℓ_0 it would appear that causality was violated to distances of ℓ_0. However, as long as the usual dispersion relation fits the experiments, Eq. (9) is of no use in bounding a fundamental length.

In conclusion, the only believable limit on a fundamental length at this time is given by the dimensional argument that since dispersion relations work at energies up to 20 BeV, a fundamental acausal length is unlikely to be much larger than $\hbar c/20\ \mathrm{BeV} = 10^{-15}\mathrm{cm}$. It should be understood that this is a purely dimensional argument and should be viewed with the appropriate caution.

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