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NONSINGULAR SCATTERING EQUATIONS*

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ABSTRACT

We obtain nonsingular integral equations for the two-body potential scattering problem. In momentum space our integral equations have square integrable kernels and require only a finite range of integration. We use our integral equation to obtain bounds on the convergence of the Born series.

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I. INTRODUCTION

The solution to the nonrelativistic potential theory scattering problem is obtained from the Lippmann-Schwinger equation for the t-matrix. Solving the Lippmann-Schwinger integral equation is difficult because the kernel is singular. We obtain here a nonsingular Fredholm equation whose solution is the t-matrix (or K-matrix). This equation has only a finite range of integration in momentum space and the kernel is square integrable. We then use our equation to give estimates on the rate of convergence of the Born series.

We begin our analysis by reviewing the known singularity characteristics of the Lippmann-Schwinger equation. In momentum space, the Lippmann-Schwinger equation for the nonrelativistic two-body scattering problem has the form

$$\langle\vec{p}|t(k^2 \pm i\epsilon)|\vec{p}\,'\rangle = \langle\vec{p}|v|p'\rangle - \int \frac{\langle\vec{p}|v|\vec{p}\,''\rangle\langle\vec{p}\,''|t(k^2 \pm i\epsilon)|\vec{p}\,'\rangle\, d\vec{p}\,''}{\frac{\vec{p}\,''^2}{2m} - \frac{k^2}{2m} \mp i\epsilon} \tag{1}$$

Here $\vec{p}$ and $\vec{p}\,'$ are the center-of-mass momentum before and after scattering. The center-of-mass energy is $k^2/2m$, where m is the reduced mass of the two-particle system. The potential (local or nonlocal) is denoted by v, and $\langle\vec{p}|t(k^2 \pm i\epsilon)|\vec{p}\,'\rangle$ is the t-matrix. The singularity of the kernel of Eq. (1) is manifest in the $i\epsilon$ prescription. Although the kernel of Eq. (1) is singular, Faddeev[1] has proved that for sufficiently well behaved potentials the solutions, $\langle p|t(k^2 \pm i\epsilon)|p'\rangle$, are unique and that the $i\epsilon \to 0$ limit is well defined. The restrictions that Faddeev imposes on the potential, to obtain these results, are that the potential have a boundedness property and a smoothness property. The boundedness property is expressed by

$$|v(\vec{p}-\vec{p}\,')| \le \frac{C_1}{(1+|\vec{p}-\vec{p}\,'|)^{1+\theta}}, \quad \theta > 1/2 \tag{2}$$

for all $\vec{p}$ and $\vec{p}'$, where C_1 is a constant. This property is constructed to ensure that all the integrals over momentum are convergent. The second limitation Faddeev imposes on v is that v satisfy the Hölder condition,

$$|v(\vec{p}-\vec{p}') - v(\vec{p}+\Delta\vec{p}-\vec{p}')| \leq \frac{C_1|\Delta\vec{p}|^{\mu}}{(1+|\vec{p}-\vec{p}'|)^{1+\theta}}, \qquad |\Delta\vec{p}| < 1,\ \mu > 0. \tag{3}$$

This smoothness in the momentum dependence of v is required so that the $i\epsilon$ prescription in the Lippmann-Schwinger equation be well defined. The only assumptions about the potential that our method uses are the properties given in Eqs. (2) and (3).

II. NONSINGULAR EQUATIONS FOR PARTIAL WAVE AMPLITUDES

In order to illustrate our approach in detail let us consider the partial wave decomposed form of Eq. (1) valid for spherically symmetric potentials,

$$t_\ell(p,p';k^2 \pm i\epsilon) = v_\ell(p,p') - c\int_0^\infty \frac{v_\ell(p,p'')\,t_\ell(p'',p';k^2 \pm i\epsilon)\,p''^2\,dp''}{p''^2 - k^2 \mp i\epsilon} \tag{4}$$

Here t_ℓ and v_ℓ are the ℓ^{th} partial wave projections of t and v respectively, and c is a constant dependent on one's choice of constants in the partial wave expansion. Since our approach gives a particularly simple result when only a principal-value type singularity occurs in the integral equation let us consider the K-matrix companion to Eq. (4).

$$K_\ell(p,p';k^2) = v_\ell(p,p') - c\,⨍_0^\infty \frac{v_\ell(p,p'')\,K_\ell(p'',p';k^2)\,p''^2\,dp''}{p''^2 - k^2} \tag{5}$$

One can recover the t-matrix from the K-matrix by the Heitler transformation

$$t(p,p';k^2 \pm io) = K(p,p';k^2) + \frac{\frac{ci\pi}{2k}\left\{K(p,p';k^2)\,K(k,k;k^2) - K(p,k;k^2)\,K(k,p';k^2)\right\}}{1 + \frac{ci\pi}{2k}\,K(k,k;k^2)} \quad . \tag{6}$$

We omit in Eq.(6) and in all the equations that follow the ℓ-index.

We now simplify Eq. (5) by changing the variables of integration. Consider a mapping (which exists only for $k^2 > 0$) defined by

$$p(x) = k\left(\frac{1+x}{1-x}\right), \qquad x\epsilon[-1, +1] \tag{7}$$

with the inverse

$$x(p) = \frac{p-k}{p+k}\,, \qquad p\epsilon[0,\infty] \quad . \tag{8}$$

In the x-coordinate system the K-matrix equation becomes, with the notation $K(x,x') \equiv K(p(x),\ p(x');\ k^2)$ and $v(x,x') \equiv v(p(x), p(x'))$,

$$K(x,x') = v(x,x') - \frac{c}{2k}⨍_{-1}^{+1} \frac{v(x,x'')\,K(x'',x')}{x''}\left(k\,\frac{1+x''}{1-x''}\right)^2 dx'' \quad . \tag{9}$$

Let us define even and odd projection operators in the x-space by

$$\mathscr{E}(x)\ f(x) = \frac{1}{2}\left[f(x) + f(-x)\right]$$

$$\mathscr{O}(x)\ f(x) = \frac{1}{2}\left[f(x) - f(-x)\right] \tag{10}$$

where f is an arbitrary function. We now exploit the fact that

$$⨍_{-1}^{+1} \frac{dx''}{x''}\,\mathscr{E}(x'')\ f(x'') = 0 \quad , \tag{11}$$

and

$$⨍_{-1}^{+1} \frac{dx''}{x''}\,\mathscr{O}(x'')\ f(x'') = 2\int_{-1}^{0} \frac{dx''}{x''}\,\mathscr{O}(x'')\ f(x'') \quad . \tag{12}$$

The right-hand side of Eq. (12) is an ordinary Riemann integral, since the integrand is integrable at $x'' = 0$, if $f(x'')$ satisfies the Hölder continuity condition.

Using identities (11) and (12) we have

$$K(x,x') = v(x,x') - \frac{c}{k} \int_{-1}^{0} \frac{dx''}{x''} \mathcal{O}(x'') \left[k^2\left(\frac{1+x''}{1-x''}\right)^2 v(x,x'') K(x'',x')\right] . \qquad (13)$$

Mapping this equation back to the p-space gives

$$K(p,p') = v(p,p') - 2c\int_{0}^{k} \frac{dp''}{p''^2 - k^2} \mathcal{O}(p'') \left[p''^2 v(p,p'') K(p'',p')\right] , \qquad (14)$$

where $\mathcal{O}(p'')$ is the transformation of the odd projection operator given in Eq. (10). In momentum space these projection operators are

$$\mathcal{E}(p) f(p) = \frac{1}{2} \left[f(p) + f(p_r)\right] ,$$

$$\mathcal{O}(p) f(p) = \frac{1}{2} \left[f(p) - f(p_r)\right] , \qquad (15)$$

where $p_r = k^2/p$.

We can obtain from Eq. (14) an integral equation by expanding $K(p,p')$ in terms of its even and odd parts. Defining $K_0(p,p')$ and $K_e(p,p')$ by

$$K_0(p,p') = \mathcal{O}(p) K(p,p'), \qquad K_e(p,p') = \mathcal{E}(p) K(p,p') , \qquad (16)$$

Equation (14) takes the form

$$K(p,p') = v(p,p') - 2c\int_{0}^{k} \frac{dp''}{p''^2 - k^2}\Big\{\left[\mathcal{O}(p'') V(p,p'')\right] K_e(p'',p')$$

$$+ \left[\mathcal{E}(p'') V(p,p'')\right] K_0(p'',p')\Big\} \qquad (17)$$

with $V(p,p'') = p''^2 v(p,p'')$. Multiplying by $\mathcal{E}(p)$ and $\mathcal{O}(p)$ will now give us two coupled equation for K_0 and K_e. However we note that the second term in the integrand on the right-hand side of Eq. (17) is nonsingular by virtue of the fact that K_0 has a zero at the point $p'' = k$ which multiplies the $(p''^2 - k^2)^{-1}$ pole in the remaining portion of the integrand. This suggests that we represent $K(p,p')$ by

$$K(p,p') = K_e(p,p') + \phi(p) \widetilde{K}_0(p,p')$$

$$K_0(p,p') = \phi(p) \widetilde{K}_0(p,p') \qquad (18)$$

where $\phi(p)$ is an odd function in p. We will choose $\phi(p) = (p-k/p+k)$. If we now obtain a coupled integral equation for the components K_e and $\widetilde{K}_0$, all the cancellations of the zero's and poles will occur only in the kernels of the equation. Explicitly multiplying Eq. (17) by $\mathscr{E}(p)$, and employing Eq. (18) yields

$$K_e(p,p') = V_e(p,p') - 2\int_0^k \left\{A_1(p,p'')\, K_e(p'',p') + A_2(p,p'')\, K_0(p'',p')\right\} dp'' \quad (19)$$

Where the kernels A_1 and A_2 are given below. Likewise multiplying Eq. (17) by $\phi^{-1}(p)\,\mathscr{O}(p)$, and using Eq. (18) again gives,

$$\widetilde{K}_0(p,p') = V_0(p,p') - 2\int_0^k \left\{A_3(p,p'')\, K_e(p'',p') + A_4(p,p'')\, \widetilde{K}_0(p'',p')\right\} dp''. \quad (20)$$

Taken together Eqs. (19) and (20) are a two-component integral equation for the functions $K_e, \widetilde{K}_0$. Once K_e amd $\widetilde{K}_0$ are determined then $K(p,p')$ is given by Eq. (18).

The kernels appearing in Eqs. (19) and (20) are easily determined to be,

$$A_1(p,p'') = \frac{c}{4(p''+k)^2}\left\{V(p,p'') + V(p_r,p'') + V(p,p_r'') + V(p_r,p_r'')\right\},$$

$$A_2(p,p'') = \frac{c}{4(p''^2-k^2)}\left\{V(p,p'') + V(p_r,p'') - V(p,p_r'') - V(p_r,p_r'')\right\},$$

$$A_3(p,p'') = \frac{c(p+k)}{4(p-k)(p''+k)^2}\left\{V(p,p'') - V(p_r,p'') + V(p,p_r'') - V(p_r,p_r'')\right\},$$

$$A_4(p,p'') = \frac{c(p+k)}{4(p-k)(p''^2-k^2)}\left\{V(p,p'') - V(p_r,p'') - V(p,p_r'') + V(p_r,p_r'')\right\}. \quad (21)$$

We can see how the Hölder condition Eq. (3) and the boundedness condition Eq. (2) will ensure that the A_i are integrable. The constraint that $\theta > 1/2$ in the boundedness condition will guarantee that terms like $V(p_r,p_r'') = k^4/p''^2\left[v(k^2/p, k^2/p'')\right]$ are finite in the neighborhood of $p''=0$. If we require $\mu_0 > 1/2$ in the Hölder condition it follows that the L_2 norms of A_i will be finite, i.e.,

$$\|A_i\| = \left(\int_0^k \int_0^k dp\, dp''\, |A_i(p,p'')|^2\right)^{1/2} < \infty \quad (22)$$

When the A_i are square integrable then the integral equations given by Eqs. (19) and (20) are the simplest kind of Fredholm integral equations.[2]

II. SOME GENERALIZATIONS

The derivation given above for the partial wave form of the K-matrix equation can be applied without change to fully angularly dependent K-matrix equation,

$$\langle \vec{p}|K(k^2)|\vec{p}'\rangle = \langle \vec{p}|v|\vec{p}'\rangle - c \fint \frac{\langle \vec{p}|v|\vec{p}''\rangle \langle \vec{p}''|K(k^2)|\vec{p}'\rangle \, p''^2 \, dp'' \, d\Omega_{\hat{p}''}}{p''^2 - k^2} \tag{23}$$

where $d\Omega_{\hat{p}''}$ indicates the angular integration. The only property of Eq. (5) exploited to obtain the reduction was that the principal value singularity was a fixed singularity in the variable p''. If we replace $v(p, p'')$ by $v(p, p''; \hat{p}, \hat{p}'')$ in the expression (21) for the kernels and increase the variables of integration in Eqs. (19) and (20) to $\int dp'' \, d\Omega_{\hat{p}''}$ then we have a nonsingular equation for the angle dependent K-matrix amplitude $\langle \vec{p}|K(k^2)|\vec{p}'\rangle$.

Another, desirable generalization of our method would be to treat the Lippmann-Schwinger equation for the t-matrix directly. If we start from the partial wave form, Eq. (4) then this can be done as follows. Expand the Green function in Eq. (4) with the representation

$$\frac{1}{p''^2 - k^2 \mp io} = \text{P.V.} \frac{1}{p''^2 - k^2} \pm i\pi\,\delta(p''^2 - k^2) \tag{24}$$

Now Eq. (4) has the form

$$t(p, p'; k^2 \pm io) = v(p, p') - c \fint_0^\infty \frac{v(p, p'')\, t(p'', p'; k^2 \pm io)\, p''^2 dp''}{p''^2 - k^2} \mp \frac{ic\pi k}{2} \, v(p, k)\, t(k, p'; k^2 \pm io) \tag{25}$$

If we set p = k we obtain

$$t(k,p';k^2 \pm io) = \frac{v(k,p')}{D^{\pm}(k)} - \frac{c}{D^{\pm}(k)} ⨍_0^{\infty} \frac{v(k,p'')\, t(p'',p';k^2 \pm io)\, p''^2\, dp''}{p''^2 - k^2} \tag{26}$$

where

$$D^{\pm}(k) = 1 \pm \frac{ic\pi k}{2} v(k,k) \tag{27}$$

and division by $D^{\pm}(k)$ is always permitted since for v(k, k) real $D^{\pm}(k)$ will have no zeros. Substituting Eq. (26) into Eq. (25) gives us an integral equation for t involving only a principal-value integration, viz.

$$t(p,p';k^2 \pm io) = U(p,p';k^2 \pm io) - c ⨍_0^{\infty} \frac{U(p,p'';k^2 \pm io)\, t(p'',p';k^2 \pm io)\, p''^2\, dp''}{p''^2 - k^2} \tag{28}$$

where the driving term v is,

$$U(p,p';k^2 \pm io) = v(p,p') \mp \frac{ic\pi k/2\ v(p,k)\, v(k,p')}{D^{\pm}(k)}$$

$$= \frac{v(p,p') \pm \frac{ci\pi k}{2}\left[v(k,k)\, v(p,p') - v(p,k)\, v(k,p')\right]}{D^{\pm}(k)} \ . \tag{29}$$

This driving term is just a unitarized first Born term. This is easily seen by setting $K(p,p';k^2)$ equal to v(p, p') in Eq. (6). We note the discontinuity structure associated with unitarity in Eq. (28) is quite different from the conventional Lippmann-Schwinger equation. Here all the discontinuity in going from $t(p,p';k^2+io)$ to $t(p,p';k^2-io)$ arises from the discontinuity in the unitarized Born term $v(p,p';k^2 \pm io)$. The principle value Green function is of course continuous across the scattering cut. Expanding the right-hand side of Eq. (25) in a Born series gives a simple picture of how the scattering amplitude t is built up. The n^{th} Born term consists of a product of the n unitarized interactions U connected by free principal-value Green function propagators.

With Eq. (28) we can now derive a nonsingular equation for $t(p, p';k^2 \pm io)$. Clearly the appropriate integral equation for $t_0, \tilde{t}_0$ is given by Eqs. (19) and (20) if we replace $v(p, p')$ by $U(p, p;k^2 \pm io)$ everywhere. Before leaving this subject we observe that the method of getting rid of the delta function in going from Eq. (4) to Eq. (28) is simple only for the partial form of the Lippmann-Schwinger equation. To use the same method on angular dependent Lippmann-Schwinger Eq. (1) would require inverse operators involving the potentials. So, although the method here allows us to extract the principal-value singularities in Eq. (1) the delta function singularities will still be present.

IV. NORMS AND BORN SERIES CONVERGENCE

In this section we want to show how to exploit the L_2 characteristics of our nonsingular equations. We will imbed our equation in a Hilbert space and then use the norms the space induces to give an estimate of the rate of convergence of the Born series. Let us define an inner product for functions of the momentum variable as

$$(f, g) = \int_0^\infty \overline{f(p)}\, g(p)\, dp, \tag{30}$$

and let us denote by $\mathscr{H}$ the Hilbert space that is associated with this inner product. The space $\mathscr{H}$ consists of all functions with finite norm, $\| f \| = (f, f)^{1/2}$. In defining our Hilbert spaces it is convenient to have the same space for all values k of the incoming momenta. So when we study the reduced representation Eq. (19) and (20) we shall treat it as an equation over the entire domain of $p \in [0, \infty]$. This is done by rewriting the right-hand integral term as

$$2\int_0^k A(p, p'')\, K(p'', p')\, dp'' = \int_0^\infty A(p, p'')\, K(p'', p')\, dp'' \tag{31}$$

where Eq. (31) is valid for any of the four integral terms in Eqs. (19) and (20). On the Hilbert space $\mathscr{H}$ each kernel A(p, p'') generates a bounded linear operator A defined by g = Af where

$$Af = \int_0^\infty A(\cdot, p'')\, f(p'')\, dp'' \quad . \tag{32}$$

The operator A is bounded since A has a finite Hilbert-Schmidt norm

$$\|A\|_s = \left(\int_0^\infty \int_0^\infty A(p, p'')^2\, dp\, dp''\right)^{1/2} \quad . \tag{33}$$

The natural Hilbert space to analyze our coupled equations in is a product Hilbert space, $\mathscr{H} \otimes \mathscr{H}$, containing two-component vector-valued functions, i.e.,

$$\vec{f}(p) = \begin{pmatrix} f_1(p) \\ f_2(p) \end{pmatrix} \qquad \vec{f} \in \mathscr{H} \otimes \mathscr{H} \tag{34}$$

with the inner product $(\vec{f}, \vec{q}) = (f_1, g_1) + (f_2, g_2)$. In this language our equation is symbolically

$$\vec{K} = \vec{b} - L\vec{K} \tag{35}$$

where in component form the operator L on $\mathscr{H} \otimes \mathscr{H}$ is

$$L = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} , \tag{36}$$

and the matrix elements of A_i of L are operators on $\mathscr{H}$ in the sense of Eq. (32).

Given bounds or the values of $\|A_i\|_s$ it is straightforward to bound the operator norm of L. Using the matrix structure of L given in Eq. (36) and the definition of operator norm[3] it follows that

$$\|L\| \leq \max\left\{\left(\|A_1\|^2 + \|A_3\|^2\right)^{1/2},\ \left(\|A_2\|^2 + \|A_4\|^2\right)^{1/2}\right\} \tag{37}$$

The operator norms, $\| A_i \|$, are bounded by the Hilbert-Schmidt norms so that

$$\| L \| \leq \max\left\{\left(\| A_1\|_s^2 + \| A_3\|_s^2\right)^{1/2}, \left(\| A_2\|_s^2 + \|A_4\|_s^2\right)^{1/2}\right\} . \tag{38}$$

The Hilbert-Schmidt norms needed in Eq. (38) can be obtained from doing the integrals in Eq. (33) numerically or estimating them analytically.

We shall now show how to obtain estimates on convergence of the Born series. Suppose we have obtained bounds for $\| A_i\|_s$ and that Eq. (38) ensures us that $\| L \| < 1$. Then from Eq. (35) the difference between $\vec{K}$ and $\vec{b}$ is

$$\| \vec{K} - \vec{b} \| = \|(1 + L)^{-1} L\vec{b} \| \leq (1 - \| L \|)^{-1} \| L \| \| \vec{b} \| . \tag{39}$$

In order to recover the physical amplitude K from $\vec{K}$ we must use relation (18). However a norm for $\vec{K}$ leads to a bound for the norm of K. From $K = K_1 + \phi K_2$ it follows that,

$$\| K \| \leq \| K_1\| + \| \phi K_2\| \leq \| K_1\| + B(\phi) \| K_2\| \leq (1 + B(\phi)) \| \vec{K} \| , \tag{40}$$

where

$$B(\phi) = \sup_{p \in [0,\infty]} |\phi(p)| < \infty , \tag{41}$$

For the form of $\phi(p)$ given after Eq. (18), $B(\phi) = 1$. Combining Eq. (40) with Eq. (39) gives us

$$\|K - v\| \leq (1 - \| L \|)^{-1} \| L \| (1 + B(\phi)) \| \vec{b} \| \tag{42}$$

where

$$b_1 = v_e(\cdot , p'), \ b_2 = \tilde{v}_0(\cdot , p').$$

IV. COMPARISON WITH OTHER APPROACHES TO NONSINGULAR SCATTERING EQUATIONS

In this concluding section we contrast the characteristics of our reduced equations with those of some of the alternate approaches available in the literature. Probably the formulation closest in spirit to the one given here for the two-body problem is the Kowalski-Noyes[4] representation. One attractive feature the Kowalski-Noyes integral equation shares with results derived here is that its kernel is known[5] to be a Hilbert-Schmidt kernel for same weak conditions on the potential given here. However, the Kowalski-Noyes approach has some drawbacks. The representation for the partial wave t-matrix has nonphysical poles[5] (albeit cancelling) not appearing in t.

Another frequently used technique is to distort (or rotate) the contour of integration[6] in Eq. (1). By allowing p'' to become complex we may distort the path of integration so that $p''^2 - k^2$ never vanishes. This is a powerful technique and works for higher dimensional integral equations as well as equations with moving point singularities. The difficulties which sometimes attend this method are that the analytic structure of the equation for p'' complex may not be easy to determine so that proving one hasn't crossed poles or branch points in distorting the integration contour becomes troublesome. Clearly if the kernel is only known in numerical form then the method isn't applicable. Also contour deformation requires knowing stronger analyticity properties than we have needed for our reduction.

We mention in connection with the contour deformation method[7] a similar approach which treats the $i\epsilon$ as nonzero. For $\epsilon > 0$, Eq. (1) is analytic in ϵ. Solutions obtained for nonzero ϵ are then continued to $\epsilon = 0$. However the difficulty of this method is that given a finite number of solutions for nonzero ϵ's there is no unique continuation onto the axis. Finally, Broido and Taylor[8] have recently studied the construction of nonsingular equations for the Bethe-Salpeter equation. Basically they expand the solution in a Taylor expansion about the point of the fixed singularity. This procedure used for the Lippman-Schwinger equation studied here will certainly give nonsingular equations, but of a more complicated construction than the ones we have given here.

We note obvious applications of our results. If we expand the operator L in terms of its eigenfunctions then we will be lead to a separable expansion for t (or K). Fixed-point singular equations in two or more variables of integration (such as Faddeev's equations) can be simplified to the extent of removing all of the principal-value type integrations, but with delta functions remaining the kernels. These delta functions would prevent us from carrying out a simple Hilbert space norm analysis as in Section IV but may not prove too difficult to handle numerically.[9]

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