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# LONGITUDINAL EFFECTS OF COLLIDING BEAM SPACE CHARGE FORCES IN ELECTRON-POSITRON STORAGE RINGS WITH CROSSING ANGLES*

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## ABSTRACT

In electron-positron storage rings in which the orbits of the colliding beams cross at a substantial angle, a phase-dependent longitudinal force is exerted on the particles of each beam by the bunches of the other. The effects of this force on both incoherent and coherent longitudinal particle motion are investigated. These effects impose a limit on the number of particles which can be stored in a bunch in such storage rings and the limit is independent of the crossing angle (provided the crossing angle is sufficiently large) and of the radiofrequency accelerating system parameters. The limit is imposed by the requirement for stability of coherent particle motions.

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## Introduction

In electron-positron storage rings the electromagnetic forces exerted by the colliding beams on each other are very strong by comparison with the self-forces of each beam, and these colliding-beam forces give rise to a variety of effects which limit the performance of the storage rings. The forces are highly nonlinear as functions of the particle coordinates, and characteristically they lead to avoidance of one beam by the other when either or both beams are too strong, with consequent reduction of the storage ring luminosity. (The luminosity is defined as the interaction rate per unit reaction cross section.) Generally speaking the particles congregate in traveling bunches, each one of which is spread around a position of synchronous phase on the radiofrequency accelerating wave. When the beams are not colliding so that the strong colliding-beam forces are absent, the unperturbed beams assume gaussian density distributions in both transverse and longitudinal coordinates under the influence of radiation fluctuations and damping. When the beams collide these equilibrium distributions may be disturbed, and as a result, there may occur an optimum beam current at which maximum luminosity occurs. In this paper, we study these effects as they apply to the longitudinal or phase coordinate. As we shall see, such effects arise in the longitudinal coordinate only if the colliding beams cross at an angle, and in that case there is a limit on the number of electrons or positrons which can be stored in a bunch.

In the past in storage rings using head-on collisions both incoherent and coherent transverse beam instabilities have been observed and studied extensively because of their importance to achievable reaction rates. In the incoherent transverse instability, the weaker beam is disrupted by the stronger beam so that the central density of the weaker beam is diminished and the luminosity accordingly reduced. This effect was first described by Amman and Ritson who characterized the onset of serious disruption by the shift in the vertical betatron frequency for small amplitude oscillations of the weak beam particles due to the strong beam forces.[1] Both experience and computation have shown that, when this shift $\Delta Q_y$ exceeds a value in the range 0.05 to 0.025, disruption ensues.[2] These transverse disruptions of the weak beam have been interpreted as being due to the strongly nonlinear form of the (largely transverse) electromagnetic field of the gaussian distribution. Coherent transverse instabilities caused by the collisions between the beams have also been observed and have been explained in terms of the mutual electromagnetic forces between two strong beams.[3]

In the case of head-on collisions, no net longitudinal impulse is imparted to a particle by the counter-rotating bunch regardless of its longitudinal coordinate. Recently however electron-positron storage rings have been designed employing rather large crossing angles.[4] In these, the trajectories of the colliding bunches cross each other at an angle, so that the collisions are not head-on. Such designs have been evolved in efforts to achieve very high luminosities by storing many bunches in each beam. With many bunches it is necessary to avoid interactions between counter-moving bunches at places other than the interaction regions where the detection equipment is located, and this is done by storing the beams in separate storage rings which intersect at an angle.

With the introduction of a crossing angle, the impulses imparted to the particles of one beam by the bunches of the opposite beam are no longer purely transverse relative to the equilibrium orbits of the particles and no longer independent of their longitudinal (phase) coordinates. The electric field of an oppositely moving bunch exerts on a particle a longitudinal force which depends on the phase of the particle. Thus Augustin has called attention to consequent potential longitudinal beam instabilities in crossing angle storage rings which are analogous to the transverse instabilities described above.[5] The present paper describes an investigation of these effects.

## Form of the Longitudinal Impulse

We describe the longitudinal motion of a particle in the crossing angle storage ring system in terms of its energy deviation $\epsilon$ from the synchronous energy E and its phase deviation $\phi$ from the synchronous phase $\psi_s$. We ignore transverse motion and consider each particle to move on its equilibrium orbit. The energy increment $\delta\epsilon$ given a particle due to the impulse imparted by one passage through a counter-rotating gaussian bunch of N particles is

$$\delta\epsilon = - \frac{2^{3/2} r_e mc^2 Nk}{R\sigma_\phi} \int_0^y e^{t^2 - y^2} dt, \qquad (1)$$

where $r_e$ is the classical radius and $mc^2$ the rest energy of the electron, k is the harmonic order of the radiofrequency accelerating system, R is the gross radius of the orbit, $\sigma_\phi$ is the standard deviation of the gaussian longitudinal density distribution of the bunch, and $y = \phi/\left(2^{1/2}\sigma_\phi\right)$.[5] The minus sign holds for electron-positron collisions; it is reversed for electron-electron collisions. In Eq. (1) it is assumed that the product of the bunch length and the crossing angle is large compared to the bunch height in the case of a vertical crossing, or compared to the beam width in the case of a horizontal crossing. Under these assumptions the impulse is independent of the crossing angle. For small $\phi$

$$\delta\epsilon \approx - \frac{2r_e mc^2 Nk}{R\sigma_\phi^2} \phi. \qquad (2)$$

The equations of longitudinal motion of a particle under the influence of the radiofrequency (rf) system and the radiation damping, neglecting for the moment radiation fluctuations, are

$$\dot{\epsilon} = eV \sin(\phi + \psi_s) - \frac{2}{D}\epsilon \approx (eV \cos \psi_s)\, \phi - \frac{2}{D}\epsilon \tag{3}$$

$$\dot{\phi} = \left(\frac{2\pi k \alpha_m}{E}\right)\epsilon \tag{4}$$

where eV is the peak energy gain per revolution available from the rf system and $\alpha_m$ is the momentum compaction coefficient. The superscript dots denote differentiation with respect to the dimensionless variable ft where f is the orbital frequency and t the time. This variable counts the number of revolutions of the particle. D is the damping time in units of this variable. These equations treat the discrete impulses given to the particle by the rf system as being spread out uniformly around the orbit. This is a good approximation for the characteristically low frequencies of the phase oscillations. We consider a storage ring system with the orbits of the colliding beams separated so that the bunches collide at only one point, and each particle of one beam encounters only one bunch of the other beam on each revolution, a situation readily realized in practice. If we then consider the collision impulse (Eq. (1)) as similarly spread out around the orbit, we can write the linearized, small-amplitude equation of motion for energy oscillations as follows.

$$\ddot{\epsilon} + \frac{2}{D}\dot{\epsilon} + \left[\omega_0^2 - \left(\frac{4\pi k^2 \alpha_m r_e N}{R\gamma\sigma_\phi^2}\right)\right]\epsilon = 0 \tag{5}$$

where $\omega_0^2 = -(2\pi k \alpha_m\ eV \cos \psi_s/E)$ is the approximate angular frequency of the oscillations in units of ft in the absence of collisions ($\omega_0$ is dimensionless), and $\gamma = E/mc^2$. The effect of the collisions is of course to shift the longitudinal oscillation frequency and, in the case of electrons hitting positrons, the shift is downward. We may define a new frequency

$$\omega^2 = \omega_0^2 - \left(\frac{4\pi k^2 \alpha_m r_e N}{R\gamma\sigma_\phi^2}\right) \tag{6}$$

A convenient beam-strength parameter is defined

$$\Delta = 1 - \frac{\omega^2}{\omega_0^2} = \frac{4\pi k^2 \alpha_m r_e N}{R\gamma\sigma_\phi^2 \omega_0^2}\ , \tag{7}$$

and we may eliminate some of the parameters of Eq. (1) in favor of $\Delta$ and $\omega_0$.

$$\delta\epsilon = -\Delta\left(\frac{E\omega_0^2}{2\pi k\alpha_m}\right)\left(2^{1/2}\ \sigma_\phi\right)\int_0^y e^{t^2 - y^2}\ dt. \tag{8}$$

Incoherent Instability

The longitudinal incoherent instability can be most clearly understood by studying the behavior of a weak beam under the influence of a strong one. Under these conditions we can assume that the strong-beam distribution is undisturbed by the weak beam and the impulse is given correctly by Eq. (8). Because of the typically low frequencies of phase oscillations in electron storage rings, we can visualize the influence of the impulse in terms of its (averaged) effect on the potential well in which a particle of the weak beam moves.* The radiofrequency system creates a potential well which, in the region of phase which would be occupied by a damped, unperturbed weak beam, is adequately approximated by

*Strictly speaking the notion of a potential well is not applicable to the system in a real storage ring; because both the radiofrequency accelerating system and the collision impulses act at certain discrete, highly localized places around the orbit. However, the impulses are weak enough that the characteristic frequencies (e.g., $\omega_0$) are sufficiently low that the concept of an approximate potential is useful.

a parabola whose curvature is proportional to the unperturbed frequency $\omega_0$. The introduction of the strong beam impulse alters the shape of the potential well, adding a potential proportional to an integral over $\phi$ of Eq. (8) and reducing the curvature at $\phi = 0$. At strong beam strength $\Delta = 1$, the curvature at $\phi = 0$ has been reduced to zero, and so correspondingly has the small amplitude frequency. At beam strengths greater than one ($\Delta > 1$), a hump appears in the center of the potential well, producing two separated potential minima. The weak beam always distributes itself in the well so as to maintain a fixed rms energy spread determined by the radiation constants and independent of the shape of the well and therefore of the strong beam strength. Thus we anticipate that the weak beam should be broadened or dispersed in phase when the strong beam distorts the potential well and that, beyond unity strong beam strength as defined above, the weak beam should begin to separate into two parts.

To test this picture we coded a simulation program to run on the SLAC IBM 360/91 computer. We chose computer simulation to study the problem primarily because we have been engaged recently in similar work, so the tools and techniques were readily at hand. In the program the weak beam distribution is developed by following the motion of a single test particle for many damping times and sampling its state of motion once each damping time. The samples are sorted into histograms which, by appeal to the ergodic hypothesis, are considered to be the weak beam bunch distribution functions in $\phi$ and $\epsilon$. Also certain statistical properties of the distributions are computed.

The radiofrequency accelerating system is treated according to Eq. (3) and Eq. (4) as a linear restoring force and average radiated power is considered to vary linearly with energy, so the particle dynamics between impulses are those of a damped harmonic oscillator. Radiation fluctuations are simulated by introducing instantaneous increments in the variable $\epsilon$ distributed randomly and uniformly in the interval between $-W/2$ and $+W/2$. Such a spectrum has rms value $f_{\epsilon rms} = W/12^{1/2}$. It was found unnecessary to randomize the phase at which these fluctuations were applied in the range of oscillator frequencies of interest. The impulse, simulating the collision with the strong beam, Eq. (8), is applied once each turn.

For purposes of computation, we measure energy in units of $(E\omega_0/2\pi k\alpha_m)$ so that for $\Delta = 0$, $\sigma_\epsilon = \sigma_\phi$ where $\sigma_\epsilon$ is the rms energy deviation of the test particle. Also we expect

$$\sigma_\epsilon = f_{\epsilon rms}\, n_{fd}^{1/2}/2 = W(n_{fd}/48)^{1/2}, \tag{9}$$

where $n_{fd}$ is the number of fluctuations occuring in a damping time. The width W of the rectangular random-increment spectrum is adjusted to give, for $\Delta = 0$,

$$\sigma_\epsilon = 2^{-1/2} \text{ energy unit, } \sigma_\phi = 2^{-1/2}, \tag{10}$$

which, together with the choice of energy unit, results in a simplified form for Eq. (8) to speed computation. The program was tested to verify that Eq. (10) held for the weak beam distributions generated in the absence of a strong beam $(\Delta = 0)$ over a range of oscillator frequencies and damping times covering the region of interest.

For simulation of the influence of the strong beam, the results of a numerical integration of Eq. (8) were tabulated in memory as a function of $\phi$ at intervals of 0.1 unit. During the simulation, the impulse as a function of the coordinate $\phi$ was obtained by interpolation in the table, a very fast procedure. The speed of the code was faster than 20 microseconds per interaction.

The results of several typical runs are shown in Fig. 1. In each run the particle is started at $\phi = 0$, $\epsilon = 0$ and run for 1000 damping times. The coordinates $\phi$ and $\epsilon$ are sampled at intervals of a damping time and sorted into the histograms shown in Fig. 1. The sampled coordinates are also used to compute their rms values and the overlap integral of the strong beam and weak beam distributions. The overlap integral $\Theta$, which is the integral over $\phi$ of the product of the weak beam distribution and the (unperturbed) strong beam distribution, is computed by summing contributions of the form $e^{-\left(\phi^2/2\sigma_\phi^2\right)}$ where $\phi$ is the sampled coordinate and $\sigma_\phi^2 = 0.5$ as in Eq. (10). At the end of the simulation computation the sum is normalized so that it is equal to one for complete overlap, which occurs for $\Delta = 0$. As a measure of the statistical fluctuation in the results of the simulation, we find an rms spread of the order of $1000^{-1/2}$ in the values of the rms value of $\epsilon$ for different runs with the same parameters, indicating that samples taken at intervals of a damping time are statistically independent.

The distributions of Fig. 1 show that there is little disruption or dilution of the weak beam up to $\Delta = 1$. At $\Delta = 2$ serious broadening in $\phi$ has begun, and at $\Delta = 3$, the weak beam has separated into two lobes. The overlap integral is plotted in Fig. 2 as a function of $\Delta$. The luminosity is proportional to the beam strength and to the overlap integral and we define their product as the relative luminosity which is plotted against $\Delta$ in Fig. 3. It is clear that, from the point of view of incoherent disruption of the weak beam by the strong beam, beam strengths well above $\Delta = 1$ (the value for which the small amplitude frequency $\omega$ goes to zero) are permissible.

Coherent Instability

Since the integrity of the colliding bunches is preserved to rather large values of the beam strength parameter, it is reasonable to study the coherent longitudinal motion in the approximation of rigid bunches in the range $\Delta < 1$.

This is a strong-beam-strong-beam case. For this purpose, we assume that each bunch of each beam is stable under the influence of its own radiofrequency system in the absence of the other beam and that each bunch of one beam collides with only one bunch of the opposing beam. Again spreading out the interactions and, in addition, linearizing the impulse as in Eq. (2), we write for the motion of bunch 1 and bunch 2,

$$\ddot{\phi}_1 + 2\alpha_1\dot{\phi}_1 + \omega_1^2\phi_1 = K_2(\phi_1 - \phi_2)$$

$$\ddot{\phi}_2 + 2\alpha_2\dot{\phi}_2 + \omega_2^2\phi_2 = K_1(\phi_2 - \phi_1), \tag{11}$$

where $\alpha_1$ and $\alpha_2$ are both positive damping rates and include both the effects of radiation damping and those of the beam-rf-system interaction.[6]

$$K_1 = \frac{4\pi k^2 \alpha_m r_e N_1}{R\gamma\sigma_{\phi 1}^2} = \omega_1^2 \Delta_1 , \tag{12}$$

and similarly for $K_2$. These are coupled, damped-oscillator equations. The free phase-oscillation frequencies for the uncoupled oscillators ($K_1 = K_2 = 0$) are $\left(\omega_1^2 - \alpha_1^2\right)^{1/2}$ and $\left(\omega_2^2 - \alpha_2^2\right)^{1/2}$, respectively. The characteristic equation for the normal modes $e^{\beta ft}$ of Eq. (11) is

$$\beta^4 + \beta^3[2(\alpha_1 + \alpha_2)] + \beta^2\left[4\alpha_1\alpha_2 + \left(\omega_1^2 - K_2\right) + \left(\omega_2^2 - K_1\right)\right]$$

$$+ \beta\left[2\alpha_1\left(\omega_2^2 - K_1\right) + 2\alpha_2\left(\omega_1^2 - K_2\right)\right] + \left(\omega_1^2 - K_2\right)\left(\omega_2^2 - K_1\right) - K_1K_2 = 0 \tag{13}$$

The conditions of the Routh-Hurwitz criterion that there be no unstable normal modes reduce to the single inequality

$$\left(\omega_1^2 - K_2\right)\left(\omega_2^2 - K_1\right) - K_1K_2 = \omega_1^2\omega_2^2 - \omega_1^4\Delta_1 - \omega_2^4\Delta_2 \geq 0. \tag{14}$$

At this point, it is important to note that the beam strength parameter $\Delta$ is independent of radiofrequency system parameters (although that fact is not explicitly evident from its definition, Eq. (7)). For the gaussian equilibrium distributions we are considering,

$$\omega_0 \sigma_\phi = \frac{2\pi k \alpha_m \sigma_\epsilon}{E} , \tag{15}$$

from which we can write

$$\Delta = \frac{r_e (mc^2)^2 \gamma N}{\pi R \alpha_m \sigma_\epsilon^2} \tag{16}$$

which is obviously independent of rf parameters.

The luminosity is proportional to $\left(\Delta_1 \ \Delta_2 / \left(\sigma_1^2 + \sigma_2^2\right)^{1/2}\right)$ where $\sigma_1$ and $\sigma_2$ are the rms phase spreads, and Eq. (14) can be rewritten as

$$\frac{\sigma_2^2 \Delta_1}{\sigma_1^2} + \frac{\sigma_1^2 \Delta_2}{\sigma_2^2} \leq 1 \tag{17}$$

It follows that $\sigma_1$ and $\sigma_2$ should be made as small as possible by increasing the accelerating voltage as much as possible. This, of course, results in the highest possible synchrotron oscillation frequencies. Then assuming the accelerating voltages to be equal we obtain the restriction

$$\Delta \leq \frac{1}{2} \tag{18}$$

for each bunch. This restriction required to retain coherent stability is obviously more stringent than that imposed by the incoherent instability treated in the previous section as can be seen with reference to Fig. 3.

Physically this limit represents the beam strength at which one of the normal mode frequencies has been pushed to zero. Beyond that point, that mode would be characterized by exponentially growing separation of the bunches. For large separations the linearized analysis breaks down and nonlinearities play an important role. It is probable therefore, that Eq. (18) gives a conservative estimate of the effective limit.

## Conclusions

The coherent longitudinal interaction between bunches in an electron-positron storage ring system employing a crossing angle places a limit on the charge which may be stored in each bunch. At this limit, there will be no substantial incoherent disruption of either bunch by the other. The limit is such that the frequency of small-amplitude incoherent phase oscillations of individual particles is reduced by the factor $2^{-1/2}$ from its unperturbed value. The limit is conservatively estimated from the small-amplitude analysis to be

$$N \leq \frac{\pi R \alpha_m \sigma_\epsilon^2}{2 r_e m c^2 E} . \tag{19}$$

The actual limit in practice may be considerably higher owing to the stabilizing influence of the intrinsic nonlinearities. As an example, for the intersecting storage rings presently under design at SLAC, $R = 35\text{m}$, $\alpha_m = 0.03$ and at $E = 2$ GeV, $\sigma_\epsilon = 0.96$ MeV. For these parameters $N \leq 0.5 \times 10^{12}$.

## Acknowledgments

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## REFERENCES AND FOOTNOTES


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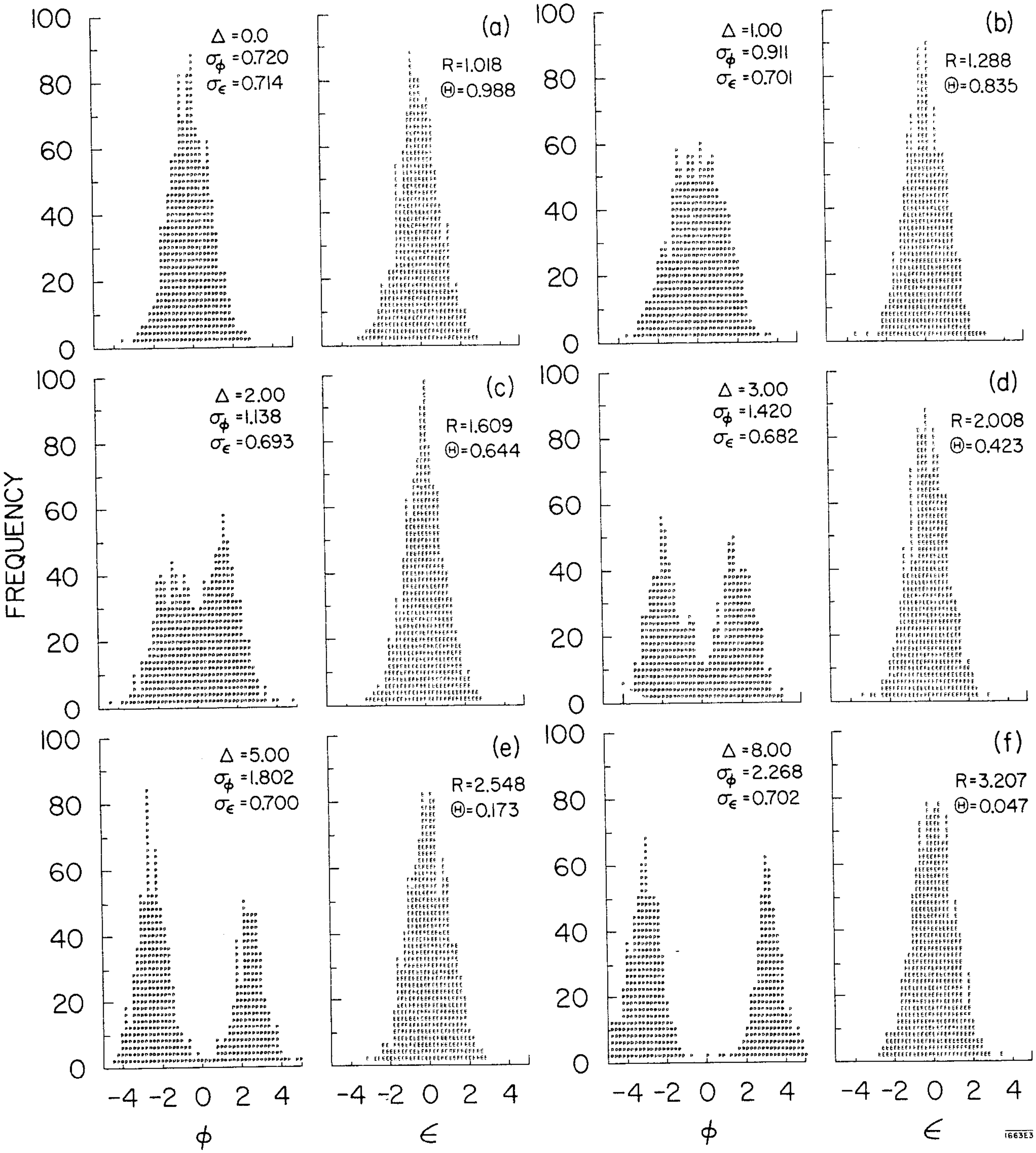
## FIGURE CAPTIONS

1. Equilibrium distributions in phase and energy at different values of the beam strength parameter $\Delta$. For all distributions $\omega_0/2\pi = 5 \times 10^{-3}$, the damping time is 2000 revolutions and there are 1000 samples in the histogram. On each histogram is shown the corresponding values of $\Delta$, the strong beam strength parameter, $\sigma_\phi$, the rms spread in phase, $\sigma_\epsilon$, the rms spread in energy, R, the ratio of $\sigma_\phi$ to its unperturbed value of 0.707 and, $\Theta$, the overlap integral.

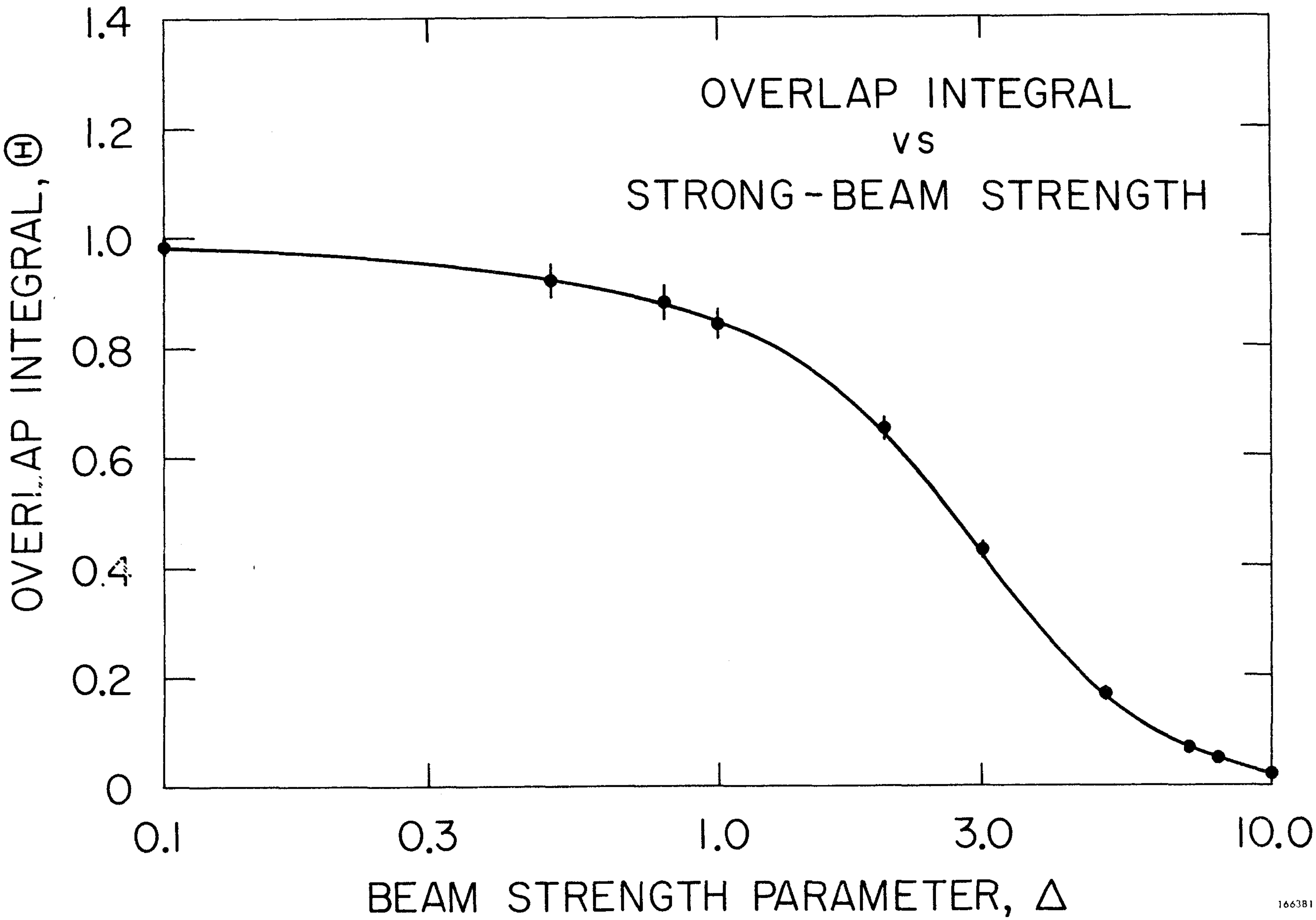
2. The overlap integral $\Theta$ of the weak beam and strong beam distributions as a function of the strong beam strength parameter $\Delta$. The integral is normalized to one for complete overlap which occurs for $\Delta = 0$. Error flags indicate estimated rms statistical spread.

3. Relative luminosity as a function of the strong beam strength parameter $\Delta$. The straight, dashed line shows the luminosity which would be achieved if there were no disruption due to the strong beam. Error flags indicate estimated rms statistical spread.



FREQUENCY
(a)
Δ = 0.0
σ_φ = 0.720
σ_ε = 0.714
R = 1.018
Θ = 0.988
(b)
Δ = 1.00
σ_φ = 0.911
σ_ε = 0.701
R = 1.288
Θ = 0.835
(c)
Δ = 2.00
σ_φ = 1.138
σ_ε = 0.693
R = 1.609
Θ = 0.644
(d)
Δ = 3.00
σ_φ = 1.420
σ_ε = 0.682
R = 2.008
Θ = 0.423
(e)
Δ = 5.00
σ_φ = 1.802
σ_ε = 0.700
R = 2.548
Θ = 0.173
(f)
Δ = 8.00
σ_φ = 2.268
σ_ε = 0.702
R = 3.207
Θ = 0.047
φ
ε
1663E3


Fig. 1



OVERLAP INTEGRAL
vs
STRONG-BEAM STRENGTH
OVERLAP INTEGRAL, Θ
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
0.1
0.3
1.0
3.0
10.0
BEAM STRENGTH PARAMETER, Δ
166381


Fig. 2

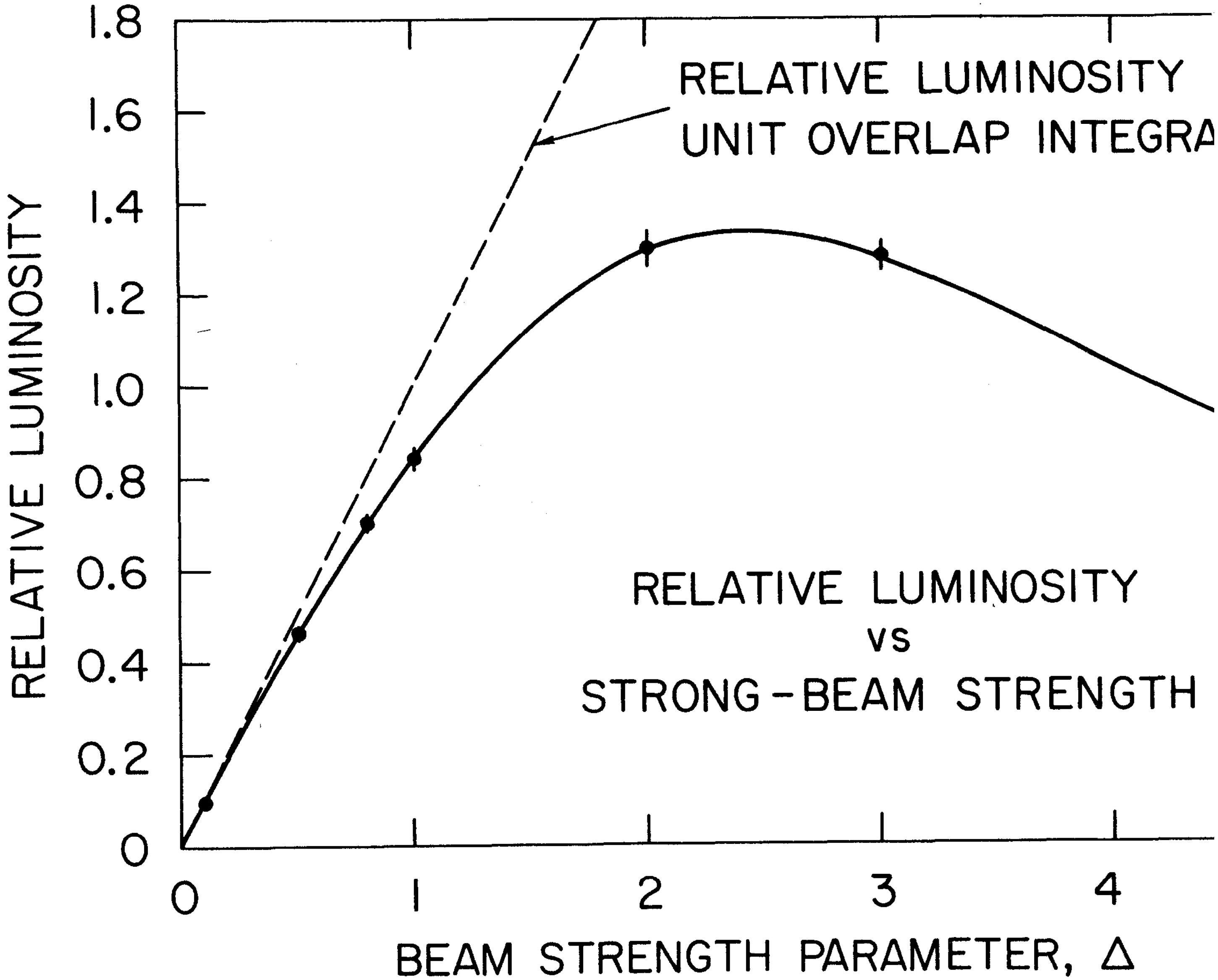

1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
RELATIVE LUMINOSITY
RELATIVE LUMINOSITY
UNIT OVERLAP INTEGRA
RELATIVE LUMINOSITY
vs
STRONG-BEAM STRENGTH
0
1
2
3
4
BEAM STRENGTH PARAMETER, Δ

Fig. 3