SLAC-PUB-747
May 1970
(TH)

METHODS FOR THE BETHE-SALPETER EQUATION I: SPECIAL FUNCTIONS AND EXPANSIONS IN SPHERICAL HARMONICS*

David Kershaw and Herschel Snodgrass
University of California, Berkeley, California

Charles Zemach
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

ABSTRACT

Elements of the structure of the Bethe-Salpeter equation are studied. Properties of useful special functions are obtained and free particle solutions to the truncated expansion of the equation in four-dimensional spherical harmonics are derived in terms of known special functions. Validity of the truncation approximation is examined in terms of a convenient representation of the Green's function. In particular, it is shown that the method of truncating the differential Bethe-Salpeter equation cannot succeed for scattering. The development of alternative procedures is deferred to the paper following. As a by-product, a simply computational technique for the approximation of integrals by Gaussian quadrature is derived.

(Submitted to Phys. Rev.)

---

*Work supported in part by the U. S. Atomic Energy Commission and by the Air Force Office of Scientific Research.

I. INTRODUCTION

The two-body relativistic equation known as the Bethe-Salpeter equation (BSE) received its first applications in quantum electrodynamics more than fifteen years ago, but a substantial period elapsed before quantative studies relevant to, or at least preliminary to, strong interaction calculations were taken up. Following the first calculations in the ladder approximation for bound states[1] and scattering,[2] obtained from variational principles in coordinate space, many results and a diversity of methods have been obtained for the two-body relativistic problem, both in the bound[3] and scattering regions.[4]

If one takes seriously, as we certainly do, the view that these relativistic wave equations and their extensions offer a fruitful approach to the dynamics of strong interactions, one must anticipate a development of both theoretical structure and of calculational techniques of which the references above are only the beginning, a development with a larger perspective than the desire to match an experimental number immediately, and in fact, more extensive and varied than the work done on the Schroedinger equation in proportion as the relativistic problem is more complex than the non-relativistic problem.

The present work is the first in a series in which we explore the properties of the BSE in coordinate space, its special functions, asymptotic behavior, approximation schemes, some new calculational methods, shapes and nodal properties of wave functions, significance of the so-called "abnormal" states, and so on, with intent to develop an intuition about the equation as well as analytic behavior.

We surmise that many of these questions are more aptly understood in coordinate, rather than momentum, space because of certain intuitional advantages and because this is distinctly true for non-relativistic equations, but this is only a surmise. Problems arising from inelasticity may be more amenable to analysis in momentum space and in fact, most of the calculations of Refs. 3 and 4 follow momentum space formulations.

In this paper, we treat the free-particle Bethe-Saltpeter equation in Wick-rotated form,

$$\left[\nabla^2 + \left(\frac{\partial}{\partial \tau} - \omega_1\right)^2 - m_1^2\right]\left[\nabla^2 + \left(\frac{\partial}{\partial \tau} + \omega_2\right)^2 - m_2^2\right] \psi(\underline{r}, \tau) = 0 \tag{I. 1}$$

with two principal objectives.

First, we explore the family of free-particle solutions, their Green's functions, and special functions related to them, both for a deeper understanding of the structure of the BSE and as a preliminary to calculations of bound state, scattering, and bootstrap parameters, to be described in a subsequent paper.

Secondly, we study approximations based on expansion of the wave function in four-dimensional spherical harmonics (Gegenbauer polynomials) and truncation of the series after a finite number of terms. The truncated function satisfies either (a) the differential equation formed by truncation of the BS differential operator or (b) the integral equation obtained upon truncation of the free-particle Green's function. Either approach transforms an equation in two continuous variables r, τ into a matrix equation in one continuous variable x, $x = (r^2 + \tau^2)^{1/2}$. One hopes to find convergence in calculated results as the matrix size, i.e., the number of retained terms in the spherical expansion, is increased. The first scheme – let's call it the differential method – has been applied to bound state calculations and the second, or integral method, to scattering as well, usually in momentum space.

In this paper, we obtain a correspondence between the two: the Green's function, and other functions characteristic of the differential method can be cast as sums over a certain parameter α, while analagous functions in the integral method appear as integrals over α. The relation of sum to integral is that of a Gaussian quadrature approximation based on the orthogonal family of four-dimensional spherical harmonics. The truncated integral equation is thus inherently more precise.

For bound states with energies not close to threshold, the differences are minimal. But as the energy moves up into the scattering region, an integrable singularity, of the inverse square root type, moves into the interval of integration over α. The inability of a Gaussian sum to approximate integrals of this type dooms the differential truncation method for scattering problems. Not only does unitarity fail at any state of truncation, but there is no convergence to unitarity.

The same analysis which exposes the difficulty in the differential method provides a remedy. Certain methods closely related to it can be made to work in the scattering case if the boundary conditions imposed on the wave function at infinity are inferred from the exact BSE which, in this respect, differs essentially from the truncated differential BSE. This will be done in the paper to follow, entitled "Methods for the Bethe-Salpeter Equation II," to which we shall refer as MBS II. By a slight variation of the spherical harmonic expansion procedure, we achieve a unified approach to bound state and scattering problems, unitary at each level of approximation, and founded on the differential equation, that is, the Wick-rotated differential BSE in coordinate space. We believe it to be a strong contender, among methods investigated till now, for strong interaction calculations involving derivative couplings, non-local interactions, and electromagnetic perturbations. Of course, the advantages anticipated for the coordinate space methods over momentum space methods need to be tested on meaningful physical problems.

The next section contains the relations among the spherical harmonic functions $R_{n,\ell}(\theta)$, and the imaginary Bessel functions $I_n(z)$, $K_n(z)$ which are fundamental to the special functions of the BSE. The asymptotic properties of solutions to the truncated free-particle BSE are considered in Section III, and explicit solutions, in terms of what we call vector Bessel functions, are constructed. In the fourth section, the notion of a bracket $[\phi, \psi]$ of two functions is defined, and brackets among the free-particle solutions are computed as a step in the derivation of the Green's function for the BSE. The importance of the bracket will be explored in considerably more detail in MBS II. An array of Green's functions which are useful in one way or another is marshalled in Section V. Finally, the results on the relation between the integral and differential approaches to truncation and the inadequacy of the differential method are given in Section VI.

As a by-product, an elementary and, apparently, new method for deriving and applying the rule for Gaussian quadrature is found. When the relevant orthogonal polynomials are normalized, the two or three term recursion relation obtained by multiplying one of them by the variable defines a tridiagonal symmetric matrix. The data required for an Nth order Gaussian quadrature are obtained from the eigenvalues and eigenvectors of the N $\times$ N truncation of the matrix which are easily calculated even for large N by current computer techniques.

Generally speaking, our notation follows that of SZ, Ref. 2. Here is a preliminary outline: the two interacting particles have masses m_1, m_2, and space-time coordinates

$$x_{1\mu} = (\underline{r}_1, t_1), \; x_{2\mu} = (\underline{r}_2, t_2) \quad . \tag{I. 2}$$

The two-particle system has a definite energy-momentum vector P_μ and is referred to the center-of-mass frame: $P_\mu = (\underline{0}, E)$. The space minus time convention is used for scaler products. For energy above threshold, $E = m_1 + m_2$, the relative momentum k of the system is calculated from

$$E = (k^2 + m_1^2)^{1/2} + (k^2 + m_2^2)^{1/2} \quad , \tag{I.3}$$

whence the "particle energies" $\underline{\omega_1}$, $\underline{\omega_2}$ are given by

$$\omega_1 = (k^2 + m_1^2)^{1/2} \; ; \; \omega_2 = (k^2 + m_2^2)^{1/2} \; , \; E = \omega_1 + \omega_2 \quad . \tag{I.4}$$

In the bound state region, $|m_1^2 - m_2^2|^{1/2} \leq E \leq m_1 + m_2$. The momentum is positive imaginary, so put $k = i\kappa$ and

$$\omega_1^2 = m_1^2 - \kappa^2 ; \; \omega_2^2 = m_2^2 - \kappa^2 \quad . \tag{I.5}$$

For the purposes of this paper, we take the center-of-mass and relative co-ordinates as

$$X_\mu = (\underline{R}, T) = (\omega_1 x_1 + \omega_2 x_2)/E; \quad x_\mu = (\underline{r}, t) = x_1 - x_2 \; , \tag{I.6}$$

allowing the separation

$$\Psi(x_1, x_2) = \exp(iP_\mu X_\mu) \, \psi(x) = \exp(-iET) \, \psi(x) \quad . \tag{I.7}$$

The momentum transformation conjugate to Eq. (I.6) is

$$P_\mu = (\underline{0}, E) = p_1 + p_2 ; \; p_\mu = (\underline{p}, p_0) = (\omega_2 p_1 - \omega_1 p_2)/E \quad . \tag{I.8}$$

The Wick rotation in the complex planes of the relative time and relative energy variables is carried out by setting

$$t = \tau e^{-i\phi}, \; p_0 = -p_4 e^{+i\phi}, \; \phi \text{ goes from } 0 \text{ to } \pi/2 \; , \tag{I.9}$$

with the result, for $\phi = \pi/2$,

$$t = -i\tau ; \; p_0 = -ip_4, \; x_\mu p_\mu = r \cdot p - tp_0 = r \cdot p + \tau p_4 \quad . \tag{I.10}$$

When this rotation is introduced into the BSE for two spinless particles,

$$\left(\nabla_1^2 - \frac{\partial^2}{\partial t_1^2} - m_1^2\right)\left(\nabla_2^2 - \frac{\partial^2}{\partial t_2^2} - m_2^2\right)\psi - I\psi = 0 \quad , \tag{I.11}$$

the result is

$$\mathscr{D}_1 \mathscr{D}_2 \psi - V\psi = 0 \quad , \tag{I.12}$$

where $\psi = \psi(\underline{r}, \tau)$ is the (Wick-rotated) relative wave function, V is the interaction, non-local in general, and

$$\mathscr{D}_1 = \nabla^2 + \left(\frac{\partial}{\partial \tau} - \omega_1\right)^2 - m_1^2 = \Box^2 - 2\omega_1 \frac{\partial}{\partial \tau} + k^2 \quad , \tag{I.13a}$$

$$\mathscr{D}_2 = \nabla^2 + \left(\frac{\partial}{\partial \tau} + \omega_2\right)^2 - m_2^2 = \Box^2 + 2\omega_2 \frac{\partial}{\partial \tau} + k^2 \quad . \tag{I.13b}$$

Note that $\mathscr{D}_1 \to \mathscr{D}_2$ when $m_1 \to m_2$ and $\omega_1 \to$ minus ω_2. Spherical coordinate notation is summarized by

$$r = x \sin\theta, \quad \tau = x\cos\theta \ , \quad 0 \leqq \theta \leqq \pi \quad , \tag{I.14}$$

and

$$d^4x = d\underline{r}d\tau = x^3 dx \sin^2\theta \, d\theta d\Omega = x^3 dx \sin^2\theta d\theta \sin\theta_3 d\theta_3 d\phi \quad . \tag{1.15}$$

The relative momentum k_μ of two free particles of momenta k_1, k_2 is

$$k_\mu = (\omega_2 k_1 - k_2\omega_1)_\mu / E = (\underline{k}, 0) \quad , \tag{I.16}$$

with $|\underline{k}|$ defined by Eq. (I.3). The resolution into partial waves of this state is

$$e^{ik_{1\mu}x_{1\mu}} e^{ik_{2\mu}x_{2\mu}} = e^{-iET} e^{i\underline{k}\cdot\underline{r}} = e^{-iET} \sum (i)^\ell (2\ell+1) P_\ell (\hat{k}\cdot\hat{r}) \, j_\ell(kr) \tag{I.17}$$

In this paper, we treat only spinless particles and shall suppose that the system has a definite angular momentum ℓ. Then the relative wave function takes the form

$\psi(r,\tau)\, Y_\ell^m(\theta_3, \phi)$. For brevity, we drop explicit mention of the Y_ℓ^m factor. For example, we say that $j_\ell(kr)$ is a solution of (I.11) for V = 0 when we mean that $j_\ell(kr)\, Y_\ell^m(\theta_3, \phi)$ is a solution.

II. SPHERICAL HARMONICS IN FOUR DIMENSIONS AND BESSEL FUNCTIONS

Let L^2 and $\mathscr{L}^2$ be the angular parts of the three dimensional and four dimensional Laplacian, respectively. Thus

$$L^2 = -\frac{1}{\sin\theta_3}\frac{\partial}{\partial\theta_3}\left(\sin\theta_3\frac{\partial}{\partial\theta_3}\right) - \frac{1}{\sin^2\theta_3}\frac{\partial^2}{\partial\phi^2}\ , \tag{II.1}$$

$$\mathscr{L}^2 = -\frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin^2\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}L^2\ , \tag{II.2}$$

$$\Box^2 = \nabla^2 + \frac{\partial^2}{\partial\tau^2} = \frac{d^2}{dx^2} + \frac{3}{x}\frac{d}{dx} - \frac{1}{x^2}\mathscr{L}^2\ . \tag{II.3}$$

The eigenvalues and eigenfunctions of L^2 are, of course, $\ell(\ell+1)$ and $Y_\ell^m(\theta_1, \phi_3)$. For $\mathscr{L}^2$, we have

$$\mathscr{L}^2\left\{R_{n,\ell}(\theta)\, Y_\ell^m(\theta_3,\phi)\right\} = n(n+2)\, R_{n,\ell}(\theta)\, Y_\ell^m(\theta_3,\phi),\ n=\ell, \ell+1, \ell+2, \ell+3, \ldots, \tag{II.4}$$

introducing a notation for the spherical harmonics in four dimensions, $R_{n,\ell}(\theta)$. More precisely, we define them in terms of the Gegenbauer polynomials $C_n^\nu(z)$ by

$$R_{n,\ell} = \left(\frac{2}{\pi}\right)^{1/2}\left\{2^\ell \ell!^2\ \frac{(n-\ell)!}{(n+\ell+1)!}\ (n+1)\right\}^{1/2} (\sin\theta)^\ell\, C_{n-\ell}^{1+\ell}(\cos\theta)\ . \tag{II.5}$$

Their elementary properties can be inferred, after translation of notation, from standard references.[5] They are mutually orthogonal and normalized to unity:

$$\int_0^\pi R_{n,\ell}(\theta)\, R_{n',\ell'}(\theta)\, \sin^2\theta\, d\theta = \delta_{n,n'} \tag{II.6}$$

At $\theta = \pi/2$, we have

$$C_{n-\ell}^{1+\ell}(0) = 0 \quad , \; n-\ell \text{ odd} \quad ,$$
$$= (-1)^{(n-\ell)/2} \frac{\left(\frac{n+\ell}{2}\right)!}{\ell! \left(\frac{n-\ell}{2}\right)!} \quad , \; n-\ell \text{ even.} \tag{II.7}$$

In the case ℓ=0,

$$R_{n,0} = \left(\frac{2}{\pi}\right)^{1/2} C_n^1(\cos\theta) = \left(\frac{2}{\pi}\right)^{1/2} \frac{\sin(n+1)\theta}{\sin\theta} \tag{II.8}$$

The addition theorem reads

$$(n+1)\, C_n^1(\cos\gamma) = \frac{1}{2}\pi \sum_{\ell=0}^{n} (2\ell+1)\, R_{n,\ell}(\theta)\, R_{n,\ell}(\theta')\, P_\ell(\cos\theta_3) \;,$$
$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\theta_3 \;. \tag{II.9}$$

The equation

$$\left(\Box^2 - \lambda^2\right)\psi = 0 \quad , \tag{II.10}$$

has particular solutions

$$\mathscr{I}_n(\lambda x, \theta) = \frac{I_{n+1}(\lambda x)}{\lambda x} R_{n,\ell}(\theta) \quad , \tag{II.11a}$$

and

$$\mathscr{K}_n(\lambda x, \theta) = (-1)^{n-\ell} \frac{K_{n+1}(\lambda x)}{\lambda x} R_{n,\ell}(\theta) \tag{II.11b}$$

where $I_{n+1}(\lambda x)$, $K_{n+1}(\lambda x)$ are the imaginary Bessel functions. The $\mathscr{I}_n$ and $\mathscr{K}_n$ will be useful later.

The function $x^n R_{n,\ell}(\theta)$ is a homogeneous polynomial in r, τ of degree n and contains the "centrifugal" factor $(x\sin\theta)^\ell = r^\ell$. While ℓ is a good quantum member for the spinless BSE, different n's are mixed by the operator $\partial/\partial\tau$,

$$\frac{\partial}{\partial\tau} = \frac{\partial}{\partial x}\cos\theta - \frac{1}{x}\sin\theta\frac{\partial}{\partial\theta} \;. \tag{II.12}$$

A solution of the BSE possesses the expansion

$$\psi(x_\mu) \equiv \psi(x,\theta) = \sum_{n=\ell}^{\infty} f_n(x)\, R_{n,\ell}(\theta) \quad . \tag{II.13}$$

Substitution of Eq. (II.13) into Eq. (I.11) and application of the orthonormality of the $R_{n,\ell}$'s in the usual way yields a matrix differential equation of fourth order for the "vector" $\{f_n(x)\}$ and this, ultimately, is the equation we wish to solve. Note that if $\tau \to -\tau$, then $\theta \to \pi - \theta$ and, moreover,

$$R_{n,\ell}(\pi-\theta) = (-1)^{n-\ell} R_{n,\ell}(\theta) \quad . \tag{II.14}$$

Functions $\psi(x_\mu)$ even under $\tau \to -\tau$ will be composed from $R_{n,\ell}$'s with $n=\ell$, $\ell+2$, $\ell+4$, ..., and will be said to have "even time parity." Functions with "odd time parity" change sign under $\tau \to -\tau$ and are composed of harmonics with $n = \ell+1$, $\ell+3$, $\ell+5$, ... In the equal mass case, $m_1 = m_2$, the operator $\mathscr{D}_1\mathscr{D}_2$ is invariant under $\tau \to -\tau$, and time-parity is a good quantum number for the BSE.

Hereafter, we assume the choice of ℓ is fixed. The ℓ subscript on the R's will be dropped.

The recursion formulas are

$$\cos\theta\, R_n = A_{n+1} R_{n+1} + A_n R_{n-1} \quad , \tag{II.15a}$$

$$-\sin\theta \frac{\partial}{\partial\theta} R_n = -n\, A_{n+1}\, R_{n+1} + (n+2) A_n\, R_{n-1} \quad , \tag{II.15b}$$

$$\cos^2\theta\, R_n = \alpha_{n+2} R_{n+2} + \beta_n R_n + \alpha_n R_{n-2} \quad , \tag{II.15c}$$

where

$A_\ell = \alpha_\ell = \alpha_{\ell+1} = 0$, and in general,

$$A_n = \frac{1}{2}\left\{1 - \frac{\ell(\ell+1)}{n(n+1)}\right\}^{1/2} \quad , \tag{II.16a}$$

$$\alpha_n = A_n A_{n-1} \quad , \tag{II.16b}$$

$$\beta_n = A_n^2 + A_{n+1}^2 = \frac{1}{2}\left\{1 - \frac{\ell(\ell+1)}{n(n+2)}\right\} . \tag{II.16c}$$

Let $\partial/\partial\tau$ act on the solutions $\mathscr{I}_n$, $\mathscr{K}_n$ of the wave equation (II.10). The Bessel relations

$$\frac{2n}{z} I_n(z) = I_{n-1}(z) - I_{n+1}(z), \; 2I_n'(z) = I_{n-1}(z) + I_{n+1}(z) \; , \tag{II.17a}$$

$$-\frac{2n}{z} K_n(z) = K_{n-1}(z) - K_{n+1}(z), \; -2K_n'(z) = K_{n-1}(z) + K_{n+1}(z) \; , \tag{II.17b}$$

together with Eqs. (II.15) and (II.12) imply

$$\frac{\partial}{\partial\tau} \mathscr{I}_n = \lambda(A_{n+1}\mathscr{I}_{n+1} + A_n\mathscr{I}_{n-1}) \quad , \tag{II.18a}$$

$$\frac{\partial}{\partial\tau} \mathscr{K}_n = \lambda(A_{n+1}\mathscr{K}_{n+1} + A_n\mathscr{K}_{n-1}) \quad . \tag{II.18b}$$

That is, $\mathscr{I}_n$ and $\mathscr{K}_n$ satisfy, with respect to $\partial/\partial(\lambda\tau)$, the same recursion relation that R_n does with respect to $\cos\theta$. This is crucial for the construction, in the next section, of explicit "vector Bessel functions" which solve the truncated, free-particle BSE.

Another formula uniting Bessel functions and the R functions is

$$j_\ell(-i\lambda \sin\alpha\, r)\, e^{\lambda \cos\alpha\,\tau} = \pi\sum_{n=\ell}^{\infty} \frac{I_{n+1}(\lambda x)}{\lambda x} R_n(\alpha)\, R_n(\theta) \tag{II.19}$$

$$= \pi\sum \mathscr{I}_n(\lambda x, \alpha)\, R_n(\theta) = \pi\sum \mathscr{I}_n(\lambda x, \theta)\, R_n(\alpha) \quad .$$

As a special case, take $\alpha = \pi/2$, $\lambda = -ik$, and note that $I_n(iz) = i^n J_n(z)$. Then

$$j_\ell(kr) = \pi \sum_{n-\ell \text{ even}} (-1)^{(n-\ell)/2} \frac{J_{n+1}(kx)}{kx} R_n(\pi/2)\, R_n(\theta) \quad , \tag{II. 20}$$

which is the harmonic expansion of a free-particle standing wave. Equation (II. 19) may look imposing, but its proof lies near the surface of our considerations. Thus, since

$$\frac{\partial^2}{\partial \tau^2} e^{\lambda \cos\alpha\, \tau} = (\lambda \cos \alpha)^2 e^{\lambda \cos \alpha\, \tau} \tag{II. 21a}$$

and

$$\nabla^2 j_\ell(-i\lambda \sin \alpha\, r) = (\lambda \sin \alpha)^2 j_\ell(-i\lambda \sin \alpha\, r) \quad , \tag{II. 21b}$$

the left side of Eq. (II. 19) satisfies Eq. (II. 10). Because it is regular at $x = 0$, it must be a sum over the functions $\mathcal{J}_n(\lambda x, \theta)$. Further the n^{th} coefficient of this sum is proportional to $R_n(\alpha)$ because the left side of Eq. (II. 19) is symmetric with respect to λ, α and x, θ. The proportionaly constant must be independent of n as the left and hence the right side are eigenfunctions of $\partial/\partial\tau$ with eigenvalue $\lambda\cos \alpha$ (see Eq. (III. 26), below). Finally, the limit $x \to 0$ determines the value of the constant.

Let γ be the angle between x_μ, x'_μ. A familiar addition theorem reads

$$\frac{K_1'(Q|x_\mu - x'_\mu|)}{Q|x_\mu - x'_\mu|} = 2 \sum_{n=0}^{\infty} (n+1)\, C_n^1 (\cos \gamma) \frac{I_{n+1}(Qx_<)}{Qx_<} \frac{K_{n+1}(Qx_>)}{Qx_>} \tag{II. 22}$$

More important to us is a result obtained by differentiating Eq. (II.22):

$$K_0(Q|x_\mu - x'_\mu|) = \frac{1}{Q}\frac{\partial}{\partial Q} Q \frac{K_1(Q|x_\mu - x'_\mu|)}{|x_\mu - x'_\mu|}$$

$$= -2 \sum_{n=0}^{\infty} (n+1)\, C_n^1(\cos\gamma) \left\{ I'_{n+1}(Qx_<) \frac{K_{n+1}(Qx_>)}{Qx_>} + \frac{I_{n+1}(Qx_<)}{(Qx_<)} K'_{n+1}(Qx_>) \right\} , \qquad \text{(II.23)}$$

or, equivalently, via Eq. (II.17),

$$K_0(Q|x_\mu - x'_\mu|) = \sum C_n^1(\cos\gamma) \left\{ I_n(Qx_<)\, K_n(Qx_>) - I_{n+2}(Qx_<)\, K_{n+2}(Qx_>) \right\} . \qquad \text{(II.24)}$$

In Section V, the resolution of $K_0(Q|x_\mu - x'_\mu|)$ into partial waves is required. Thus, we set

$$K_0(Q|x_\mu - x'_\mu|) = \sum_\ell \frac{(2\ell+1)}{4\pi} P_\ell(\hat{r}\cdot\hat{r}')\, K^{(\ell)}(x_\mu, x'_\mu, Q) \qquad \text{(II.25)}$$

and apply Eq. (II.9) to get

$$K^{(\ell)}(x_\mu, x'_\mu, Q) = 2\pi^2 \sum_{n=\ell}^{\infty} (n+1)^{-1} \left\{ I_n(Qx_<)\, K_n(Qx_>) - I_{n+2}(Qx_<)\, K_{n+2}(Qx_>) \right\} R_n(\theta) R_n(\theta') . \qquad \text{(II.26)}$$

Lastly, we mention some further Bessel relations:

$$K_n(z)\, I'_n(z) - K'_n(z)\, I_n(z) = \frac{1}{z} , \qquad \text{(II.27a)}$$

$$K_{n+1}(z)\, I_n(z) - K_n(z)\, I_{n+1}(z) = \frac{1}{z} , \qquad \text{(II.27b)}$$

$$K_n(z) = (-1)^{n+1} I_n(z) \log z + z^{-n} \times (\text{entire function of } z^2) . \qquad \text{(II.28)}$$

The cut of $K_n(z)$, like the cut of log z, is taken along the negative real z axis. Equation (II.28) implies that if $-\pi \leqq \arg z \leqq 0$, and -z means $e^{i\pi}z$,

$$(-1)^n K_n(z) - K_n(-z) = i\pi I_n(z) \quad . \tag{II.29}$$

Therefore, if $\mathrm{Im}(\lambda) \leq 0$, x real and positive,

$$(-1)^n \mathscr{K}_n(-\lambda x, \theta) - \mathscr{K}_n(\lambda x, \theta) = (-1)^\ell \, i\, \pi\, \mathscr{I}_n(\lambda x, \theta) \quad . \tag{II.30}$$

III. SOLUTIONS OF THE FREE BSE: VECTOR BESSEL FUNCTIONS

A. Truncation

Let $\psi(x, \theta)$ be a solution of the BSE whose expansion in R-functions is

$$\psi(x, \theta) = \sum_{n=\ell}^{\infty} f_n(x)\, R_n(c) \ , \tag{III.1}$$

and let N be a fixed positive integer. If the series (III.1) be cut off after the N^{th} term, we speak of an approximation to ψ by truncation with truncation parameter N. The truncated ψ is still a function of x, θ, but may also be interpreted as a vector function $\{f_n(x)\}$ in an N dimensional space. The operators $\cos\theta$, $\partial/\partial\tau$, $\mathscr{D}_1$, etc., when restricted to this space, are "truncated" operators. They may also be considered as $N \times N$ matrix operators, with matrix elements given by

$$\langle \mathscr{D}_1 \rangle^N_{nm} = \int_0^\pi R_n(\theta)\, \mathscr{D}_1 R_m(\theta)\, \sin^2\theta \, d\theta, \text{ etc.}$$

Both $\langle\cos\theta\rangle^N, \langle\cos^2\theta\rangle^N$, are Hermitian, of course, and their only non-zero matrix elements are (see Eq. (II.15))

$$\langle\cos\theta\rangle^N_{n,n-1} = \langle\cos\theta\rangle^N_{n-1,n} = A_n \quad ; \tag{III.2a}$$

$$\langle \cos^2\theta \rangle^N_{n,n-2} = \langle \cos^2\theta \rangle^N_{n-2,n} = \alpha_n \quad ; \tag{III. 2b}$$

$$\langle \cos^2\theta \rangle^N_{n,n} = \beta_n \; ; \; n = \ell, \ell+1, \ \ldots, \ N+\ell-1 \ . \tag{III. 2c}$$

Now $\langle \cos^2\theta \rangle^N$ is not the square of $\langle \cos\theta \rangle^N$; rather, in virtue of Eq. (III. 2a), the correct statement relates $\langle \cos^2\theta \rangle^N$ to the square of $\langle \cos\theta \rangle^{N+1}$.

Two approximations to the differential BSE can now be formulated, the truncated BSE

$$\left\{ \langle \mathscr{D}_1 \mathscr{D}_2 \rangle^N - V \right\} \psi = 0 \tag{III. 3a}$$

and the doubly truncated BSE

$$\left\{ \langle \mathscr{D}_1 \rangle^N \langle \mathscr{D}_2 \rangle^N - V \right\} \psi = 0 \tag{III. 3b}$$

The interaction V is presumed to be rotation invariant here. Equation (III. 3a) is doubtless a truer approximation to the exact equation because it is the condition obtained from the Rayleigh-Ritz type variational principle[1] when the trial function is truncated. But Eq. (III. 3b) is the equation we shall treat analytically because the free solutions satisfy the second order equations

$$\langle \mathscr{D}_1 \rangle^N \psi = 0 \ , \langle \mathscr{D}_2 \rangle^N \psi = 0 \quad , \tag{III. 4}$$

and these solution's can be written down analytically. The discussion is still relevant to Eq. (III. 3a) because in the equal mass case, the two formulations are equivalent, as we now show.

B. <u>The Equal Mass Case ($m_1 = m_2$)</u>

In the equal mass case, $\mathscr{D}_1 \mathscr{D}_2$ is invariant under $\tau \to -\tau$. Then $\langle \mathscr{D}_1 \mathscr{D}_2 \rangle^N$ contains $\langle \partial^2/(\partial\tau)^2 \rangle^N$, but not $\langle \partial/\partial\tau \rangle^N$. Its matrix connects terms of like parity but not of unlike time-parity. Any truncated operator $\langle O \rangle^N$ of this character can be divided into two parts,

$$\langle O \rangle^N = \left\{ \langle O \rangle^N \right\}_+ + \left\{ \langle O \rangle^N \right\}_- . \qquad \text{(III. 5)}$$

The first part connects terms of even-time parity only; that is, it has the matrix elements of type $\langle O \rangle^N_{nn'}$ with both n-ℓ and n'-ℓ even. It is a matrix of dimension $\frac{1}{2}$N or $\frac{1}{2}$(N+1) depending on whether N is even or odd. The second part does the same service for odd-time parity, being a matrix of dimension $\frac{1}{2}$N or $\frac{1}{2}$(N-1) for N even or odd. It is easy to verify that

$$\left\{ \langle \mathscr{D}_1 \rangle^{N+1} \langle \mathscr{D}_2 \rangle^{N+1} \right\}_{\pm} = \left\{ \langle \mathscr{D}_1 \mathscr{D}_2 \rangle^N \right\}_{\pm} = \left\{ \langle \mathscr{D}_1 \mathscr{D}_2 \rangle^{N+1} \right\}_{\pm} , \qquad \text{(III. 6)}$$

with the ± pertaining to N odd or even, respectively.

C. The Asymptotic Behavior of $\langle \mathscr{D}_1 \rangle^N \psi = 0$

We now search for solutions of $\langle \mathscr{D}_1 \rangle^N \psi = 0$ where

$$\psi(x, \theta) = \sum_{n=\ell}^{\ell+N-1} f_n(x)\, R_n(\theta) . \qquad \text{(III. 7)}$$

The matrix equation for the f's is

$$\left(\frac{d^2}{\partial x^2} + \frac{3}{x} \frac{\partial}{\partial x} - \frac{n(n+2)}{x^2} + k^2 \right) f_n(x) - 2\omega_1 \sum_m \left\{ \frac{\partial}{\partial x} \langle \cos\theta \rangle^N_{n,m} - \frac{1}{x} \langle \sin\theta \frac{\partial}{\partial \theta} \rangle_{n,m} \right\} f_m(x) = 0 . \qquad \text{(III. 8)}$$

General theory[6] tells us that in the asymptotic region, $x \to \infty$, solutions exist of the form

$$f_n(x) \sim \frac{e^{-\lambda x}}{x^s} \left(a_n + O(x^{-1}) \right) \qquad \text{(III. 9)}$$

where the possible λ's are eigenvalues of a certain matrix, the $\{a_n\}$ are the associated eigenvectors, and the exponent s is determined from a certain consistency requirement. Because the BSE describes waves spreading out in four-space, we anticipate s = 3/2. In the asymptotic limit, (III. 8) reduces to

$$\left(\frac{\partial^2}{\partial x^2} + k^2\right) f_n(x) - 2\omega_1 \sum \langle \cos \theta \rangle^N_{n,m} \frac{\partial f_n(x)}{\partial x} = 0 \tag{III. 10}$$

Substitution of the lead term of (III. 9) into (III. 10) yields

$$\sum_m \left\{ (\lambda^2 + k^2)\, \delta_{n,m} - 2\omega_1 \lambda \langle \cos\theta \rangle^N_{n,m} \right\} a_m = 0 \tag{III. 11}$$

which is the desired eigenvalue equation.

It is easy to see that the eigenvalues of the Hermitian matrix $\langle \cos\theta \rangle^N$, which can be represented by cos α, all lie in the interval

$$-1 \leq \cos \alpha \leq 1, \qquad 0 \leq \alpha \leq \pi \ . \tag{III. 12}$$

An insight can be obtained by observing what happens in the limit as $N \to \infty$. Then the operator $\langle \cos\theta \rangle^N$ is just multiplication by cos θ , and the eigenvalues are cos α for any α; thus the spectrum becomes continuous and occupies the whole interval (III. 12). The eigenfunctions are δ functions, $\delta(\theta - \alpha)$, and have the harmonic expansion

$$\frac{\delta(\theta-\alpha)}{\sin\theta \sin\alpha} = \sum_{n=\ell}^{\infty} R_n(\alpha)\, R_n(\theta) \ . \tag{III. 13}$$

Returning to the general case, consider an eigenvalue cos α with eigenvector $\{a_n\}$ and let λ_α be determined by

$$\lambda_\alpha^2 - 2\omega_1 \lambda_\alpha \cos \alpha + k^2 = 0 \tag{III. 14}$$

One root of this equation is

$$\lambda_\alpha = \omega_1 \cos \alpha + (\omega_1^2 \cos^2\alpha - k^2)^{1/2} \tag{III. 15}$$

In the bound state region, $k = i\kappa$, and $0 < \kappa \leq m_1$. Then the λ_α of (III. 15) are all positive and lie in the interval

$$m_1 - \omega_1 \leq \lambda_\alpha \leq m_1 + \omega_1 \quad . \tag{III. 16}$$

In the scattering region, $\omega_1^2 \cos^2 \alpha$ may be less than k^2, and the rule $k^2 \to k^2 - i\epsilon$ (outgoing wave boundary condition) instructs us to take the square root as follows in this case:

$$\lambda_\alpha = \omega_1 \cos \alpha - i \mid k^2 - \omega_1^2 \cos^2 \alpha \mid^{1/2} \tag{III. 17}$$

Both positive and negative values of λ_α also occur, but always,

$$-k \leq \text{Re}\ (\lambda_\alpha) \leq m_1 + \omega_1 \tag{III. 18}$$

This scheme identifies N solutions with asymptotic behaviour (III. 9); they are the solutions regular at ∞ for the BSE. They are so identified from the asymptotic form of the BS integral equation (see SZ, Ref. 2) which in turn derives from the causality $i\epsilon$ prescription in the definition of the Green's function.

There are also N solutions singular at ∞, obtained with the opposite sign of the square root in (III. 15), (III. 17). These λ's are negative in the bound state region. They lie in the interval

$$-(m_1 + \omega_1) \leq \lambda \leq -(m_1 - \omega_1) \tag{III. 19}$$

and the associated wave functions diverge exponentially as $x \to \infty$. The gap between the regular and singular λ spectra vanishes as the bound state energy approaches threshold, and the spectra overlap in the scattering region.

To proceed further, we must solve the eigenvalue problem for $\langle \cos \theta \rangle^N$ and, more generally, find eigenfunctions of $\partial / \partial \tau$ for all x.

D. The Eigenvalue Problem

Let α be any angle between 0 and π. Let λ be any constant. Consider the function

$$\sum_{n=\ell}^{\ell+N-1} R_n(\alpha)\, R_n(\theta) \quad , \tag{III. 20a}$$

or, equivalently, the vector

$$S_n(\alpha) = R_n(\alpha), \quad \ell \le n \le \ell + N - 1 \quad . \tag{III. 20b}$$

We have, from (III. 2a) ,

$$\sum_m \langle \cos\theta \rangle^N_{n,m} S_m(\alpha) = A_{n+1}R_{n+1}(\alpha) + A_n R_{n-1}(\alpha) \; , \quad n < \ell + N - 1 \; , \tag{III. 21a}$$

$$= A_n R_{n-1}(\alpha) \; , \; n = \ell + N - 1 \; .$$

But from (II. 15a),

$$\cos\alpha \; S_n(\alpha) = A_{n+1}R_{n+1}(\alpha) + A_n R_{n-1}(\alpha) \; . \tag{III. 21b}$$

Therefore

$$\sum_m \left\{ \langle \cos\theta \rangle^N - \cos\alpha \right\}_{n,m} S_m(\alpha) = 0 \; , \tag{III. 22}$$

provided that

$$R_{N+\ell}(\alpha) = 0 \; . \tag{III. 23}$$

Now if $\ell \ge 1$, then $R_{N+\ell}(\alpha)$ has at least one factor of $\sin\alpha$, so that $\alpha = 0$ and $\alpha = \pi$ are roots of (III. 23). But these roots are uninteresting as all $R_n(\alpha)$ vanish at these values. The other roots of (III. 23) we term the non-trivial zeros of $R_{N+\ell}$; they are, in fact, the roots of

$$C_N^{1+\ell}\,(\cos\alpha) = 0 \; . \tag{III. 24}$$

The Gegenbauer polynomial $C_N^{1+\ell}(z)$ is of degree N and even or odd according as N is even or odd. It has N roots, all real. They are found in the open interval $-1 < z < 1$.

We have established, then, that the eigenvalues of $\langle\cos\theta\rangle^N$ are precisely the roots of (III. 24) and that the associated eigenvectors are $\{R_n(\alpha)\}$. Or, in function language, the eigenfunction is (III. 20a). In the limit $N \to \infty$ this structure goes directly over to (III. 13), which served to motivate the analysis. Thus as $N \to \infty$, the roots of (III. 24), become a continuous distribution on the interval $0 \leq \alpha \leq \pi$, as indicated in (III. 13).

Consider now the functions

$$S_I = \sum_{n=\ell}^{\ell+N-1} \mathscr{I}_n(\lambda x, \theta)\, R_n(\alpha), \tag{III. 25a}$$

$$S_K = \sum_{n=\ell}^{\ell+N-1} \mathscr{K}_n(\lambda x, \theta)\, R_n(\alpha)\ . \tag{III. 25b}$$

We have already remarked that $\mathscr{I}_n$, $\mathscr{K}_n$ satisfy the same recursion relations with respect to $\partial/\partial\tau$ that $R_n(\theta)$ satisfies with respect to $\cos\theta$. Pursuing the same line as in the above paragraphs, we obtain the eigenfunctions and eigenvalues for $\langle\partial/\partial\tau\rangle^N$, namely,

$$\langle\partial/\partial\tau\rangle^N\, S_I = \lambda\cos\alpha\ S_I \tag{III. 26a}$$

$$\langle\partial/\partial\tau\rangle^N\, S_K = \lambda\cos\alpha\ S_K \tag{III. 26b}$$

with the same α's as before, namely the roots of (III. 23).

For even N, the zeros of $C_N^{1+\ell}(\cos\alpha)$ occur in pairs, one zero being the negative of another. When α is one angle of a pair, the other is $\pi-\alpha$. For N odd, N-1 of the zeros occur in pairs and the remaining one is $\cos\alpha = 0$, $\alpha = \frac{1}{2}\pi$.

Define normalized eigenvectors by

$$\overline{R}_n(\alpha) = (h_\alpha)^{1/2} R_n(\alpha); \; (h_\alpha)^{-1} = \sum_{n=\ell}^{\ell+N-1} \left[R_n(\alpha)\right]^2. \qquad \text{(III. 27)}$$

Then, for n, $m \leq \ell + N-1$, and the N eigenvalues α,

$$\Sigma_\alpha \overline{R}_n(\alpha)\, \overline{R}_m(\alpha) = \delta_{n,m}; \Sigma_n \overline{R}_n(\alpha)\, \overline{R}_n(\alpha') = \delta_{\alpha,\alpha'} \; . \qquad \text{(III. 28)}$$

Note for later use that

$$\Sigma_n A_n \overline{R}_n(\alpha)\, \overline{R}_{n-1}(\alpha) = \frac{1}{2} \Sigma_n \overline{R}_n(\alpha) \left[A_{n+1} \overline{R}_{n+1}(\alpha) + A_n \overline{R}_{n-1}(\alpha)\right]$$
$$= \frac{1}{2} \cos \alpha \Sigma_n \left[\overline{R}_n(\alpha)\right]^2 = \frac{1}{2} \cos \alpha \; . \qquad \text{(III. 29)}$$

The eigenvalue problem for $\langle \cos^2 \theta \rangle^N$ can be solved in the same way. First, suppose N even and consider the $\frac{1}{2}$ N-dimensional subspace of even time-parity spanned by

$$\left\{R_n(\theta)\right\}, \; n = \ell, \ell + 2, \ldots \ell + N-2 \; . \qquad \text{(III. 30)}$$

The condition

$$R_{\ell+N}(\alpha) = 0 \qquad \text{(III. 31)}$$

supplies N non-trivial values of α, but only $\frac{1}{2}$ N values of $(\cos \alpha)^2$. These are the eigenvalues. They can be associated with the $\frac{1}{2}$ N values of α in the interval $(0, \frac{1}{2}\pi)$. We already know that

$$\sum_{n=\ell}^{\ell+N-1} \overline{R}_n(\alpha)\, \overline{R}_n(\alpha) = 1 \qquad \text{(III. 32)}$$

and, by (II. 14)

$$\sum_{n=\ell}^{\ell+N-1} \overline{R}_n(\alpha)\, \overline{R}_n(\alpha)\, (-1)^{n-\ell} = \delta_{\alpha, (\pi-\alpha)} = 0 \qquad \text{(III. 33)}$$

Hence, averaging (III. 32), (III. 33),

$$\sum_{(n-\ell)\ \text{even}} \overline{R}_n(\alpha)\ \overline{R}_n(\alpha) = \frac{1}{2}\ . \qquad \text{(III. 34)}$$

Therefore, the normalized eigenvector for α is

$$\left\{ \sqrt{2}\ \overline{R}_n(\alpha) \right\},\ n=\ell,\ \ell+2,\ \ldots,\ \ell+N-2 \qquad \text{(III. 35)}$$

Similarly, the $\frac{1}{2}N$ dimensional subspace of odd time-parity, spanned by

$$\left\{ R_n(\theta) \right\},\ n=\ell+1,\ \ell+3,\ \ldots,\ \ell+N-1 \qquad \text{(III. 36)}$$

provides for $\langle \cos^2\theta \rangle^N$ the eigenvalue condition

$$R_{\ell+N+1}(\alpha) = 0 \qquad \text{(III. 37)}$$

The root $\alpha = \frac{1}{2}\pi$ is irrelevant as all $R_n\left(\frac{1}{2}\pi\right) = 0$ for n in the list of (III. 36). There remain $\frac{1}{2}N$ non-zero values of $\cos^2\alpha$, associated with α's in the interval $(0, \frac{1}{2}\pi)$ which are the eigenvalues. The normalized eigenvectors are

$$\left\{ \sqrt{2}\ \overline{R}_n(\alpha) \right\},\ n=\ell+1,\ \ell+3,\ \ldots,\ \ell+N-1\ . \qquad \text{(III. 38)}$$

E. Construction of Solutions

Consider the functions

$$I^{(\alpha, i)}(x,\theta) = \sum_n \mathscr{I}_n(\lambda_{\alpha, i}\, x, \theta)\ \overline{R}_n(\alpha) \qquad \text{(III. 39a)}$$

$$K^{(\alpha, i)}(x,\theta) = \sum_n \mathscr{K}_n(\lambda_{\alpha, i}\, x, \theta)\ \overline{R}_n(\alpha) \qquad \text{(III. 39b)}$$

where the $\lambda_{\alpha, i}$ are functions of α and ω_i, i=1, 2, to be specified below. The summations are from $n=\ell$ to $n=\ell+N-1$ as before. Let α be one of the non-trivial zeros of $R_{\ell+N}(\alpha)$. Then by (II. 10), (III. 26),

$$\Box^2\, I^{(\alpha, i)} = (\lambda_{\alpha, i})^2\, I^{(\alpha, i)},\quad \Box^2\, K^{(\alpha, I)} = (\lambda_{\alpha, i})^2\, K^{(\alpha, i)}, \qquad \text{(III. 40)}$$

and

$$\frac{\partial}{\partial\tau} I^{(\alpha, i)} = \lambda_{\alpha, i} \cos\alpha\, I^{(\alpha, i)}, \qquad \frac{\partial}{\partial\tau} K^{(\alpha, i)} = \lambda_{\alpha, i} \cos\alpha\, K^{(\alpha, i)}. \tag{III.41}$$

Hence, both $I^{(\alpha, i)}$ and $K^{(\alpha, i)}$ satisfy

$$\langle \mathscr{D}_i \rangle^N \psi = 0 \tag{III.42}$$

and

$$\langle \mathscr{D}_1 \rangle^N \langle \mathscr{D}_2 \rangle^N \psi = 0 \tag{III.43}$$

provided that

$$(\lambda_{\alpha, 1})^2 - 2\omega_1 \cos\alpha\, \lambda_{\alpha, 1} + k^2 = 0 \quad , \tag{III.44a}$$

$$(\lambda_{\alpha, 2})^2 + 2\omega_2 \cos\alpha\, \lambda_{\alpha, 2} + k^2 = 0 \quad . \tag{III.44b}$$

This type of equation was already obtained in Section III.C in the study of asymptotic behaviour. The definition of $I^{(\alpha, i)}$, $K^{(\alpha, i)}$ is fixed by setting

$$\lambda_{\alpha, 1} = \omega_1 \cos\alpha + (\omega_1^2 \cos^2\alpha - k^2)^{1/2} \tag{III.45a}$$

$$\lambda_{\alpha, 2} = -\omega_2 \cos\alpha + (\omega_2^2 \cos^2\alpha - k^2)^{1/2} \tag{III.45b}$$

As before, the square root is taken positive when real, and negative imaginary when not real. The N values of α lie in the interval $(0,\pi)$. For bound states, $\omega_i < m_i$ and the $\lambda_{\alpha, i}$ are real, positive, with

$$m_i - \omega_i \le \lambda_{\alpha, i} \le m_i + \omega_i \quad . \tag{III.46}$$

For scattering, $\omega_i \geq m_i$ and there are three cases:

Case 1 $\quad k \le \omega_i \cos\alpha$. Then $\lambda_{\alpha, i}$ is real and

$$k \le \lambda_{\alpha, i} < m_i + \omega_i \tag{III.47}$$

Case 2 $\quad -k < \omega_i \cos\alpha < k$. Define $\omega_i \cos\alpha = k \cos(\phi_{\alpha,i})$,

with $0 \le \phi_{\alpha,i} \le \pi$.

Then $\lambda_{\alpha,i} = k\, e^{-i\phi_{\alpha,i}}$. $\qquad$ (III.48)

In particular, $\lambda_{\alpha,i} = -ik$ if $\alpha = \frac{1}{2}\pi$. In this case, $I^{(\alpha,i)}$ is, apart from a constant factor, a truncation of the expansion of $j_\ell(kr)$ in Eq. (II.20).

Case 3 $\quad \pm\omega_i \cos\alpha \le -k$. Again $\lambda_{\alpha,i}$ is real and

$$-(\omega_i - m_i) \le \lambda_{\alpha,i} \le -k \qquad \text{(III.49)}$$

We have found 2N solutions to $\langle \mathscr{D}_1 \rangle^N \langle \mathscr{D}_2 \rangle^N \psi = 0$ of type $I^{(\alpha,i)}$, regular at $x \to 0$, and 2N solutions of type $K^{(\alpha,i)}$, regular at $x \to \infty$. A fourth order equation for an N component vector function has precisely 4N independent solutions so we have a complete set.

Let us now specialize to equal masses. The $\lambda_{\alpha,1}$ and $\lambda_{\alpha,2}$ are related, so write

$$\lambda_\alpha = \lambda_{\alpha,1} = \lambda_{\pi-\alpha,2} \qquad \text{(III.50)}$$

The solutions may be classified as even or odd under time-parity. Set

$$I^{(\alpha,e)}(x,\theta) = \left(I^{(\alpha,1)} + I^{(\pi-\alpha,2)}\right)/\sqrt{2}$$

$$= \textstyle\sum_n \mathscr{I}_n(\lambda_\alpha x, \theta)\sqrt{2}\,\overline{R}_n(\alpha)\,, \qquad (n-\ell) \text{ even.} \qquad \text{(III.51a)}$$

Similarly,

$$I^{(\alpha,o)}(x,\theta) = \left(I^{(\alpha,1)} - I^{(\pi-\alpha,2)}\right)/\sqrt{2}$$

$$= \textstyle\sum_n \mathscr{I}_n(\lambda_\alpha x, \theta)\sqrt{2}\,\overline{R}_n(\alpha)\,, \qquad (n-\ell) \text{ odd.} \qquad \text{(III.51b)}$$

There is a peculiarity for N odd and time-parity even. Then N+1 solutions of type $I^{(\alpha, e)}$ are required, but only N delivered by the above prescription. In fact $\alpha = \pi/2$ is one of the roots here. For this case,

$$I^{(\alpha, 1)}(x,\theta) = \sum_n \mathscr{I}_n(-ikx,\theta)\sqrt{2}\,\overline{R}_n\left(\frac{1}{2}\pi\right), \quad n-\ell \text{ even} \tag{III.52}$$

solves both $\langle \mathscr{D}_1 \rangle^N \psi = 0$ and $\langle \mathscr{D}_2 \rangle^N \psi = 0$.

Therefore

$$\frac{\partial}{\partial k}\left(kI^{\left(\frac{1}{2}\pi,\,1\right)}\right) = \sum_n I'_{n+1}(-ikx)\, R_n(\theta)\, \overline{R}_n\left(\frac{1}{2}\pi\right), \quad n-\ell \text{ even} \tag{III.53}$$

is the $(N+1)^{\text{st.}}$ solution.

The definition of $K^{(\alpha, e)}(x,\theta)$, $K^{(\alpha, o)}(x,\theta)$ follows the same lines. These even and odd I's and K's were constructed to solve the double truncated BSE, but as indicated in III.B, the even (or odd) time-parity solutions of the singly truncated BSE, Eq. (III.3a), having $\frac{1}{2}$ N spherical harmonic components are the same as the $I^{(\alpha, e)}$ and $K^{(\alpha, e)}$ or $I^{(\alpha, o)}$ and $K^{(\alpha, o)}$ solutions of (III.3b) with truncation parameter N (or N+1). The peculiarity mentioned in the previous paragraph does not arise for these solutions.

Construction of vector Bessel functions for (III.3a) when $m_1 \neq m_2$ is , more complex. The operator to be diagonalized is a linear combination of $\langle \cos\theta \rangle^N$ and $\langle \cos^2\theta \rangle^N$ with coefficients depending on ω_1, ω_2, and λ . It appears unnecessary to go into this because the inadequacy of the truncated BSE for scattering can be made clear enough in the equal mass case. The promised remedy does not, in any case, depend on explicit knowledge of these functions.

IV. BRACKETS

A. The Transposed Equations

Let $\widetilde{\mathscr{D}}_1, \widetilde{\mathscr{D}}_2$ be the transposes of $\mathscr{D}_1, \mathscr{D}_2$. Thus

$$\widetilde{\mathscr{D}}_1 = \Box^2 + k^2 + 2\omega_1 \ \partial/\partial\tau ; \ \widetilde{\mathscr{D}}_2 = \Box^2 + k^2 - 2\omega_2 \ \partial/\partial\tau \quad . \tag{IV. 1}$$

The only change is in the sign of τ. Similarly, we define a transpose operation on functions of r, τ by

$$\widetilde{f}(r, \tau) = f(x, \tau) = f(r, -\tau) \tag{IV. 2}$$

or, equivalently, for functions of the polar coordinates,

$$\widetilde{f}(x,\theta) = f(x, \pi-\theta) . \tag{IV. 3}$$

Then we have transposed vector Bessel functions $I_{(\alpha, i)}$, $K_{(\alpha, i)}$ given by

$$I_{(\alpha, i)}(x, \theta) = \widetilde{I}^{(\alpha, i)}(x, \theta) = \sum_{n=\ell}^{\ell+N-1} (-1)^{n-\ell} \mathscr{I}_n(\lambda_{\alpha, i} x, \theta) \, \overline{R}_n(\alpha) \quad , \tag{IV. 4a}$$

$$K_{(\alpha, i)}(x, \theta) = \widetilde{K}^{(\alpha, i)}(x, \theta) = \sum_{n=\ell}^{\ell+N-1} (-1)^{n-\ell} \mathscr{K}_n(\lambda_{\alpha, i} x, \theta) \, \overline{R}_n(\alpha) \quad . \tag{IV. 4b}$$

These functions satisfy $\langle\widetilde{\mathscr{D}}_1\rangle^N \langle\widetilde{\mathscr{D}}_2\rangle^N \ \phi = 0$.

B. Definition of the Brackets

Given functions $\psi(x,\theta)$ and $\phi(x,\theta)$, which may or may not be solutions of the BSE and the adjoint BSE, there exists a bilinear function which we term a bracket of ϕ, ψ . It is related to the notion of a matrix element of flux.

We shall use brackets $(\phi,\psi)^{(1)}$ and $(\phi,\psi)^{(2)}$ associated with the operators $\mathscr{D}_1, \mathscr{D}_2$ respectively, and a higher order bracket $[\phi,\psi]$ associated with $\mathscr{D}_1\mathscr{D}_2$.

The brackets are functions of x only. The discussion here is valid if $\mathscr{D}_1, \mathscr{D}_2$ are replaced by $\langle\mathscr{D}_1\rangle^N, \langle\mathscr{D}_2\rangle^N$. The definitions begin from

$$\frac{d}{dx}(\phi,\psi)^{(i)} = \int x^3 \sin^2\theta \, d\theta \left\{\phi\mathscr{D}_i\psi - (\tilde{\mathscr{D}}_i\,\phi)\psi\right\} \quad , \; i = 1, 2 \tag{IV.5}$$

$$\frac{d}{dx}[\phi,\psi] = \int x^3 \sin^2\theta \, d\theta \left\{\phi\,\mathscr{D}_1\mathscr{D}_2\,\psi - (\tilde{\mathscr{D}}_1\tilde{\mathscr{D}}_2\,\phi)\psi\right\} \; . \tag{IV.6}$$

Then the brackets can be written explicitly as

$$(\phi,\psi)^{(1)} = \int x^3 \sin^2\theta \, d\theta \left\{\phi\frac{\partial\psi}{\partial x} - \psi\frac{\partial\phi}{\partial x} - (2\,\omega_1 \cos\theta)\;\phi\psi\right\} , \tag{IV.7a}$$

$$(\phi,\psi)^{(2)} = \int x^3 \sin^2\theta \, d\theta \left\{\phi\frac{\partial\psi}{\partial x} - \psi\frac{\partial\phi}{\partial x} + (2\omega_2 \cos\theta)\;\phi\psi\right\} \tag{IV.7b}$$

$$[\phi,\psi] = (\phi,\mathscr{D}_2\psi)^{(1)} + (\tilde{\mathscr{D}}_1\,\phi,\psi)^{(2)} \tag{IV.8a}$$

$$= (\phi,\mathscr{D}_1\psi)^{(2)} + (\tilde{\mathscr{D}}_2\,\phi,\psi)^{(1)} \; . \tag{IV.8b}$$

Furthermore, if ϕ,ψ are expanded in spherical harmonics,

$$\psi = \Sigma f_n(x)\; R_n(\theta) \quad , \quad \phi = \Sigma\; g_n(x)\; R_n(\theta) \quad , \tag{IV.9}$$

then

$$(\phi,\psi)^{(1)} = x^3\Sigma\left\{g_n f'_n - f_n g'_n - 2\omega_1\, A_n(g_n f_{n-1} + g_{n-1}f_n)\right\} \tag{IV.10a}$$

$$(\phi,\psi)^{(2)} = x^3\Sigma\left\{g_n f'_n - f_n g'_n + 2\omega_2\, A_n(g_n f_{n-1} + g_{n-1}f_n)\right\} \tag{IV.10b}$$

The proofs of these relations will be given in the following paper, where the topic of brackets is covered more thoroughly. For our present purposes, it is sufficient to note the following:

(a) If $\mathscr{D}_i = 0$ and $\widetilde{\mathscr{D}}_i \phi = 0$, then by (IV.5), $(\phi,\psi)^{(i)}$ is a constant independent of x. If $\mathscr{D}_1\mathscr{D}_2 \psi = 0$, $\widetilde{\mathscr{D}}_1\widetilde{\mathscr{D}}_2 \psi = 0$, then $[\phi, \psi]$ is a constant.

(b) If $\mathscr{D}_1 \psi = 0$, $\widetilde{\mathscr{D}}_2 \phi = 0$ or $\mathscr{D}_2 \psi = 0$, $\widetilde{\mathscr{D}}_1 \phi = 0$, then by (IV.8), the constant $[\phi, \psi]$ is zero.

(c) If $\mathscr{D}_1 \psi = 0$, $\widetilde{\mathscr{D}}_1 \phi = 0$, then $\mathscr{D}_2 \psi = 2E\,\partial\psi/\partial\tau$ and by (IV.8a), the constant bracket is

$$[\phi,\psi] = 2E\left(\phi, \frac{\partial \psi}{\partial \tau}\right)^{(1)} . \tag{IV11a}$$

Also, if $\mathscr{D}_2 \psi = 0$, $\widetilde{\mathscr{D}}_1 \phi = 0$, then $\mathscr{D}_1 \psi = -2E\,\partial\psi/\partial\tau$ and

$$[\phi,\psi] = -2E\left(\phi, \frac{\partial \psi}{\partial \tau}\right)^{(2)} . \tag{IV.11b}$$

C. <u>Brackets of the Vector Bessel Functions</u>

First we calculate $(\phi,\psi)^{(i)}$ for ϕ,ψ in the $I^{(\alpha, i)}$, $K^{(\alpha, i)}$ family. Recall that as $x\to\infty$, the $I^{(\alpha, i)}$ and $K^{(\alpha, i)}$ have asymptotic parts that go like $\exp(\pm\lambda_{\alpha, i} x)$. Therefore, only

$$(I_{(\alpha, i)}, K^{(\alpha, i)})^{(i)} , \quad (K_{(\alpha, i)}, I^{(\alpha, i)})^{(i)}$$

are nonzero; the other combinations can succeed in being x independent for large x, as required by point (a) above only by vanishing. The ingredients for calculating the nonzero brackets are all in hand. We apply the definitions (III.39) and (IV.4) of the functions, the definitions (II.11), of $\mathscr{I}_n$ and $\mathscr{K}_n$, the bracket formulas (IV.10), the Bessel identities (II.27), and the $\overline{R}_n$ normalizations (III.28), (III.29) in that order. The result is

$$(K_{(\alpha, i)}, I^{(\alpha, i)})^{(i)} = -(I_{(\alpha, i)}, K^{(\alpha, i)})^{(i)} = (\lambda_{\alpha, i})^{-3} (\omega_i^2 \cos^2 \alpha - k^2)^{1/2}$$

$$i = 1, 2 . \tag{IV.12}$$

The complete set of formulas for the fourth-order bracket, deduced with the aid of (IV.11), (III.41), and (IV.12) is

$$\left[I_{(\alpha,i)},\ I^{(\beta,j)}\right] = \left[k_{(\alpha,i)},\ K^{(\beta,j)}\right] = 0\ , \qquad \text{(IV.13a)}$$

$$\left[K_{(\alpha,i)},\ I^{(\beta,j)}\right] = -\left[I_{(\alpha,i)},\ K^{(\beta,j)}\right] = (\sigma_{\alpha,i})^{-1}\,\delta_\alpha^\beta\,\delta_i^j\ ;\ i,j = 1,2 \qquad \text{(IV.13b)}$$

where

$$\sigma_{\alpha,i} = \frac{\pm(\lambda_{\alpha,i})^2}{2E\cos\alpha\,(\omega_i^2\cos^2\alpha - k^2)^{1/2}}\ . \qquad \text{(IV.14)}$$

The plus sign in (IV.14) is for i=1, and the minus sign for i=2. The square root in (IV.14) is either positive real or negative imaginary. The BSE is symmetric (equal to its transpose) when $m_1 = m_2$ and $I_{(\alpha,e)} = I^{(\alpha,e)}$, $I_{(\alpha,o)} = -I^{(\alpha,o)}$, etc. The only nonzero brackets in the equal mass case are (with $\sigma_\alpha = \sigma_{\alpha,1} = +\sigma_{\pi-\alpha,2}$)

$$\left[K_{(\alpha,e)},\ I^{(\alpha,e)}\right] = \left[K_{(\alpha,o)},\ I^{(\alpha,o)}\right] = -\left[I_{(\alpha,e)},\ K^{(\alpha,e)}\right] = -\left[I_{(\alpha,o)},\ K^{(\alpha,o)}\right] = (\sigma_\alpha)^{-1} \qquad \text{(IV.15)}$$

V. FREE-PARTICLE GREEN'S FUNCTIONS

A. Green's Functions for the Exact Equation

The representations for which we have specific application are listed below in (V.6), already derived in SZ, and the partial wave reductions, (V.12) and (V.23). We begin by listing the more basic forms from which these derive, in order to fix conventions and normalizations.

The one particle free Green's functions $G_1(x-x')$, $G_2(x-x')$ satisfy

$$(p_1^2 + m_1^2)G_1 = \delta^4(x_1 - x_1')\quad ,\quad (p_2^2 + m_2^2)\,G_2 = \delta^4(x_2 - x'_2) \qquad \text{(V.1)}$$

and the outgoing wave (causal) boundary condition. The-two particle propagator is then G_1G_2, and the relative Green's function is

$$G(x_\mu, x'_\mu) = \int e^{iPX}\, G_1(x_1-x'_1)\, G_2(x_2-x'_2)\, d^4X \ . \qquad (V.2)$$

With $P_\mu = (\underline{0}, E)$, we have

$$\left\{\nabla^2 - (i\frac{\partial}{\partial t} + \omega_1)^2 - m_1^2\right\}\left\{\nabla^2 - (i\frac{\partial}{\partial t} - \omega_2)^2 - m_2^2\right\} G = \delta^4(x-x') \ . \quad (V.3)$$

Then we perform the Wick rotation, (I.10). The property of the time δ-function to be maintained is $\delta(t-t')\, dt \rightarrow \delta(\tau - \tau')\partial\tau$ so that[7]

$$\delta(t-t') \rightarrow + i\, \delta(\tau - \tau') \ . \qquad (V.4)$$

We retain some of our notation, but with modified meaning:

$$\text{new } G(x_\mu, x'_\mu) = (-i) \times \text{old } G(x_\mu, x'_\mu)\Big|_{t \rightarrow -i\tau} \qquad (V.5a)$$

$$\text{new } \delta^4(x-x') = (-i) \times \text{old } \delta^4(x-x')\Big|_{t \rightarrow -i\tau} = \delta(\underline{r}-\underline{r}')\, \delta(\tau - \tau') . \qquad (V.5b)$$

The new (Wick-rotated) Green's function has the representations[2]

$$G(x_\mu, x'_\mu) = \int_C \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x-x')}}{\left[\underline{p}^2 + (p_4 + i\omega_1)^2 + m_1^2\right]\left[\underline{p}^2 + (p_4 - i\omega_2)^2 + m_2^2\right]} \qquad (V.6a)$$

$$= \int_{-\omega_2}^{\omega_1} \frac{d\beta}{8\pi^2 E}\, e^{\beta(\tau-\tau')}\, K_o(Q|x_\mu - x'_\mu|) \qquad (V.6b)$$

$$= \frac{1}{8\pi E} \frac{e^{ik|r-r'|}}{|r-r'|} - \left(\int_{-\infty}^{-\omega_2} + \int_{\omega_1}^{\infty}\right) \frac{d\beta}{8\pi^2 E}\, e^{\beta(\tau-\tau')}\, K_o(Q|x_\mu - x'_\mu|) , \qquad (V.6c)$$

where $Q = (\beta^2 - k^2 - i\epsilon)^{1/2}$. The contour C runs from $p_4 = -\infty$ to $p_4 = +\infty$ passing above the poles in the p_4 plane at

$$i\left[-\omega_1 - (\underline{p}^2 + m_1^2)^{1/2}\right], \quad i\left[+\omega_2 - (\underline{p}^2 + m_2^2)^{1/2}\right] \qquad \text{(V. 7a)}$$

and below the p_4-poles at

$$i\left[-\omega_1 + (\underline{p}^2 + m_1^2)^{1/2}\right] \quad , \quad i\left[+\omega_2 + (p^2 + m_2^2)^{1/2}\right] \;. \qquad \text{(V. 7b)}$$

In the bound state region, $\omega_i < m_i$ and C can be taken along the real p_4 axis.

The rotated Green's function obeys

$$\mathscr{D}_1 \mathscr{D}_2 \, G = \delta^4(x-x') \qquad \text{(V. 8a)}$$

and the equations for the wave function,

$$(\mathscr{D}_1 \mathscr{D}_2 - V) \quad = 0 \;, \qquad \text{(V. 8b)}$$

$$\psi = \psi_o + GV\psi \;, \qquad \text{(V. 8c)}$$

keep the form and signs they had before rotation. This arrangement means, perhaps unfortunately, that V and G have signs opposite to the nonrelativistic potential and propagator, as usually defined (the V in (V. 8b) is positive for an attractive force).

Let G be resolved into partial waves:

$$G(x_\mu, x'_\mu) = \Sigma_{\ell, m} \, Y_\ell^m(\theta_3, \phi) \, Y_\ell^{m*}(\theta'_3, \phi') \, G^{(\ell)}(x, \theta; x', \theta')$$

$$= \Sigma_\ell \left(\frac{2\ell+1}{4\pi}\right) P_\ell(\hat{r} \cdot \hat{r}') \, G^\ell(x, \theta; x', \theta') \;. \qquad \text{(V. 9)}$$

Then, since

$$\delta^4(x-x') = \frac{\delta(x-x')}{x^3} \frac{\delta(\theta-\theta')}{\sin^2\theta} \delta(\Omega-\Omega')$$

$$= \frac{\delta(x-x')}{x^3} \frac{\delta(\theta-\theta')}{\sin^2\theta} \sum_{\ell, m} Y_\ell^m(\theta_3, \phi) Y_\ell^{m*}(\theta_3', \phi'), \tag{V.10}$$

the equation for $G^{(\ell)}$ is

$$\mathscr{D}_1 \mathscr{D}_2 G^{(\ell)} = \frac{\delta(x-x')}{x^3} \frac{\delta(\theta-\theta')}{\sin^2\theta} \quad . \tag{V.11}$$

The explicit form is obtained via (II.25), (II.26), and (V.6b):

$$G^{(\ell)}(x, \theta; x', \theta') = \int_{-\omega_2}^{\omega_1} \frac{d\beta}{4E} e^{\beta(\tau-\tau')} \sum_{n=\ell}^{\infty} (n+1)^{-1} \Big[I_n(Qx_<) K_n(Qx_>) \tag{V.12}$$

$$- I_{n+2}(Qx_<) K_{n+2}(Qx_>) \Big] R_n(\theta) R_n(\theta') \ .$$

This equation will turn out to be more useful than it looks.

We now develop an alternative to (V.12) which begins from (V.6a). Suppose first that E is in the bound state region; then C lies along the real p_4 axis, and we can transform the momentum variables to spherical coordinates $(p, \alpha, \theta_p, \phi_p)$:

$$p_4 = p \cos\alpha, \qquad |\underline{p}| = p \sin\alpha,$$
$$d^4p = p^3 dp \sin^2\alpha \, d\alpha \, d\Omega_p \tag{V.13}$$

The analytic continuation to scattering energies may be done at the end of the calculation.

To obtain partial waves, we make the projection

$$\Sigma_m \int d\Omega d\Omega' \, Y_\ell^m (\theta_3, \phi) \, Y_\ell^{m*} (\theta_3' \, \phi') \, e^{i\underline{p}\cdot r} \, e^{-i\underline{p}\cdot \underline{r}'}$$

$$= \Sigma_m (4\pi)^2 \, Y_\ell^m (\theta_p \phi_p) \, Y_\ell^{m*} (\theta_p, \phi_p) \, j_\ell (|\underline{p}| r) \, j_\ell(|p| r) \qquad \text{(V.14)}$$

$$= 4\pi \, (2\ell+1) \, j_\ell (|\underline{p}| r) \, j_\ell (|\underline{p}| r).$$

Then

$$G^{(\ell)}(x, \theta ; x' \theta') = \pi^{-2} \int p^3 dp \, \sin^2\alpha d\alpha \, \ell^{ip_4(\tau - \tau')} \, \frac{j_\ell (|\underline{p}| r) \, j_\ell(|\underline{p}| r')}{D_1 D_2} \qquad \text{(V.15)}$$

where

$$D_1 = p^2 + 2i \, p\omega_1 \cos \alpha + \kappa^2 = (p + i\lambda_{\alpha, 1})(p - i\lambda_{\pi-\alpha, 1}) \quad , \qquad \text{(V.16a)}$$

$$D_2 = p^2 - 2i \, p\omega_2 \cos \alpha + \kappa^2 = (p + i\lambda_{\alpha, 2}) \, (p - 1\lambda_{\pi-\alpha, 2}) \quad . \qquad \text{(V.16b)}$$

The λ's defined in (III.43), (III.44) which figure in the asymptotic behaviour of the wave functions now appear as poles in the momentum propagator. We pass to the spherical harmonic expansion

$$G^\ell(x, \theta \, ; \, x', \theta') = \Sigma_{n, n'} \, g_{n, n'} \, (x, x') \, R_n(\theta) \, R_n(\theta') \qquad \text{(V.17)}$$

and apply (II.19):

$$j_\ell(|\underline{p}| r) \, e^{ip_4\tau} = \pi\Sigma \mathscr{I}_n(ipx, \alpha) \, R_n(\theta) \qquad \text{(V.18a)}$$

$$j_\ell(|\underline{p}| r) \, e^{-ip_4\tau} = \pi\Sigma \mathscr{I}_n(-ipx', \alpha) \, R_n(\theta') \qquad \text{(V.18b)}$$

Hence,

$$g_{n, n'}(x, x') = \int_0^\pi \sin^2\alpha \, d\alpha \int_0^\infty p^3 dp \, \frac{\mathscr{I}_n(ipx, \alpha) \, \mathscr{I}_{n'}(-ipx', \alpha)}{D_1 D_2} \qquad \text{(V.19)}$$

First consider the case $x < x'$. By (II. 30), (II. 14), with $\lambda x \to -ipx'$, $\theta \to \alpha$,

$$i\pi\, \mathscr{I}_{n'}(-ipx', \alpha) = (-1)^{\ell} \left[(-1)^{n'} \mathscr{K}_{n'}(ipx', \alpha) - \mathscr{K}_{n'}(-ipx', \alpha) \right]$$
$$= \mathscr{K}_{n'}(ipx', \pi-\alpha) - (-1)^{n'} \mathscr{K}_{n'}(-ipx', \pi-\alpha) \tag{V. 20}$$

Substitution of (V. 20) into (V. 19) leaves $g_{n,n'}(x, x')$ as the sum of two terms. In the term containing $\mathscr{K}_{n'}(-ipx', \pi-\alpha)$ only, make the transformation $\alpha \to \pi-\alpha$, $p \to -p$. This implies

$$\int_0^\infty p^3 dp \to -\int_{-\infty}^0 p^3 dp$$

$$D_1 D_2 \to D_1 D_2 \quad , \tag{V. 21}$$

$$(-1)^{n'} \mathscr{I}_n(ipx, \alpha)\, \mathscr{K}_{n'}(-ipx', \pi-\alpha) \to \mathscr{I}_n(ipx, \alpha)\, \mathscr{K}_{n'}(ipx', \pi-\alpha) \quad ,$$

in view of (II. 11), (II. 14), and $I_n(-z) = (-1)^n I_n(z)$. Therefore, still for the case $x < x'$,

$$g_{nn'}(x, x') = (-i/\pi) \int_0^\pi \sin^2\alpha\, d\alpha \int_{-\infty}^{\infty} p^3 dp\, \frac{\mathscr{I}_n(ipx, \alpha)\mathscr{K}_{n'}(ipx', \pi-\alpha)}{D_1 D_2} \tag{V. 22}$$

The p contour of integration can now be closed in the lower half complex plane. The branch cut in the $\mathscr{K}_{n'}$ function due to the cut in $K_{n'+1}(ipx')$ runs along the positive imaginary p axis and causes no trouble. The integral is evaluated in terms of the residues at the two poles at $p = -i\lambda\alpha, i$. The case $x' < x$ is treated by resolving $\mathscr{I}_n(ipx, \alpha)$ into $\mathscr{K}_n$'s in place of (V. 20). The complete result is

$$G^{\ell}(x,\theta;x'\theta') = \sum_{n,n'} g_{nn'}(x,x')\, R_n(\theta)\, R_n(\theta') ,$$

$$g_{nn'} = -\sum_{i=1,2} \int \sin^2\alpha\, d\alpha\, \mathscr{I}_n(\lambda_{\alpha,i} x, \alpha)\, \sigma_{\alpha,i} \mathscr{K}_{n'}(\lambda_{\alpha,i} x', \pi-\alpha), \quad x < x',$$

$$= -\sum_{i=1,2} \int \sin^2\alpha\, d\alpha\, \mathscr{K}_n(\lambda_{\alpha,i} x, \alpha)\, \sigma_{\alpha,i}\, \mathscr{I}_{n'}(\lambda_{\alpha,i} x', \pi-\alpha), \quad x < x', \tag{V.23}$$

where $\sigma_{\alpha,i}$ was defined in Eq. (IV.14). The integrand in (V.23) has a singularity at $\alpha = \frac{1}{2}\pi$, but there is no singularity from the combined contribution of i=1,2. One may consider (V.22) as a principal value integral at $\alpha = \frac{1}{2}\pi$; this is harmless as (V.22) has no singularity. Then the separate contributions to (V.23) are also principal value integrals and each is finite by itself.

The continuation of (V.23) into the scattering region requires merely the continuation of the $\lambda_{\alpha,i}$, $\sigma_{\alpha,i}$, which has already been discussed.

B. Green's Function's for the Truncated BSE

Consider the doubly truncated equation which has 2N solutions $I_{(\alpha,1)}$, $I_{(\alpha,2)}$ regular at the origin and 2N solutions $K_{(\alpha,1)}$, $K_{(\alpha,2)}$ regular at ∞. The propagator equation is

$$\langle \mathscr{D}_1 \rangle^N \langle \mathscr{D}_2 \rangle^N G^{(\ell),N}(x,\theta;x',\theta') = \left\langle \frac{\delta(x-x')}{x^3} \frac{\delta(\theta-\theta')}{\sin^2\theta} \right\rangle^N$$

$$= \frac{\delta(x-x')}{x^3} \sum_{n=\ell}^{N+\ell-1} R_n(\theta)\, R_n(\theta') . \tag{V.24}$$

The solution with regular boundary conditions must have the form

$$G_N^{(\ell)} = \sum_{\alpha,i} I^{(\alpha,i)}(x,\theta)\, c_{\alpha,i} \qquad x < x' ,$$

$$= \sum_{\alpha,i} K^{(\alpha,i)}(x,\theta)\, d_{\alpha,i} \qquad x > x' . \tag{V.25}$$

with coefficients $c_{\alpha, i}$ and $d_{\alpha, i}$, dependent on x', θ', to be determined. Suppose $\phi(x, \theta)$ obeys

$$\langle \widetilde{\mathscr{D}}_1 \rangle^N \langle \widetilde{\mathscr{D}}_2 \rangle^N \phi = 0 \,. \tag{V. 26}$$

Multiply (V. 24) by ϕ and (V. 26) by $G_N^{(\ell)}$; then take the difference and integrate over $x^3 dx \sin^2\theta \, d\theta$. By (IV. 6), we have

$$\left[\phi, G_N^{(\ell)} \right]_{x=\infty} - \left[\phi, G_N^{(\ell)} \right]_{x=0} = \phi(x', \theta') \quad . \tag{V. 27}$$

But the explicit form (V. 25) and the bracket formulas (IV. 13) tell us that

$$\left[\overline{K}_{(\alpha, i)}, \, G_N^{(\ell)} \right]_{x=\infty} = \left[\overline{I}_{(\alpha, i)}, \, G_N^{(\ell)} \right]_{x=0} = 0 \quad , \tag{V. 28}$$

and hence

$$K_{(\alpha, i)}(x', \theta') = - \left[K_{(\alpha, i)}, \, G_N^{(\ell)} \right]_{x=0} = - (\sigma_{\alpha, i})^{-1} \, c_{\alpha, i} \quad , \tag{V. 29a}$$

$$I_{(\alpha, i)}(x', \theta) = + \left[I_{(\alpha, i)}, \, G_N^{(\ell)} \right]_{x=\infty} = - (\sigma_{\alpha, i})^{-1} \, d_{\alpha, i} \quad . \tag{V. 29b}$$

Therefore,

$$\begin{aligned} G_N^{(\ell)}(x, \theta ; x', \theta') &= -\sum_{\alpha, i} I^{(\alpha, i)}(x, \theta) \;\; \sigma_{\alpha, i} \, K_{(\alpha, i)}(x', \theta'), \quad x < x' \\ &= -\sum_{\alpha, i} K^{(\alpha, i)}(x, \theta) \;\; \sigma_{\alpha, i} \, I_{(\alpha, i)}(x', \theta'), \quad x > x' \,. \end{aligned} \tag{V. 30}$$

For a comparison of this result with the $N \to \infty$ limit, derived in the previous section, we rephrase (V. 30) using the factor h_α of (III. 27).

$$G_N^{(\ell)}(x, \theta ; x', \theta') = \; g^N_{nn'}(x, x') \, R_n(\theta) \, R_n(\theta') \; ,$$

$$\begin{aligned} g^N_{nn'}(x, x') &= -\sum_{\alpha, i} h_\alpha \;\; \mathscr{I}_n(\lambda_{\alpha, i} x, \alpha) \;\; \sigma_{\alpha, i} \mathscr{K}_n(\lambda_{\alpha, i} x', \pi - \alpha), \qquad x < x' \\ &= -\sum_{\alpha, i} h_\alpha \, \mathscr{K}_n(\lambda_{\alpha, i} x, \alpha) \;\; \sigma_{\alpha, i} \;\; \mathscr{I}_n(\lambda_{\alpha, i} x', \pi - \alpha), \qquad x > x' . \end{aligned} \tag{V. 31}$$

IV. COMPARISON OF THE TRUNCATED AND EXACT BSE

A. Gaussian Quadrature

We turn again to the eigenvalue problem of Section III. D, and consider the N roots of

$$C_N^{1+\ell}(\cos\alpha) = 0 \quad . \tag{VI. 1}$$

By (II. 6), (III. 27), (III. 29), we have

$$\int_0^\pi R_n(\alpha)\, R_m(\alpha)\, \sin^2\alpha d\alpha = \sum_\alpha h_\alpha\, R_n(\alpha)\, R_m(\alpha) \tag{VI. 2}$$

where the sum is over just these roots, provided that

$$\ell \leq n \leq N + \ell - 1 \quad , \quad \ell \leq m \leq N + \ell \, , \tag{VI. 3}$$

because both sides of (VI. 2) are then equal to $\delta_{n,m}$. Observe that if $F(\alpha)$ is any polynomial in $\cos\alpha$ of degree $\leq 2N+1$, then $F(\alpha)\sin^{2\ell}\alpha$ can be written as a linear combination of the terms $R_n(\alpha)\, R_m(\alpha)$ with n, m within the limits of (VI. 3). Therefore

$$\int_0^\pi F(\alpha)\, \sin^{2\ell+2}\alpha d\alpha = \sum_\alpha h_\alpha\, \sin^{2\ell+2}\alpha\, F(\alpha) \tag{VI. 4}$$

is an exact relation for this class of $F(\alpha)$. Furthermore, the sum of (VI. 4) approximates the integral if $F(\alpha)$ is approximately equal to such a polynomial.

This is,essentially, the well-known rule for Gaussian quadrature[8] adapted here for the orthogonal family of $R_n(\alpha)$'s. The argument, when abstracted from the context of our general discussion, is at least as simple as other derivations. It applies to any system of orthogonal polynomials because the existence of a recursion relation like (II. 15a) is common to all. More precisely, let $\{\phi_n(x)\}$ be any complete, orthonormal family of polynomials, with ϕ_n of degree n, n=0, 1, 2, . . . , defined on $a \leq x \leq b$ by the weight function w(x), so that

$$\int_a^b \phi_n(x)\ \phi_m(x)\ w(x)\, dx = \delta_{nm} \quad . \tag{VI. 5}$$

Then the matrix M_{nm} ,

$$M_{nm} = \int_a^b \phi_n(x)\ x\ \phi_m(x)\ w(x)\ dx\ , \tag{VI. 6}$$

is tridiagonal, real, symmetric, with nonzero elements made up of the coefficients A_n, B_n that occur in

$$x\phi_n(x) = A_{n+1}\ \phi_{n+1}(x) + B_n\ \phi_n(x) + A_n\ \phi_{n-1}(x)\ . \tag{VI. 7}$$

The N × N truncation of M_{mn}, with indices restricted to $0 \leq n, m \leq N-1$ will have N eigenvalues λ .

Then the methods of Section III. D show that

(1) The λ's are the solutions of $\phi_N(x) = 0$,

(2) The components of the normalized eigenvectors $\underline{v}^{(\lambda)}$ are

$$v_n^{(\lambda)} = \frac{\phi_n(\lambda)}{\left\{ \sum_{n=0}^{N-1} \left[\phi_n(\lambda)\right]^2 \right\}^{1/2}}\ , \quad 0 \leq n \leq N-1\ , \tag{VI. 8}$$

(3) The Gaussian weight factors are

$$h_\lambda = \sum_{n=0}^{N-1} \left[\phi_n(\lambda)\right]^2 = \left[v_0^{(\lambda)} / \phi_0(\lambda)\right]^2\ , \tag{VI. 9}$$

(4) The Gaussian quadrature rule

$$\int_a^b F(x)\ w(x)\ dx \approx \sum h_\lambda\ F(\lambda) \tag{VI. 10}$$

is exact if F(x) is a polynomial of degree not greater than 2N+1. Thus the practical task of computing the zeros and the factors h_λ is reduced to the diagonalization problem for an elementary type of matrix.

For our purposes, the point of interest is the relation between (V. 23) and (V. 31). In particular, $g^N_{n,m'}$ is the Gaussian approximant of order N to $g_{n,n'}$. This confirms, formally, at least, that as $N \to \infty$,

$$G_N^{(\ell)} (x, \theta ; x', \theta) \to G^{\ell}(x, \theta ; x', \theta') . \tag{VI. 11}$$

Whatever the accuracy of the integral truncation method for the BSE, this formulation is more accurate than the differential method insofar as an integral is not exactly equal to its Gaussian approximant. Calculations of bound state-energies with the truncated differential BSE do, in fact, give good results, i.e., rapid convergence (to the correct answer) for increasing N, if the binding energy is not small. For example, our own calculations in the ladder approximation with all masses equal (to be described later) yielded good convergence for $(K/m) \gtrsim .1$, i.e., $(E/2m) \lesssim .995$. Calculation of the residue of the scattering amplitude at the bound state pole did less well, unless $(K/m) \lesssim .4$.

The quantity $\sigma_{\alpha, i}$ which enters into the integrals and sums has a denominator which includes

$$\cos \alpha (\omega_1^2 \cos^2 \alpha - k^2)^{1/2} \tag{VI. 12}$$

giving a simple pole at $\alpha = \frac{1}{2}\pi$ and (integrable) square root singularities at $\omega_i^2 \cos^2 \alpha = k^2$. The simple pole, treated as a principal value is not necessarily fatal to the Gaussian sum if the discrete α's are arranged symmetrically about the pole, as they are. However, the square root singularities, which appear at the edges of the integration interval at k=0, and move into the interior of the interval for $k > 0$, cause the convergence difficulty for bound states near threshold to which we have referred, and are fatal for scattering. What this means is that regardless of the auxiliary functions or methods used in the differential approach the boundary conditions at $x = \infty$ inherent in that method do not approximate or converge to the true asymptotic boundary conditions. One may expect – and this was the case in a test calculation we performed – that the calculated scattering phase shift oscillates about the correct result without convergence as $N \to \infty$, being either too large or too small depending on how the discrete α's for the relevant N (whether or not the calculation explicitly uses them) lie in the interval $(0, \pi)$ with respect to the singular points. A modification of the differential approach which circumvents this difficulty is given in the following paper.

We gratefully acknowledge the hospitality of the Lawrence Radiation Laboratory, Berkeley, and also of the Aspen Summer Institute of Physics where portions of this work were done.

REFERENCES AND FOOTNOTES

1. C. Schwartz, Phys. Rev. 137, B1717 (1965).
2. C. Schwartz and C. Zemach, Phys. Rev. 141, 1454 (1966), hereafter called SZ. We use the representations of the Green's function derived in this paper and the corresponding formulation of the scattering problem. A general introduction to the theory of the BSE is contained in C. Zemach, Selected Topics in Solid State and Theoretical Physics, Latin American Summer School of Physics, 1966 (Gordon and Breach, New York, 1968).
3. K. D. Rothe, Phys. Rev. 170, 1548 (1967);
   A. Pagnamenta, Nuovo Cimento 53A, 30 (1968);
   P. Narayanaswamy and A. Pagnamenta, Nuovo Cimento 53A, 635 (1968);
   K. Ladányi, Nuovo Cimento 56A, 173 (1968);
   W. B. Kaufmann, Phys. Rev. 172, 1697 (1968);
   E. zur Linden and H. Mitter, Nuovo Cimento 61B, 389 (1969).
4. M. Levine, J. Tjon, and J. Wright, Phys. Rev. 154, 1433 (1967);
   R. M. Saenger, J. Math. Phys. 8, 2366 (1967);
   R. Haymaker, Phys. Rev. Letters 18, 968 (1967), Phys. Rev. 165, 1790 (1968);
   K. Ladányi, Nuovo Cimento 61A, 173 (1969);
   B. C. McInnis, Phys. Rev. 183, 1474 (1969).
5. For example, W. Magnus, F. Oberhettinger, and R. P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, Third Edition (Springer Verlag, New York, 1966).
6. E. Coddington and N. Levinson, Theory of Ordinary Differential Equations (McGraw-Hill, New York, 1955).
7. If the δ-function is represented by $2\pi i\,\delta(t) = (t-i\epsilon)^{-1} - (t+i\epsilon)^{-1}$, its analytic continuation can be handled routinely, as was remarked by R. Saenger, Ref. 4.
8. For example, Z. Kopal, Numerical Analysis (John Wiley, New York, 1961).