UNITARY THREE-PARTICLE ON-SHELL T-MATRIX†

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ABSTRACT

A one-variable integral equation is derived whose solution gives a unitary description of the physical three-particle scattering matrix once phenomenological parameters describing the interior three-particle region are specified. This equation remains convergent, and the result unitary, in the zero-range limit, thus providing a unique construction of the three-particle T matrix from two particle phase shifts and binding energies as a first approximation.

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In this letter we prove that a unitary description of the on-shell three-particle T-matrix, comparable to the phase shift prescription for two-particle amplitudes, is provided by the solution of a one-variable integral equation. The phenomenological parameters are coefficients of the expansion of the interior[1] wave function in terms of any complete set which does not contribute in the asymptotic region and which has no discontinuities across the two- and three-particle cuts. The equation is convergent for all two-particle interactions bounded by decreasing exponentials in configuration space. This interaction region may be allowed to shrink to zero without destroying either convergence or unitarity, giving a unitary description of any three-particle system at a single energy in terms of two-particle phase shifts and binding energies plus (if desired) three-particle parameters.

The amplitudes $M_{\alpha\beta}$ defined by Faddeev[2] can be shown to satisfy two-variable equations[3,4] by introducing the coordinates $\underline{p}$ for the interacting pair and $\underline{q}$ for the free particle defined by Lovelace[5] (with the on-shell restriction $p^2 + q^2 = z$) and making the partial wave decomposition

$$< \underline{pq} \mid M_{\alpha\beta}(z) \mid \underline{p}_0 \underline{q}_0 > \doteq \sum_{\ell\lambda} M_{\ell\lambda}^{\alpha\beta}(p,q;z)\, Y_{J\ell\lambda}^{M}(\hat{p},\hat{q}) \tag{1}$$

To obtain the analog to the interior wave function discussed in CS, we define (J and M fixed)

$$I_{\ell\lambda}^{\alpha\beta}(p,q;z) \equiv M_{\ell\lambda}^{\alpha\beta}(p,q;z) - M_{\ell\lambda}^{\alpha\beta}(p,(z-p^2)^{\frac{1}{2}};z) \equiv (p^2+q^2-z)\, F_{\ell\lambda}^{\alpha\beta}(p,q;z) T_{\ell\lambda}^{\alpha\beta}(p;z) \tag{2}$$

and find that if $I_{\ell\lambda}^{\alpha\beta}$ is known, the three-particle on-shell T matrices satisfy the one variable equations

$$M^{\alpha\beta}_{\ell\lambda}(p,(z-p^2)^{\frac{1}{2}};z) \equiv T^{\alpha\beta}_{\ell\lambda}(p;z) = t^{\ell}_{\alpha}(p,p_0;p^2)\,\frac{2}{\sqrt{z-p^2}}\,\delta(p^2+q_0^2-z)\,\delta_{\alpha\beta}\,\delta_{\ell\ell_0}\,\delta_{\lambda\lambda_0}$$

$$-\frac{1}{\sqrt{z-p^2}}\int_0^{\infty} dp'^2 \int_{q_-^2}^{q_+^2} dq'^2\, t^{\ell}_{\alpha}(p,\bar{p};p^2) \sum_{\gamma\neq\alpha}\sum_{\ell'\lambda'} K^{\alpha\gamma}_{\ell\lambda\ell'\lambda'} F^{\gamma\beta}_{\ell'\lambda'}(p',q';z)\, T^{\alpha\beta}_{\ell\lambda}(p';z) \qquad (3)$$

$$-\frac{1}{\sqrt{z-p^2}}\int_0^{\infty} dp'^2 \int_{q_-^2}^{q_+^2} dq'^2\, \frac{t^{\ell}_{\alpha}(p,\bar{p};p^2)}{p'^2+q'^2-z} \sum_{\gamma\neq\alpha}\sum_{\ell'\lambda'} K^{\alpha\gamma}_{\ell\lambda\ell'\lambda'}\, T^{\alpha\beta}_{\ell'\lambda'}(p';z)$$

where

$$\bar{p}^2 = p'^2 + q'^2 - z + p^2$$

$$q_{\pm} = p'\tan\mu_{\alpha\gamma} \pm \sqrt{z-p^2}\,\sec\mu_{\alpha\gamma} \qquad (4)$$

$$\cos\mu_{\alpha\gamma} = \left[m_\alpha m_\gamma / (m_\alpha + m_\gamma)(m_\gamma + m_{\gamma'})\right]^{\frac{1}{2}}$$

and $K^{\alpha\gamma}_{\ell\lambda\ell'\lambda'}$ are purely geometrical recoupling coefficients which can be found, for example, from Ref. 3.

This equation for $T^{\alpha\beta}$ is precisely of the same form as the operator equation for $M_{\alpha\beta}$

$$\sum_{\gamma} K_{\alpha\gamma}(z) M_{\gamma\beta}(z) = \delta_{\alpha\beta} t_{\alpha}(z) = \delta_{\alpha\beta} t_{\beta}(z) = \sum_{\gamma} M_{\alpha\gamma}(z)\overline{K}_{\gamma\beta}(z) \qquad (5)$$

with

$$K_{\alpha\gamma}(z) = \delta_{\alpha\gamma} + (1-\delta_{\alpha\gamma}) t_{\alpha}(z) G_0(z) \text{ and } \overline{K}_{\gamma\beta}(z) = \delta_{\gamma\beta} + (1-\delta_{\gamma\beta}) G_0(z) t_{\beta}(z) \qquad (6)$$

if we add appropriate terms $\Delta K_{\alpha\gamma}$ and $\Delta\overline{K}_{\gamma\beta}$ in Eq. (6). The unitarity of Eq. (5) follows algebrically from the substitution of the definitions of K and $\overline{K}$ (Eq. (6)) to the left hand side of

$$V_{\alpha'}\sum_{\alpha\beta}K_{\alpha'\alpha}(z_1)(M_{\alpha\beta}(z_1)-M_{\alpha\beta}(z_2))\overline{K}_{\beta\beta'}(z_2)V_{\beta'} = V_{\alpha'}t_{\alpha'}(z_1)\overline{K}_{\alpha'\beta'}(z_2)V_{\beta'} - V_{\alpha'}K_{\alpha'\beta'}(z_1)t_{\beta'}(z_2)V_{\beta'} \tag{7}$$

Substituting Eq. (5) for K and $\overline{K}$ in the right hand side of Eq. (7) and making an obvious rearrangement, this same expression becomes

$$= V_{\alpha'}t_{\alpha'}(z_1)(G_0(z_2)-G_0(z_1))t_{\beta'}(z_2)V_{\beta'} + \delta_{\alpha'\beta'}\left\{t_{\alpha'}(z_1)[V_{\alpha'}-G_0(z_2)t_{\alpha'}(z_2)V_{\alpha'}]V_{\alpha'} - [V_{\alpha'}-t_{\alpha'}(z_1)G_0(z_2)V_{\alpha'}]t_{\alpha'}(z_2)V_{\alpha'}\right\} \tag{8}$$

The term proportional to $\delta_{\alpha'\beta'}$ vanishes because of the (convergent) two-particle Lippmann-Schwinger equations for $t_{\alpha'}$ and we can replace $t_{\alpha'}(z_1)$ in the first term by $\sum_{\gamma}\delta_{\alpha'\gamma}t_{\gamma}(z_1)$; this plus the corresponding replacement on the right allow us to use Eq. (5) to re-express the result as

$$= -V_{\alpha'}\sum_{\alpha\gamma}K_{\alpha'\alpha}(z_1)M_{\alpha\gamma}(z_1)(G_0(z_1)-G_0(z_2))\sum_{\gamma\beta}M_{\gamma\beta}(z_2)K_{\beta\beta'}(z_2)V_{\beta'} \tag{9}$$

Since Faddeev has shown[2] that inverses to $V_{\alpha'}K_{\alpha'\alpha}$ and $\overline{K}_{\beta\beta'}V_{\beta'}$ exist, the equality of Eqs. (7) and (9) then established the full unitarity relation

$$M_{\alpha\beta}(z_1)-M_{\alpha\beta}(z_2) = -\sum_{\gamma}M_{\alpha\gamma}(z_1)(G_0(z_1)-G_0(z_2))\sum_{\gamma}M_{\gamma\beta}(z_2) \tag{10}$$

This proof of unitarity would also suffice for $T_{\alpha\beta}$ if the additional term $V_{\alpha'}t_{\alpha'}(z_1)\Delta\overline{K}_{\alpha'\beta'}(z_2)V_{\beta'} - V_{\alpha'}\Delta K_{\alpha'\beta'}(z_1)t_{\beta'}(z_2)V_{\beta'}$ vanished. Since we are interested here only in a formalism applicable to the analysis of physical three-particle systems, we need not require general off-shell three-particle unitarity, but only that this term vanish in the limit $z_1 \to z + i0$, $z_2 \to z - i0$ with $z = p^2 + q^2$. This happens automatically for all terms which are independent of the interior function F, since these contribute to ΔK and $\Delta\overline{K}$ only terms proportional to

$$(p^2 + q^2 - z)^{-1}\left[t^{\ell}_{\alpha}(p,\bar{p};z-q^2) - t^{\ell}_{\alpha}(p,\bar{p};p^2)\right] \tag{11}$$

and Kowalski has shown[6] that the difference between these full and half off-shell two-particle t-matrices is always proportional to $(p^2 + q^2 - z)$ times a remainder function which itself vanishes on shell. The remaining terms are proportional to the difference $F^{\alpha'\beta'}_{\ell'\lambda'}(p',q';z_1) - F^{\alpha'\beta'}_{\ell'\lambda'}(p',q';z_2)$, since the singularity due to G_0 has been removed by definition, Eq. (2). Thus the only requirement on the interior wave functions F needed for T determined from Eq. (3) to satisfy on-shell three-particle unitarity is that these functions have no discontinuity across any of the three (or two) particle branch cuts in the physical region. Hence, we can use any such complete set F_n to expand the interior wave function (provided the integrals converge), and the coefficients of these functions will serve as "phase shifts" at that energy of the three-particle system, as already discussed in CS.

This proof of unitarity still fails if the operator for inverting Eq. (3) does not exist. The existence of this operator was proved in CS, but the proof was cumbersome and required a finite range cutoff R. By using a representation of the two-particle half off-shell t matrices valid[7] for any interaction bounded by $Ae^{-\mu x}$ in configuration space, or equivalently whose Born approximation is bounded by $A/(\mu^2 + (\underline{p} - \underline{p}_0)^2)$ in momentum space, the convergence proof can be made directly from Eq. (3). This representation is

$$\begin{aligned} t^{\ell}_{\alpha}(p,k;p^2) &= \tau^{\ell}_{\alpha}(p)\int_{\mu^2}^{\infty} d\beta^2 C(\beta^2)\,\frac{\beta^2(\beta^2 + 4k^2)}{(p^2-k^2)^2 + 2\beta^2(p^2-k^2) + \beta^4} \\ &= \tau^{\ell}_{\alpha}(p)\left[1 + (k^2 - p^2)\int_{\mu^2}^{\infty} d\beta^2 C(\beta^2)\,\frac{(2\beta^2 + p^2 - k^2)}{(p^2-k^2)^2 + 2\beta^2(p^2+k^2) + \beta^4}\right] \end{aligned} \tag{12}$$

where

$$\tau^{\ell}_{\alpha}(p^2) = e^{i\delta^{\ell}_{\alpha}(p)} \sin\delta^{\ell}_{\alpha}(p)/p \quad \text{and} \quad \int_{\mu^2}^{\infty} d\beta^2 C(\beta^2) = 1 \tag{13}$$

This representation allows us to represent the kernel to be integrated over p' in the last term of Eq. (3) as

$$\frac{\tau^{\ell}_{\alpha}(p)}{\sqrt{z-p^2}}\left[\int_{L_-}^{L_+} K^{\alpha\gamma}_{\ell\lambda\ell'\lambda'} \frac{dy}{y} + \int_{\mu^2}^{\infty} d\beta^2 C(\beta^2) \int_{L_-}^{L_+} dy \frac{(2\beta^2 - y)}{(y+\beta^2)^2 + 4\beta^2 p^2} K^{\alpha\gamma}_{\ell\lambda\ell'\lambda'}\right] \equiv Q^{\alpha\gamma}_{\ell\lambda\ell'\lambda'}(p, p') \tag{14}$$

where

$$L_{\pm} = \left[(p' \pm \sqrt{z-p^2}\,\sin\mu_{\alpha\gamma})^2 - p^2\cos^2\mu_{\alpha\gamma}\right] / \cos^2\mu_{\alpha\gamma} \tag{15}$$

The asymptotic convergence of this kernel follows immediately from the fact that for $z > p^2$ the limits have complex conjugate imaginary parts and hence that the result of the y integration is bounded by $\pi/2$; by making the change of variable $(p^2 - z)^{\frac{1}{2}} = r\sin\phi$, $p' = r\cos\phi$ it then follows from the fact that $\tau(p)$ is bounded by A/p^2 that the integral $\int^{\infty} dp^2 \int^{\infty} dp'^2 Q^2(p,p')$ converges at least as well as $\int^{\infty} r^3 dr/r^6$ at the upper limit. The only singularity is therefore the three-particle scattering branch cut in the physical region, which occurs only in the on-shell term and not in the off-shell extension (cf. second form of Eq. (12)). Since the geometrical factor $K_{\ell\lambda\ell'\lambda'}$ is bounded by a constant (in fact is a constant for the $J = \ell = \lambda = 0$ amplitude), there are two logarithmic singularities at

$$p' = |\,(z-p^2)^{\frac{1}{2}} \sin\mu_{is} \pm p\cos\mu_{is}\,| \tag{16}$$

where the singularity from the larger value of p' always comes from the lower limit of integration, while the smaller comes from L_+ if $p^2 < z\sin^2\mu$ and L_- otherwise. Since these logarithmic singularities are integrable, they can be removed by any convenient trick in the numerical solution of Eq. (4) (e.g. one can add and subtract the value of $F(p')$ at the two branch points and perform the integral over the logarithm explicitly). It is important to realize that, in contrast to other treatments of this branch cut (for example, that due to D. D. Brayshaw[8]), the only contribution from the cut comes in the physical region, and is explicitly known, independent of the dynamics. This is because we have put p on shell rather than q. The usual form of the Faddeev-Lovelace (or Amado[9] or Mitra[10]) equations with separable interactions takes q^2 to $+\infty$ and hence the energy of the interacting pair gets so negative that "potential singularities" have to be taken into account. Brayshaw[8] notes that these singularities are distant in the non-relativistic case, but if we treat, for example, the nucleon as a bound state of the πN system in a $\pi N \rightarrow 2\pi N$ reaction, the "potential" singularities are just as near as the region of interest. It appears that our approach offers advantages for such calculations in that only the necessary physical scattering singularity appears whether or not we include the off-shell correction.

Since our proof of convergence given above guarantees the existence of a resolvent kernel for Eq. (3) even in the zero-range limit ($\mu^2 \rightarrow \infty$) for the two-particle interactions, in this limit Eq. (3) provides one-variable integral equations for the three-particle T matrix using only physical two-particle phase shifts and binding energies. Note that nowhere have we made any high-energy approximation, so this limiting equation is valid as a zero-range approximation at any energy. Whether it is a good <u>physical</u> approximation will depend on whether these interior effects are physically significant or not; in particular, we know that this limit <u>cannot</u> be used for three-particle bound states, since Thomas[11] has shown that the zero-range limit gives infinite binding to the ground states of such systems.

We also note that the unitarity proof breaks down for a subtle reason if the two-particle subsystems have bound states. Then the t-matrices in the kernel have poles, and the T-matrix is undefined at these points. This same difficulty exists in the ordinary Faddeev equations, but can be solved either by iteration or by explicitly separating out the coefficients of these singular terms and writing equations coupling them to the continuum terms. Explicitly, if the bound-state residues in the two-particle t matrices are given by

$$\lim_{p^2 \to -\epsilon_\alpha^2} (p^2 + \epsilon_\alpha^2)\, t_\alpha^\ell(p,k;p^2) = \phi_{\epsilon_\alpha}^\alpha(i\epsilon_\alpha)\phi_{\epsilon_\alpha}^\alpha(k) \tag{17}$$

and we make in Eq. (3) the substitution

$$T_{\ell\lambda}^{\alpha\beta}(p;z) \to \frac{P_{\ell\lambda}^{\alpha\beta}(z+\epsilon_\alpha^2)}{p^2+\epsilon_\alpha^2} + T_{\ell\lambda}^{\alpha\beta}(p;z) \tag{18}$$

the resulting equation can still be solved for $T_{\ell\lambda}^{\alpha\beta}$ if we couple it to the system

$$\begin{aligned} P_{\ell\lambda}^{\alpha\beta}(z+\epsilon_\alpha^2) &= \phi_\alpha^2(i\epsilon_\alpha)\frac{2}{\sqrt{z+\epsilon_\alpha^2}}\,\delta(q_0^2 - z - \epsilon_\alpha^2)\,\delta_{\alpha\beta}\delta_{\ell\ell_0}\delta_{\lambda\lambda_0} \\ &- \frac{\phi_{\epsilon_\alpha}^\alpha(i\epsilon_\alpha)}{\sqrt{z+\epsilon_\alpha^2}} \int_0^\infty dp'^2 \int_{\ell_-}^{\ell_+} dq'^2\, \phi_{\epsilon_\alpha}^\alpha(\bar{p}) \sum_{\gamma\neq\alpha}\sum_{\ell'\lambda'} K_{\ell\lambda\ell'\lambda'}^{\alpha\gamma}\left[F_{\ell'\lambda'}^{\alpha\gamma}(p'q';z) + \frac{1}{p'^2+q'^2-z}\right] \\ &\left[\sum_{\epsilon_\gamma} \frac{P^{\alpha\gamma}(z+\epsilon_\gamma^2)}{p'^2+\epsilon_\gamma^2} + T_{\ell'\lambda'}^{\alpha\gamma}(p';z)\right] \end{aligned} \tag{19}$$

for the $P_{\ell\lambda}^{\alpha\beta}$ and solve for both P and T at the same time; the P's are, of course, simply the amplitudes for elastic scattering and rearrangement collusions. The system is still convergent, and the unitarity relation now explicitly contains the two-particle branch cuts due to these terms. This is still true if we make the zero-range approximation for the three-particle interior region (F = 0), or the two-particle zero-range ($\mu^2 \rightarrow \infty$) approximation, which leaves only the physical phase shifts and binding energies of the two-particle subsystems as input.

This paper completes the mathematical aspects of a program, whose physical motivation has been described in more detail elsewhere[12]; the objective is to give a complete description of three-particle systems in terms of two-particle observables (phase shifts and binding energies), the physical wave functions corresponding to these observables, and phenomenological parameters describing the interior three-particle wave function; the latter can now be fitted by three-particle experiments in a unique way. As has already been emphasized[12], this demonstrates conclusively that, in principle, the wave function for two strongly interacting particles inside the range of forces can be constructed from quantum mechanical observables, given a sufficiently rich body of three-particle data. An unexpected bonus is that this description continues to be both convergent and unitary in the zero-range limit, thus giving a unique construction of the on-shell three-particle T matrix from on-shell two-particle t-matrices which is valid, as a first approximation, at <u>any</u> energy. Whether this will be a good approximation requires further investigation of the corrections for specific physical systems, which can be obtained by model calculations of the interior wave function and data analysis of three-particle systems. Finally we emphasize again that if covariant kinematics and two-particle t-matrices which satisfy a covariant Lippmann-Schwinger equation (e.g. the Blankenbecler-Sugar equation[13]) are used, this analysis is applicable at any energy, and is not restricted to non-relativistic problems.

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FOOTNOTES

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